# Extended Monotone Methods for Hyperbolic Problems in Three Variables

J. O. Adeyeye[1], R. Okojie[2], and S. G. Pandit[3] *

*Abstract*—**We consider Initial-Boundary value problems associated with nonlinear hyperbolic partial differential equation in three independent variables, in a general setting wherein the forcing function is a sum of two monotonic functions. Employing natural lower-upper and coupled lower-upper solutions, we develop iterative schemes which converge uniformly and monotonically to the minimal-maximal and coupled minimal-maximal solutions of the problem.**

***Keywords:** lower-upper solutions; coupled lower-upper solutions; minimal-maximal solutions, monotonic functions*

## 1 Introduction

Monotone iterative techniques, which yield sequences of linear monotone iterates converging uniformly to the maximal and minimal solutions of nonlinear problems have been successfully developed in [6] for problems including, but not limited to

$$u' = f(t,u), \quad u = u(t) \tag{1.1}$$

and its two dimensional analog

$$u_{xy} = f(x,y,u,u_x,u_y), \quad u = u(x,y). \tag{1.2}$$

The techniques initiated in [6, 8, 9] for the two dimensional initial-boundary value problems (IBVPs, for short) and the periodic-boundary value problems (PBVPs, for short) have recently been extended in [1, 11] using the unified approach in [7]. Similar problems for higher hyperbolic partial differential equations are treated in [2, 12]. Conlan [5] was the first to develop a numerical technique (Euler-Cauchy polygon method) for the nonlinear hyperbolic problem

$$\begin{aligned} u_{xyz} &= F(x,y,u,u_x,u_y,u_z,u_{xy},u_{xz},u_{yz}), \\ u &= u(x,y,z). \end{aligned} \tag{1.3}$$

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Monotone iterative techniques for the IBVPs and PBVPs associated with (1.3) were later derived in [4] by employing the variation of parameters formula and the comparison theorems developed in [3]. The important special case, when $F$ is non-increasing in the last seven variables, is not covered by any of the results obtained in [4]. To this end, it is apt to note [10] that the successive approximations are especially useful, even for numerical computations, when the nonlinearities are nonincreasing.

In this paper we consider the IBVP associated with (1.3) when $F$ admits a decomposition $F = f + g$, where $f$ is nondecreasing and $g$ is nonincreasing in the last seven variables. Utilizing the natural lower-upper solutions and the coupled lower-upper solutions, we derive monotone iterative schemes in a unified setting. Not only the existing results become very special cases of our unified approach, but we also obtain some new ones.

## 2 Notations and Definitions

For $a$, $b$, $c \in \mathbb{R}$, $a > 0$, $b > 0$ and $c > 0$, $I_a$, $I_b$ and $I_c$, respectively denote the intervals $[0,a]$, $[0,b]$ and $[0,c]$; $R_{ab}$, $R_{ac}$ and $R_{bc}$ respectively denote the rectangles $I_a \times I_b$, $I_a \times I_c$ and $I_b \times I_c$; and, $P = P_{abc}$ denotes the rectangular parallelopiped $I_a \times I_b \times I_c$. The three variables $x$, $y$, and $z$ are the independent variables and we always assume, without explicit mention, that $x \in I_a$, $y \in I_b$, $z \in I_c$, $(x,y) \in R_{ab}$, $(x,z) \in R_{ac}$, $(y,z) \in R_{bc}$ and $(x,y,z) \in P$. When there is no danger of ambiguity, we shall suppress the independent variable(s) and/or its (their) domain(s). The hypercylinder $\mathbb{C}$ in $\mathbb{R}^{10}$ is defined by $\mathbb{C} = \{(x,y,z,x_4,...,x_{10}); (x,y,z) \in P, \text{ and } x_i \in \mathbb{R}, 4 \le i \le 10\}$. For $u \in C^3[P,\mathbb{R}]$, $\langle u \rangle$ denotes the 7-tuple $(u,u_x,u_y,u_z,u_{xy},u_{xz},u_{yz})$ on $P$. We assume throughout that $[M] = (M_1,-M_2,-M_3,-M_4,M_5,M_6,M_7) \in \mathbb{R}^7$ denotes the constant vector. The expression $[M] \cdot \langle u \rangle = M_1u - M_2u_x - M_3u_y - M_4u_z + M_5u_{xy} + M_6u_{xz} + M_7u_{yz}$ is the usual inner product. All the inequalities among vectors are componentwise. To shorten the otherwise lengthy expressions, we shall frequently denote expressions such as

$$(u(x,y,0),\ u_x(x,y,0),\ u_y(x,y,0),\ u_{xy}(x,y,0))$$

by $(u,u_x,u_y,u_{xy})(x,y,0)$.

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Under these notations, consider the IBVP

$$u_{xyz} = f(x,y,z,\langle u\rangle) + g(x,y,z,\langle u\rangle); \tag{2.1}$$

$$\begin{aligned} u(x,y,0) &= \alpha(x,y), \\ u(x,0,z) &= \beta(x,z), \\ u(0,y,z) &= \gamma(y,z), \end{aligned} \tag{2.2}$$

$$\begin{aligned} \alpha(x,0) &= \beta(x,0), \\ \alpha(0,y) &= \gamma(y,0), \\ \beta(0,z) &= \gamma(0,z), \end{aligned} \tag{2.3}$$

$$\alpha(0,0) = \beta(0,0) = \gamma(0,0) = u_0, \tag{2.4}$$

where $f$, $g \in C[\mathbb{C},\mathbb{R}]$, $f$ is nondecreasing in $\langle u\rangle$, $g$ is nonincreasing in $\langle u\rangle$, $\alpha \in C^2[R_{ab},\mathbb{R}]$, $\beta \in C^2[R_{ac},\mathbb{R}]$ and $\gamma \in C^2[R_{bc},\mathbb{R}]$.

**Definition 2.1.** Functions $v^0$, $w^0 \in C^3[P,\mathbb{R}]$, $\langle v^0\rangle \le \langle w^0\rangle$, are said to be natural lower-upper solutions relative to (2.1)–(2.4) if

$$v^0_{xyz} \le f(x,y,z,\langle v^0\rangle) + g(x,y,z,\langle v^0\rangle);$$

$$w^0_{xyz} \ge f(x,y,z,\langle w^0\rangle) + g(x,y,z,\langle w^0\rangle);$$

$$\begin{aligned} (v^0, v^0_x, v^0_y, v^0_{xy})(x,y,0) &\le (\alpha,\alpha_x,\alpha_y,\alpha_{xy})(x,y) \\ &\le (w^0, w^0_x, w^0_y, w^0_{xy})(x,y,0); \end{aligned}$$

$$\begin{aligned} (v^0, v^0_x, v^0_z, v^0_{xz})(x,0,z) &\le (\beta,\beta_x,\beta_z,\beta_{xz})(x,z) \\ &\le (w^0, w^0_x, w^0_z, w^0_{xz})(x,0,z); \end{aligned}$$

$$\begin{aligned} (v^0, v^0_y, v^0_z, v^0_{yz})(0,y,z) &\le (\gamma,\gamma_y,\gamma_z,\gamma_{yz})(y,z) \\ &\le (w^0, w^0_y, w^0_z, w^0_{yz})(0,y,z). \end{aligned}$$

**Definition 2.2.** Functions $v^0$, $w^0 \in C^3[P,\mathbb{R}]$, $\langle v^0\rangle \le \langle w^0\rangle$, are said to be coupled lower-upper solutions of Type I relative to (2.1)–(2.4) if

$$v^0_{xyz} \le f(x,y,z,\langle v^0\rangle) + g(x,y,z,\langle w^0\rangle);$$

$$w^0_{xyz} \ge f(x,y,z,\langle w^0\rangle) + g(x,y,z,\langle v^0\rangle);$$

$$\begin{aligned} (v^0, v^0_x, v^0_y, v^0_{xy})(x,y,0) &\le (\alpha,\alpha_x,\alpha_y,\alpha_{xy})(x,y) \\ &\le (w^0, w^0_x, w^0_y, w^0_{xy})(x,y,0); \end{aligned}$$

$$\begin{aligned} (v^0, v^0_x, v^0_z, v^0_{xz})(x,0,z) &\le (\beta,\beta_x,\beta_z,\beta_{xz})(x,z) \\ &\le (w^0, w^0_x, w^0_z, w^0_{xz})(x,0,z); \end{aligned}$$

$$\begin{aligned} (v^0, v^0_y, v^0_z, v^0_{yz})(0,y,z) &\le (\gamma,\gamma_y,\gamma_z,\gamma_{yz})(y,z) \\ &\le (w^0, w^0_y, w^0_z, w^0_{yz})(0,y,z). \end{aligned}$$

**Lemma 2.1** [4] *Suppose that $m \in C^3[P,\mathbb{R}]$ and satisfies $m_{xyz} \ge [M]\cdot\langle m\rangle$ on $P$, where $M_i \ge 0$ for $5 \le i \le 7$, and*

$$\begin{aligned} &M_2 = M_5M_6,\ M_3 = M_5M_7, \\ &M_4 = M_6M_7, \ \text{ and } M_1 = M_5M_6M_7. \end{aligned} \tag{2.5}$$

*If*

$$\begin{aligned} (m, m_x, m_y, m_{xy})(x,y,0) &\ge (0,0,0,0); \\ (m, m_x, m_z, m_{xz})(x,0,z) &\ge (0,0,0,0); \\ (m, m_y, m_z, m_{yz})(0,y,z) &\ge (0,0,0,0); \text{ and} \\ m(x,0,0) = m(0,y,0) &= m(0,0,z) = 0, \end{aligned}$$

*then*

$$\begin{aligned} &(m, m_x, m_y, m_z, m_{xy}, m_{xz}, m_{yz})(x,y,z) \\ &\quad \ge (0,0,0,0,0,0,0) \text{ everywhere in } P. \end{aligned}$$

# 3 Main Results

We employ the following two iterative schemes in the development of monotone iterative schemes for the IBVP (2.1)–(2.4).

$$\begin{aligned} v^n_{xyz}(x,y,z) &= f(x,y,z,\langle v^{n-1}(x,y,z)\rangle) + \\ &\quad g(x,y,z,\langle w^{n-1}(x,y,z)\rangle), \\ v^n(x,y,0) &= \alpha(x,y),\ v^n(x,0,z) = \beta(x,z), \\ v^n(0,y,z) &= \gamma(y,z), \\ \alpha(x,0) = \beta(x,0),\ &\alpha(0,y) = \gamma(y,0),\ \beta(0,z) = \gamma(0,z), \\ \alpha(0,0) &= \beta(0,0) = \gamma(0,0) = u_0, \\ w^n_{xyz}(x,y,z) &= f(x,y,z,\langle w^{n-1}(x,y,z)\rangle) + \\ &\quad g(x,y,z),\langle v^{n-1}(x,y,z)\rangle) \\ w^n(x,y,0) &= \alpha(x,y),\ w^n(x,0,z) = \beta(x,z), \\ w^n(0,y,z) &= \gamma(y,z), \\ \alpha(x,0) = \beta(x,0),\ &\alpha(0,y) = \gamma(y,0),\ \beta(0,z) = \gamma(0,z), \\ \alpha(0,0) &= \beta(0,0) = \gamma(0,0) = u_0, \end{aligned} \tag{S$_1$}$$

and

$$\begin{aligned} v^n_{xyz}(x,y,z) &= f(x,y,z),\langle w^{n-1}(x,y,z)\rangle) + \\ &\quad g(x,y,z),\langle v^{n-1}(x,y,z)\rangle) \\ v^n(x,y,0) &= \alpha(x,y),\ v^n(x,0,z) = \beta(x,z), \\ v^n(0,y,z) &= \gamma(y,z), \\ \alpha(x,0) = \beta(x,0),\ &\alpha(0,y) = \gamma(y,0),\ \beta(0,z) = \gamma(0,z), \\ \alpha(0,0) &= \beta(0,0) = \gamma(0,0) = u_0, \\ w^n_{xyz}(x,y,z) &= f(x,y,z),\langle v^{n-1}(x,y,z)\rangle) + \\ &\quad g(x,y,z),\langle w^{n-1}(x,y,z)\rangle) \\ w^n(x,y,0) &= \alpha(x,y),\ w^n(x,0,z) = \beta(x,z), \\ w^n(0,y,z) &= \gamma(y,z), \\ \alpha(x,0) = \beta(x,0),\ &\alpha(0,y) = \gamma(y,0),\ \beta(0,z) = \gamma(0,z), \\ \alpha(0,0) &= \beta(0,0) = \gamma(0,0) = u_0. \end{aligned} \tag{S$_2$}$$

In the space $C^3[P, \mathbb{R}]$, we introduce the following norm:

$$||u|| = \max_{i,j,k}\Big\{\sup \left|\partial^{i+j+k}u(x,y,z)/\partial x^i y^j x^k\right|,\ (x,y,z) \in P\Big\},$$

where $0 \le i,j,k \le 3,$ and $i+j+k < 3$. For $v^0,\ w^0 \in C^3[P,\mathbb{R}]$, such that $\langle v^0\rangle \le \langle w^0\rangle$ on $P$, denote the closed set $\{(x,y,z,u) : (x,y,z) \in P$, and $\langle v^0\rangle \le \langle u\rangle \le \langle w^0\rangle$ on $P\}$, by $\Omega$.

Our first result, which yields natural sequences by utilizing natural lower-upper solutions in conjunction with the scheme $(S_1)$, includes and improves the earlier results in [4].

**Theorem 3.1.** *Assume that*
$(A_1)$ $v^0,\ w^0 \in C^3[P,\mathbb{R}]$ *are natural lower-upper solutions of* (2.1)–(2.4) *such that* $\langle v^0\rangle \le \langle w^0\rangle$ on $P$;

$(A_2)$ $f,\ g \in C[\mathbb{C},\mathbb{R}]$, $f$ *is nondecreasing in* $\langle u\rangle$ *and* $g$ *is nonincreasing in* $\langle u\rangle$.

*Then there exist sequences* $\{v^n\}, \{w^n\}$ *in* $\Omega$ *such that* $\{[v^n]\}$ *is nonincreasing,* $\{w^n\}$ *is nonincreasing and satisfies* $\langle v^n\rangle \to \langle v\rangle$, $\langle w^n\rangle \to \langle w\rangle$, *where* $v$ *and* $w$ *are coupled minimal and maximal solutions respectively of* $(2.1)-(2.4)$ *on* $P$*, that is* $v$ *and* $w$ satisfy

$$v_{xy} = f(x,y,z,\langle v\rangle) + g(x,y,z,\langle w\rangle), (x,y) \in P,$$
$$w_{xy} = f(x,y,z,\langle w\rangle) + g(x,y,z,\langle v\rangle), (x,y) \in P,$$

provided $\langle v^0\rangle \le \langle v^1\rangle$ and $\langle w^1\rangle \le \langle w^0\rangle$ on $P$. Also, $\langle v^0\rangle \le \langle v\rangle \le \langle w\rangle \le \langle w^0\rangle$ on $P$.

**Proof.** For $n = 1,2,3,...$, define the iterates as given by scheme $(S_1)$. It is easy to see that the solutions of these IBVPs exist and are unique for each $n = 1,2,3,....$ We prove that the sequences $\{v^n\}$ and $\{w^n\}$ satisfy the (natural) monotone behavior

$$\langle v^0\rangle \le \langle v^1\rangle \le \langle v^2\rangle \le ... \le \langle v^n\rangle \le \langle w^n\rangle \le ... \le \langle w^2\rangle \le \langle w^1\rangle \le \langle w^0\rangle \text{ on } P. \tag{3.1}$$

By assumption, we already have $\langle v^0\rangle \le \langle v^1\rangle$ and $\langle w^1\rangle \le \langle w^0\rangle$ on $P$. We assert that

$$\langle v^1\rangle \le \langle w^1\rangle \text{ on } R. \tag{3.2}$$

To this end, setting $p = w^1 - v^1$ we note that

$$(p,p_x,p_y,p_{xy})(x,y,0) = (0,0,0,0);$$
$$(p,p_x,p_y,p_{xz})(x,0,z) = (0,0,0,0);$$
$$(p,p_x,p_y,p_{yz})(0,y,z) = (0,0,0,0);$$
$$p(x,0,0) = 0;\ p(0,y,0) = 0;\ p(0,0,z) = 0,$$

and $p_{xy} = f(\langle v^0\rangle) + g(\langle w^0\rangle) - f(\langle w^0\rangle) + g(\langle v^0\rangle) \ge 0$, in view of the monotone character of $f(\langle u\rangle)$ , $g(\langle u\rangle)$ and the assumption $\langle v^0\rangle \le \langle w^0\rangle$. Hence, Lemma 2.1 yields $\langle p\rangle \ge \langle 0\rangle$, which, in turn establishes (3.2). Thus, we have $\langle v^0\rangle \le \langle v^1\rangle \le \langle w^1\rangle \le \langle w^0\rangle$ on $P$. Assume that, for some $n > 1$,

$$\langle v^{n-1}\rangle \le \langle v^n\rangle \le \langle w^n\rangle \le \langle w^{n-1}\rangle \text{ on } R. \tag{3.3}$$

We shall prove that

$$\langle v^n\rangle \le \langle v^{n+1}\rangle \le \langle w^{n+1}\rangle \le \langle w^n\rangle \text{ on } R. \tag{3.4}$$

To do this, let $p = v^{n+1} - v^n$. As before, by (3.3) and Lemma 2.1 we have $\langle v^n\rangle \le \langle v^{n+1}\rangle$ on $P$. A similar argument yields $\langle w^{n+1}\rangle \le \langle w^n\rangle$ on $P$. Inequalities in (3.4) now follow in view of (3.3), Lemma 2.1 and the monotone nature of $f(\langle u\rangle), g(\langle u\rangle)$. Hence, by induction, (3.1) is established. using the monotone character of the sequences $\{v^n\}, \{w^n\}$ in $\Omega$, together with the Ascoli-Arzela theorem, it follows by using standard arguments that $\langle v^n\rangle \to \langle v\rangle$, $\langle w^n\rangle \to \langle w\rangle$, where $v, w \in C^3[R,R]$ are coupled solutions of the IBVP (2.1)–(2.4). To show that $v, w$ are in fact the coupled extremal solutions of (2.1)–(2.4), let $u \in \Omega$ be any solution of (2.1)–(2.4). Assume for some $n > 1$, that we have

$$\langle v^n\rangle \le \langle u\rangle \le \langle w^n\rangle \text{ on } R. \tag{3.5}$$

Then letting $p = u - v^{n+1} + 1$, we again note that

$$(p,p_x,p_y,p_{xy})(x,y,0) \equiv (0,0,0,0);$$
$$(p,p_x,p_z,p_{xz})(x,0,z) \equiv (0,0,0,0);$$
$$(p,p_y,p_z,p_{yz})(0,y,z) \equiv (0,0,0,0);$$
$$p(x,0,0) = p(0,y,0) = p(0,0,z) = 0,$$

and $p_{xy} = f(\langle u\rangle) - f(\langle v^n\rangle) + g(\langle u\rangle) - g(\langle w^n\rangle) \ge 0$, because of the monotonicity of $f(\langle u\rangle)$ and $g(\langle u\rangle)$ and (3.5). Lemma 2.1 therefore implies that $\langle v^n\rangle \le \langle u\rangle$ on P. Similarly, $\langle u\rangle \le \langle w^n\rangle$ on $P$. It therefore follows by induction that $\langle v^n\rangle \le \langle u\rangle \le \langle w^n\rangle$ on $P$ for all $n = 1,2,3,\ldots$ and this implies, in turn, that $\langle v\rangle \le \langle u\rangle \le \langle w\rangle$ on $P$, proving thereby that $v$ and $w$ are coupled minimal and maximal solutions respectively of the IBVP (2.1)–(2.4). This completes the proof.

**Corollary 3.1.** *In addition to the assumptions of Theorem 3.1, suppose that* $f$, $g$ *satisfy Lipschitz conditions, then* $v \equiv u \equiv w$ is the*unique solution of* (2.1)–(2.4) *in* $\Omega$.

In the next result, we employ scheme $(S_2)$ and obtain sequences which are alternatively monotone, under the same hypotheses $(A_1)$ and $(A_2)$ as in Theorem 3.1, but by dropping the assumptions $\langle v^0\rangle \le \langle v^1\rangle$ ,$\langle w^1\rangle \le \langle w^0\rangle$ and adding the new ones $\langle v^0\rangle \le \langle v^2\rangle, \langle w^2\rangle \le \langle w^0\rangle$.

**Theorem 3.2** *Let the assumptions* $(A_1)$ *and* $(A_2)$ *of Theorem 3.1 hold. Then, for any solution* $u$ *of* (2.1)–(2.4)*, the sequences* $\{v^n\}, \{w^n\}$ *in the iterative scheme*

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(S$_2$) *satisfy the inequalities*

$$\langle v^0\rangle \le \langle v^2\rangle \le \cdots \le \langle v^{2n-2}\rangle \le \{u\} \le \langle v^{2n-1}\rangle \le \cdots \le \langle v^3\rangle \le \langle v^1\rangle$$
$$\langle w^1\rangle \le \langle w^3\rangle \le \cdots \le \langle w^{2n-2}\rangle \le \{u\} \le \langle w^{2n-2}\rangle \le \cdots \le \langle w^2\rangle \le \langle w^0\rangle$$

*on $P$, provided $\langle v^0\rangle \le \langle v^2\rangle$ and $\langle w^2\rangle \le \langle w^0\rangle$ on $R$. Moreover, the monotone sequences $\{v^{2n}\}$, $\{v^{2n-1}\}$, $\{w^{2n}\}$, $\{w^{2n-1}\}$ converge uniformly to $v$, $w$, $v^*$, and $w^*$ respectively on $P$ and satisfy the equations*

$$\begin{gathered}
w_{xyz} = f(x,y,z,\langle v^*\rangle) + g(x,y,z,\langle v\rangle);\\
w(x,y,0) = \alpha(x,y),\ w(x,0,z) = \beta(x,z),\\
w(0,y,z) = \gamma(y,z);\ w(0,0,0) = u_0,\\
v_{xyz} = f(x,y,z,\langle w^*\rangle) + g(x,y,z,\langle w\rangle);\\
v(x,y,0) = \alpha(x,y),\ v(x,0,z) = \beta(x,z),\\
v(0,y,z) = \gamma(y,z);\ v(0,0,0) = u_0,\\
w^*_{xyz} = f(x,y,z,\langle v\rangle) + g(x,y,z,\langle v^*\rangle);\\
w^*(x,y,0) = \alpha(x,y),\ w^*(x,0,z) = \beta(x,z),\\
w^*(0,y,z) = \gamma(y,z);\ w^*(0,0,0) = u_0,\\
v^*_{xyz} = f(x,y,z,\langle w\rangle) + g(x,y,z,\langle w^*\rangle).
\end{gathered}$$

**Corollary 3.1.** *In addition to the assumptions of Theorem 3.1, suppose that $f$, $g$ satisfy Lipschitz conditions, then $v \equiv u \equiv w$ is the unique solution of* (2.1)–(2.4) *in* $\Omega$.

In the next result, we employ scheme (S$_2$) and obtain sequences which are alternatively monotone, under the same hypotheses (A$_1$) and (A$_2$) as in Theorem 3.1, but by dropping the assumptions $\langle v^0\rangle \le \langle v^1\rangle$ ,$\langle w^1\rangle \le \langle w^0\rangle$ and adding the new ones $\langle v^0\rangle \le \langle v^2\rangle, \langle w^2\rangle \le \langle w^0\rangle$.

**Theorem 3.2** *Let the assumptions* (A$_1$) *and* (A$_2$) *of Theorem 3.1 hold. Then, for any solution $u$ of* (2.1)–(2.4), *the sequences $\{v^n\}, \{w^n\}$ in the iterative scheme* (S$_2$) *satisfy the inequalities*

$$\langle v^0\rangle \le \langle v^2\rangle \le \cdots \le \langle v^{2n-2}\rangle \le \{u\} \le \langle v^{2n-1}\rangle \le \cdots \le \langle v^3\rangle \le \langle v^1\rangle$$
$$\langle w^1\rangle \le \langle w^3\rangle \le \cdots \le \langle w^{2n-2}\rangle \le \{u\} \le \langle w^{2n-2}\rangle \le \cdots \le \langle w^2\rangle \le \langle w^0\rangle$$

*on $P$, provided $\langle v^0\rangle \le \langle v^2\rangle$ and $\langle w^2\rangle \le \langle w^0\rangle$ on $R$. Moreover, the monotone sequences $\{v^{2n}\}$, $\{v^{2n-1}\}$, $\{w^{2n}\}$, $\{w^{2n-1}\}$ converge uniformly to $v$, $w$, $v^*$, and $w^*$ respectively on $P$ and satisfy the equations*

$$\begin{gathered}
w_{xyz} = f(x,y,z,\langle v^*\rangle) + g(x,y,z,\langle v\rangle);\\
w(x,y,0) = \alpha(x,y),\ w(x,0,z) = \beta(x,z),\\
w(0,y,z) = \gamma(y,z);\ w(0,0,0) = u_0,\\
v_{xyz} = f(x,y,z,\langle w^*\rangle) + g(x,y,z,\langle w\rangle);\\
v(x,y,0) = \alpha(x,y),\ v(x,0,z) = \beta(x,z),\\
v(0,y,z) = \gamma(y,z);\ v(0,0,0) = u_0,\\
w^*_{xyz} = f(x,y,z,\langle v\rangle) + g(x,y,z,\langle v^*\rangle);\\
w^*(x,y,0) = \alpha(x,y),\ w^*(x,0,z) = \beta(x,z),\\
w^*(0,y,z) = \gamma(y,z);\ w^*(0,0,0) = u_0,\\
v^*_{xyz} = f(x,y,z,\langle w\rangle) + g(x,y,z,\langle w^*\rangle);\\
v^*(x,y,0) = \alpha(x,y),\ v^*(x,0,z) = \beta(x,z),\\
v^*(0,y,z) = \gamma(y,z); v^*(0,0,0) = u_0.
\end{gathered}$$

**Corollary 3.2** *In addition to the assumptions of Theorem 3.2, suppose that $f$ and $g$ satisfy Lipschitz conditions as in Corollary 3.1, then $v \equiv w \equiv v^* \equiv w^* \equiv u$ is the unique solution of* (2.1)–(2.4) *in* $\Omega$.

Our third and fourth result employs coupled lower and upper solutions of Type I together with the scheme and yields intertwined monotone sequences, without any additional requirements (S$_2$).

**Theorem 3.3** *Assume that*
(B$_1$) $v^0, w^0 \in C^3[P,\mathbb{R}]$ *are coupled lower-upper solutions of Type I of* (2.1)–(2.4) *such that*

$$\langle v^0\rangle \le \langle w^0\rangle \ \text{on } P;$$

(B$_2$) $f,\, g \in C[\mathbb{C},\mathbb{R}]$, *$f$ is nondecreasing in $\langle u\rangle$ and $g$ is nonincreasing in $\langle u\rangle$.*

*Then the sequences $\{v^n\}, \{w^n\}$ generated by the scheme satisfy the intertwined property*

$$\langle v^0\rangle \le \langle w^1\rangle \le .... \le \langle v^{2n-2}\rangle \le \langle w^{2n-1}\rangle \le \{u\} \le \langle v^{2n-1}\rangle \le \langle w^{2n-2}\rangle \le ... \le \langle v^1\rangle \le \langle v^0\rangle$$

*on $P$, where $u$ is any solution of* (2.1)–(2.4) *in* $\Omega$. *The sequences*

$$\left\{\left\langle v^{2n}\right\rangle, \left\langle w^{2n-1}\right\rangle\right\} \to \langle v\rangle$$

*and*

$$\left\{\left\langle w^{2n}\right\rangle, \left\langle v^{2n-1}\right\rangle\right\} \to \langle w\rangle$$

*uniformly where $v$ and $w$ are coupled minimal and maximal solutions respectively of* (2.1)–(2.4) *on $P$,that is $v$ and $w$ satisfy*

$$\begin{gathered}
v_{xy} = f(x,y,z,\langle v\rangle) + g(x,y,z,\langle w\rangle),\\
w_{xy} = f(x,y,z,\langle w\rangle) + g(x,y,z,\langle v\rangle).
\end{gathered}$$

*Also, $\langle v\rangle \le \langle u\rangle \le \langle w\rangle$ on $P$. Furthermore, if $f$ and $g$ satisfy the Lipschitz conditions, then $v \equiv w \equiv u$ is the unique solution of* (2.1)–(2.4) *in* $\Omega$.

Finally, we provide a simple and illuminating example to illustrate one of our results above (Theorem 3.1).

**Example 3.1**. In the cube $P = [0,1] \times [0,1] \times I[0,1]$, consider the IBVP

$$u_{xyz} = f(x,y,z,u) = \begin{cases} \sqrt{u} & \text{if} \quad 0 \le u \le 1 \\ 0 & \text{if} \quad u < 0 \\ 1 & \text{if} \quad u > 1 \end{cases} \tag{3.6}$$

$$u(x,y,0) \equiv u(x,0,z) \equiv u(0,y,z) \equiv 0 \tag{3.7}$$

Then $f$ is monotonically nondecreasing in the last variable, but does not satisfy Lipschitz condition. Indeed, equation (3.6) has two solutions $v(x,y,z) \equiv 0$, and $w(x,y,z) \equiv 2^{-6}x^2y^2z^2$, satisfying the initial-boundary conditions (3.7). The functions $v^0(x,y,z) \equiv 0$ and $w^0(x,y,z) = xyz$ form a pair of natural lower-upper solutions of (3.6)–(3.7). Also, it is easy to see that the iterates $v^1$ and $w^1$ satisfy the requirements $v^0(x,y,z) \le v^1(x,y,z)$ and $w^0(x,y,z) \ge w^1(x,y,z)$ of Theorem 3.1. Consequently, the iterates in the scheme $(S_1)$ satisfy the inequalities (3.1). In fact, the sequences $\{\langle v^n \rangle\}$ and $\{\langle w^n \rangle\}$ converge uniformly and monotonically to $\langle v \rangle$ and $\langle w \rangle$ respectively.

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