Nonstationary Inverse Source Problem of Active Shielding

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Abstract—**The problem of active shielding of some domains from the effect of the sources distributed in other domains is considered. The problem can be formulated either in a bounded domain or in an unbounded domain. The active shielding is realized via the implementation of additional sources in such a way that the total contribution of all sources leads to the desirable effect. Mathematically the problem is reduced to seeking the source terms satisfying some *a priori* described requirements to the solution and belongs to the class of inverse source problems. From the application standpoint, this problem can be closely related to the active shielding of noise, active vibration control and active scattering. It is important to note that along with undesirable field (noise) to be shielded the presence of a desirable component is accepted in the analysis. The solution of the problem requires only the knowledge of the total field on the perimeter of the shielded domain. The examples of acoustic and Maxwell equations are considered. This is the first publication where the solution of the problem is proved in a quite general nonstationary formulation.**

Keywords: inverse problem, active noise shielding, active sound control, nonstationary problem, distribution

1 Introduction

The active shielding (AS) of some domains from the effect of the field (noise) generated in other domains is realized via the implementation of additional sources in such a way that the total contribution of all sources leads to the desirable effect. Mathematically the problem is reduced to seeking the source terms satisfying some *a priori* described requirements to the solution of an appropriate boundary value problem (BVP). Thus, it belongs to the class of inverse source problems [1]. From the application standpoint, this problem can be closely related to the active noise shielding and active vibration control. The comprehensive reviews of the theoretical and experimental methods related to these subjects can be found in books [2], [3], [4] and review [5]. Most theoretical approaches assume some quite detailed information about the undesirable sources and the properties of the medium. The JMC method [6], [7], [5], based on the Huygens' construction, requires only the information on the undesirable field on the perimeter of the shielded domain. Yet this method is not used in the case if a desirable field ("friendly sound"), generated in the shielded domain, has to be taken into account. In addition, the JMC method was only used to tackle the problems formulated in unbounded domains.

A principally new solution can be reached via the application of the Difference Potential Method (DPM) [8], [9]. The solution obtained in a finite–difference formulation requires only the knowledge of the total field (both desirable and undesirable) at the grid boundary of the shielded domain. Any other information on the sources and medium is not required. It is possible to say that the solution demands, in some sense, minimal information which is *a priori* available. A comprehensive study of the general solution [9] in the application to the Helmholtz equation including its optimization can be found in [10], [11], [12], [13]. In [16] the problem of AS in composite domains is formulated for the first time and its general finite–difference solution is provided. The DPM–based solution was extended to arbitrary hyperbolic systems of equations including acoustic Euler equations with constant and variable coefficients in our paper [14]. In [15] it is shown that the control sources are capable not to disturb even the echo of the "friendly" sound component if the AS problem is considered in bounded domains. For the acoustic Euler equations in continuous spaces, the AS solution was first obtained in our paper [17] for time–harmonic waves under quite general assumptions. It is shown the equivalence between the DPM–based discrete solution and the obtained solution if the space step vanishes. In the current paper, for the first time the approach [17] is extended to substantionally nonstationary problems (non time–harmonic waves). The solution is strictly proved in the appropriate Sobolev spaces. The examples of the acoustic and Maxwell equations are considered.

2 General formulation of the AS problem

The AS problem can be formulated as follows. Let us assume that some field (sound) U is described by the

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following correct BVP in a domain $D \subseteq \mathbb{R}^m$:

$$LU = f, \tag{1}$$
$$U \in \Xi(D). \tag{2}$$

Here, the operator L is a linear differential operator, $\Xi(D)$ is some functional space specified further. In particular, the operator L can correspond to the acoustic Euler equations.

We assume that $f \in F(D)$, where $F(D) \subset L_2^{loc}(D)$ is a linear space of functions f for which the solution of BVP (1), (2) exists and unique.

Consider some bounded domain D^+: $\overline{D^+} \subset D$. It is worth noting that the domain D^+ can be composite. It is assumed that the domain D^+ has a smooth boundary Γ. The sources on the right–hand side can be distributed both on D^+ and outside D^+:

$$\begin{gathered} f = f^+ + f^-, \\ \operatorname{supp} f^+ \subset D^+, \\ \operatorname{supp} f^- \subset D^- \stackrel{def}{=} D \diagdown \overline{D^+}. \end{gathered} \tag{3}$$

Here, $f^+ \in F(D)$ is the source of a "friendly" field (sound), while f^- is the source of an "adverse" field (noise).

Suppose that we know the trace of the function U on the boundary Γ of the domain D^+. It is to be noted that only this information is assumed to be available. In particular, the distribution of the sources f on the right-hand side is unknown. The AS problem is reduced to seeking additional sources G in $\overline{D^-}$ such that the solution of the following BVP

$$\begin{gathered} LU' = f + G, \\ \operatorname{supp} G \subset \overline{D^-}, \\ U' \in \Xi(D) \end{gathered} \tag{4}$$

coincides on the domain D^+ with the solution of BVP (1), (2) if $f^- \equiv 0$:

$$\begin{gathered} LU^+ = f^+, \\ U^+ \in \Xi(D). \end{gathered} \tag{5}$$

Thus, we seek a source term G such that on the domain D^+ the functions U and U' coincide with each other:

$$U'_{D^+} = U^+_{D^+}. \tag{6}$$

It is important to emphasize that an "obvious" solution $f = -f^-$ is not appropriate here because the function f^- is unknown.

3 Solution of the stationary AS problem

First let us consider the stationary formulation of BVP (1), (2). Assume that the operator L is as follows:

$$L \stackrel{def}{=} \sum_1^m A^i \frac{\partial}{\partial x^i}, \tag{7}$$

where $\{x^i\}$ $(i = 1, ..., m)$ is a Cartesian coordinate system, $A^i(\mathbf{x}) \in C^\infty(D)$ $(i = 1, ..., m)$. We also suppose that some stationary linear boundary conditions are set on the boundary of D:

$$l_{\partial D} U = 0. \tag{8}$$

Here $l_{\partial D}$ is some differential operator.

Thus, BVP (1), (2) reduces to the following:

$$\begin{gathered} \sum_1^m A^i \frac{\partial U}{\partial x^i} = f, \\ l_{\partial D} U = 0., \\ f = f^- + f^+, \\ \operatorname{supp} f^+ \subset D^+, \quad \operatorname{supp} f^- \subset D^-, \end{gathered} \tag{9}$$

where U and f are vector–functions with the dimension of m.

Let us consider the solution of BVP (9) in the generalized sense [18], [19]. For this purpose we introduce the space of basic functions $\Phi \in C_0^\infty(D)$. Equality (1) is then considered in the weak sense: $< LU, \Phi > = < f, \Phi >$ for any $\Phi \in C_0^\infty(D)$ where $< f, \Phi >$ means a linear distribution determined on the space of the basic functions $C_0^\infty(D)$.

We define the functional space $\Xi(D)$ in such a way that the weak solution of BVP (9) coincides almost everywhere with the classical solution of this problem. Thus, we require that the functions from $\Xi(D)$ are piece–wise smooth and satisfy the boundary condition (8). Hence, we suppose that $\Xi(D) \subset H^s(D)$, where $H^s(D)$ is the Sobolev space of functions, determined on D, with $s > 1 + m/2$. We also assume the boundary conditions are such that if $W \in \Xi(D)$, then $W(\partial D) \in H^{s-1/2}(\partial D)$. These estimates immediately follow from the Sobolev embedding theorems [19].

Then, the solution of the AS problem is given by the following theorem.

Theorem 1 *A solution of the AS problem* (1), (2), (4), (9) *is given by the following one–layer distribution:*

$$G = G_0 \stackrel{def}{=} A_n U_\Gamma \delta(\Gamma), \tag{10}$$

where $U_\Gamma \stackrel{def}{=} U(\Gamma)$, $A_n \stackrel{def}{=} \sum_1^m n_i A^i$, n_i *are the coordinates of the unit vector of the external normal* $\boldsymbol{n}$ *to the boundary* ∂D, δ *is the Dirac delta–function.*

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Proof. Thus, it is required to prove that the solution of problem (4) coincides with the solution of BVP (5) in D^+: $U' = U^+$ if $\mathbf{x} \in D^+$. For this purpose, let us consider the following four additional BVPs.

BVP 1^0:

$$LU^+ = f^+, \qquad (11)$$
$$l_{\partial D} U^+ = 0.$$

It is clear that the solution of this BVP exists and unique since $f^+ \in F(D)$.

BVP 2^0:

$$LU^- = f^-, \qquad (12)$$
$$l_{\partial D} U^- = 0.$$

Similar, the solution of this BVP also exists and unique.

BVP 3^0:

$$L\overline{U} = A_n U_\Gamma \delta(\Gamma), \qquad (13)$$
$$l_{\partial D} \overline{U} = 0.$$

The solution of this problem exists, unique and is the following:

$$\overline{U}(\mathbf{x}) = \begin{cases} -U^-, & \text{if } \mathbf{x} \in D^+ \\ U^+ & \text{if } \mathbf{x} \in D^- \end{cases} \qquad (14)$$

Indeed,

$$\begin{aligned} &< L\overline{U}, \Phi >= -\textstyle\sum_1^m < \overline{U}, \nabla_i(A^i\Phi) >= \qquad (15) \\ &-\textstyle\sum_1^m \int_D (\overline{U}, \nabla_i(A^i\Phi))d\mathbf{x} = \\ &-\textstyle\sum_1^m \int_{D^-} (\overline{U}, \nabla_i(A^i\Phi))d\mathbf{x} - \\ &\textstyle\sum_1^m \int_{D^+} (\overline{U}, \nabla_i(A^i\Phi))d\mathbf{x} =< \left\{L\overline{U}\right\}, \Phi > + \\ &\int_\Gamma A_n \left[\overline{U}\right]_\Gamma \Phi d\mathbf{x} =< \left\{L\overline{U}\right\}, \Phi > + \\ &< A_n U_\Gamma \delta(\Gamma), \Phi >=< A_n U_\Gamma \delta(\Gamma), \Phi > \end{aligned}$$

Here, (a, b) denotes the scalar product of vectors a and b, $\left\{L\overline{U}\right\}$ is the regular part of $L\overline{U}$ on D, $[.]_\Gamma$ means a discontinuity across the boundary Γ. Thus,

$$[V]_\Gamma \stackrel{def}{=} \lim_{\mathbf{x} \to \Gamma \cap \mathbf{x} \in D^-} V(\mathbf{x}) - \lim_{\mathbf{x} \to \Gamma \cap \mathbf{x} \in D^+} V(\mathbf{x})$$

BVP 4^0:

$$L\overline{U}^+ = f + A_n U_\Gamma \delta(\Gamma), \qquad (16)$$
$$l_{\partial D} \overline{U}^+ = 0.$$

The solution of this BVP exists and unique because of the linearity of the problem.

Thus, from BVPs $1^0 - 3^0$ it follows that:

$$\overline{U}(\mathbf{x}) = \begin{cases} U^+, & \text{if } \mathbf{x} \in D^+ \\ U + U^+ & \text{if } \mathbf{x} \in D^- \end{cases} \qquad (17)$$

□

It is to be noted that $U_\Gamma \notin \operatorname{Ker} A_n$. Otherwise the solution of the BVP (1), (2) would not be unique since the homogeneous BVP with $f \equiv 0$ had the solution (14) apart from the trivial solution $U \equiv 0$.

It is worth noting that AS solution (10) provided by the Theorem does not explicitly depend on the boundary conditions. Although the boundary conditions are not explicitly specified, we are able to obtain the AS source term if the solution of the considered BVP is correct.

4 Nonstationary AS problem

AS solution (10) is applicable to a nonstationary AS problem in $\mathbb{R}^{m+1}$ under some additional requirements. Then, the proof of this statement mostly repeats the stationary case.

Suppose that field U is described by a correct initial–boundary value problem (IBVP) in the cylinder $K_\infty = D \times (0, \infty)$:

$$LU \stackrel{def}{=} \tfrac{\partial U}{\partial t} + \textstyle\sum_1^m A^i \tfrac{\partial U}{\partial x^i} = f, \qquad (18)$$
$$l_{\partial D} U = 0, \qquad (19)$$
$$U(\mathbf{x}, 0) = U_0(\mathbf{x}), \qquad (20)$$
$$f = f^- + f^+,$$
$$\operatorname{supp} f^+ \subset D^+, \quad \operatorname{supp} f^- \subset D^-,$$

where $U_0(x) \in H^s(D)$. As in the previous section we consider a generalized solution of BVP (18), (19), (20). In distinguish to the stationary case, distribibutions such as $< f, \Phi >$ are considered in the following sense:

$$< f, \Phi >= \int_0^\infty \int_D (f, \Phi) d\mathbf{x} dt. \qquad (21)$$

First, it is necessary to note that without the violation of generality we can suppose that the initial data are homogeneous: $U_0(\mathbf{x}) = 0$. Otherwise we can represent the solution of IBVP (18), (19), (20) as the following sum: $U = U_f + U_t$, where U_f is the solution of IBVP problem with the homogeneous initial data while U_t is the solution of IBVP with the homogeneous right–hand side. It is clear that the function U_f has nothing to do with the unwanted component of the total solution U.

Now, we consider four auxiliary problems similar to the stationary case. In contrast to the stationary case, the appropriate IBVPs are to be considered. All IBVPs are set with the homogeneous initial data. The right–hand side f is obviously a nonstationary function now. Then, the proof mostly repeats the stationary case with the only distinguish that the generalized solution of equation (18) is considered as follows:

$$\int_0^\infty \int_D (LU - f, \Phi) d\mathbf{x} dt = 0 \qquad (22)$$

for any $\Phi \in C_0^\infty(D)$.

Then, we obtain that the AS solution is represented by (10) where $U_\Gamma = U_\Gamma(t)$ now depends on time.

It is important to note that in this solution we do not take into account the influence of the AS source term on the value of $U_\Gamma(t)$.

5 Examples

Let us now consider a few examples of AS sources terms for stationary and nonstationary problems.

1^0. *Charges on the boundary of a metallic body*:

Consider a metallic bounded body in an external electric field $\mathbf{E}_{out}$. It is well known that if the problem is static, then the field in the body must equal zero. Charges in the body are redistributed on its surface in such a way that the internal electric field equals zero. Thus, the contribution of the surface charges is similar to shielding the body from the external field $\mathbf{E}_{out}$, where $\mathbf{E}_{out}$ is the external field on the perimeter of the body. From the Maxwell equations it follows that

$$div\mathbf{E} = 4\pi\rho + g_0, \tag{23}$$

$$curl\mathbf{E} = 0, \tag{24}$$

where $\mathbf{E}$ is the electric field, ρ is the density of charges, g_0 is the AS source term.

Assume that $f^- = 4\pi\rho$. Then, let us write the AS source terms in the following form: $g_0 = 4\pi\sigma_\rho\delta(\Gamma)$. From (10) we obtain that

$$g_0 = \mathbf{E}_{out} \cdot \mathbf{n}\delta(\Gamma), \tag{25}$$

$$\mathbf{n} \times \mathbf{E}_{out} = 0. \tag{26}$$

Hence,

$$\sigma_\rho = \frac{1}{4\pi}E_{out} \tag{27}$$

It fully coincides with the classical result from electrostatic on an electric field in a metallic sphere (see, e.g., [20]). It is clear that this result is valid for an arbitrary bounded metallic body having a smooth boundary.

2^0. *Bound current on the boundary of a superconductor*:

Let us now consider a magnetic field around a superconductor. It is well known that the magnetic field inside a superconductor equals zero. The magnetic external field induces a bound current with a density $\mathbf{j}$ which plays the shielding role.

Consider the Maxwell equations for a static magnetic field:

$$curl\mathbf{H} = \frac{4\pi}{c}\sigma\mathbf{E} + g_0, \tag{28}$$

where H is the magnetic field, σ is the conductivity. Suppose that $\mathbf{g}_0 = \frac{4\pi}{c}\mathbf{j}_b\delta(\Gamma)$. On the other hand, from (10) it follows that $\mathbf{g}_0 = \mathbf{n} \times \mathbf{H}\delta(\Gamma)$. Hence,

$$\frac{4\pi}{c}\mathbf{j}_b = \mathbf{n} \times \mathbf{H} \tag{29}$$

This result coincides with the well known result on the bound current [20] on the surface of a superconductor.

The next two examples are related to active noise shielding in acoustics. The solutions of these AS problems have been earlier obtained in the linear formulation.

3^0. *Helmholtz equation*:

The Helmholtz equation describes the propagation of a monochromatic wave

$$\Delta\phi + k^2\phi = s. \tag{30}$$

We can rewrite it as the system of first-order equations:

$$\nabla\mathbf{a} + k^2\phi = s, \tag{31}$$
$$\nabla\phi - \mathbf{a} = 0.$$

In $\mathbb{R}^3$, we have:

$$U = (a_1, a_2, a_3, \phi)^T, \tag{32}$$

where $a_i \;\; (i = 1, 2, 3)$ are the coordinates of the vector $\mathbf{a}$. Hence,

$$A_n = \begin{pmatrix} n_1 & n_2 & n_3 & 0 \\ 0 & 0 & 0 & n_1 \\ 0 & 0 & 0 & n_2 \\ 0 & 0 & 0 & n_3 \end{pmatrix}, \tag{33}$$

and

$$G_0(\Gamma) = (a_n, \phi n_1, \phi n_2, \phi n_3)^T\delta(\Gamma), \tag{34}$$

where $a_n = \mathbf{a} \cdot \mathbf{n}$.

Having turned back to the Helmholtz equation for the variable ϕ, we obtain

$$\Delta\phi + k^2\phi = s + g_0, \tag{35}$$

where the shielding function g_0 is as follows:

$$g_0 = \delta(\Gamma)\tfrac{\partial\phi}{\partial\mathbf{n}} + \nabla(\delta(\Gamma)\phi\mathbf{n}) \tag{36}$$

or

$$g_0 = \delta(\Gamma)\tfrac{\partial\phi}{\partial\mathbf{n}} + \tfrac{\partial\delta(\Gamma)\phi}{\partial\mathbf{n}}. \tag{37}$$

The AS term is represented via the sum of the single–layer and double–layer additional source terms. This solution fully coincides with the solution obtained in [10]. As mentioned above, the solution is applicable to the linear analogue of the Helmholtz equation with variable coefficients.

4^0. *Acoustic equations*:

Next, let us consider the acoustic equations:

$$p_t + \rho c^2 \mathbf{u}_x = f^{(p)} + \rho c^2 q_{vol}, \qquad (38)$$
$$\rho u_t + \nabla p = f^{(u)} + \mathbf{f}_{vol},$$

where q_{vol} is the volume velocity per a unit volume and $\mathbf{f}_{vol}$ is the force per a unit volume [2]. In this case, we have

$$U = (u_1, u_2, u_3, p)^T, \qquad (39)$$

where u_j $(j = 1, 2, 3)$ are the components of the velocity $\mathbf{u}$.

Then, the matrix A_n appears to coincide with the appropriate matrix (33) of the Helmholtz equation describing only time–harmonic waves.

As the result, we obtain the following AS source terms in the form of a simple layer:

$$q_{vol} = \mathbf{u} \cdot \mathbf{n}_{|\Gamma} \delta(\Gamma), \qquad (40)$$
$$\mathbf{f}_{vol} = p_{|\Gamma} \mathbf{n} \delta(\Gamma).$$

Thus, the AS solution depends on the normal component of the particle velocity and the sound pressure on the boundary of the shielded domain. These values are to be taken from measurements and based on the contribution of both desirable and undesirable sources without their factorization. This AS solution was obtained in [17] for the continuous space and in [14] for the finite–difference formulation. It also corresponds to AS sources derived in [21] via the modified JMC method in one-dimensional case under the assumption of $f^+ \equiv 0$ and formally extended to the three-dimensional formulation.

6 Conclusion

The solution of the AS problem has been obtained in the form of a simple–layer source term in the general non-stationary formulation. The solution only requires the knowledge of the total field (desirable and undesirable) on the perimeter of the shielded domain. It does not use any additional information on either the characteristics of the undesirable sources or the surrounding medium. The knowledge of the Green's function of the problem is not required either. The application of the general AS solution to the Maxwell equations, Helmholtz equation and acoustic Euler equations give us the appropriate AS source terms.

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References

[1] Isakov, V., *Inverse source problems*, Providence, R.I.: American Mathematical Society, Mathematical surveys and monographs; Vol. 34, 1990.

[2] Nelson, P.A., Elliott, S.J., *Active control of sound*, Academic Press, San Diego, CA, USA, 1992.

[3] Fuller, C.R., Nelson, P.A., Elliott, S.J., *Active control of vibration*, Academic Press, 1996.

[4] Tochi, O., Veres, S., *Active sound and vibration control. Theory and applications*, The Institution of Electrical Engineers, 2002.

[5] Uosukainen, S., Välimäki, V., *JMC actuators and their applications in active attenuation of noise in ducts*, VTT Publications, 341, VTT Buidling Technology, ESPOO, 1998, 100p.

[6] Jessel, M.J., "Some evidences for a general theory of active noise sound absorption", *Proceedings of Inter–Noise 79*, Warzaw, 1979, pp. 169–174.

[7] Mangiante, G., "The JMC Method for 3D active sound absorption: a numerical simulation", *Noise Control Engineering J.*, V41, N2, pp. 339–345, 1993.

[8] Ryaben'kii, V.S., *Method of difference potentials and its applications*, Berlin, Springer-Verlag, 2002.

[9] Ryaben'kii, V.S., "A difference shielding problem. Functional Analysis and Applications", V29, pp. 70–71, 1995.

[10] Lončarić, J., Ryaben'kii, V. S., Tsynkov, S.V., "Active shielding and control of noise", *SIAM J. Appl. Math.*, V62, 2, pp. 563–596, 2001.

[11] Lončarić, J., Tsynkov, S.V., "Optimization of acoustic sources strength in the problems of active noise control", *SIAM J. Appl. Math.*, V63, pp. 1141–1183, 2003.

[12] Lončarić, J., Tsynkov, S.V., "Optimization of power in the problem of active control of sound", *Mathematics and Computers in Simulation*, V65, N4–5, pp. 323–335, 2004.

[13] Lončarić, J., Tsynkov, S.V., "Quadratic optimization in the problems of active control of sound", *Applied Numerical Mathematics*, V52, N4, pp. 381–400, 2005.

[14] Ryaben'kii, V.S., Utyuzhnikov, S.V., "Active shielding model for hyperbolic equations", *IMA Journal of Applied Mathematics*, V71, N6, pp. 924–939, 2006.

[15] Ryaben'kii, V.S., Utyuzhnikov, S.V., Turan, A.A., "On the Application of Difference Potential Theory to Active Noise Control", *J. Advances in Applied Mathematics*, V20, N5, pp. 511–515, 2007.

[16] Ryaben'kii, V.S., Tsynkov, S.V., Utyuzhnikov, S.V., "Inverse source problem and active shielding for composite domains", *Applied Mathematics Letters*, 2006, to appear.

[17] Ryaben'kii, V.S., Utyuzhnikov, S.V., "Differential and finite-difference problems of active shielding", *Applied Numerical Mathematics*, V57, N4, pp. 374–382, 2007.

[18] Vladimirov, V.S., *Equations of Mathematical Physics*, Dekker, New York, 1971.

[19] Lions, J-L, Magenes, E., *Non-homogeneous boundary value problems and applications*, Springer, Berlin-Heidelberg-New York, 1972.

[20] Good, R.H., Jr., Nelson, T.J., *Classical Theory of Electric and Magnetic Fields*, Academic Press, New York and London, 1971.

[21] Uosukainen, S., "Active sound scatters based on the JMC method", *Journal of Sound and Vibration*, V267, pp. 979–1005, 2003.