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Two Point Fully Implicit Block Direct Integration Variable Step Method for Solving Higher Order System of Ordinary Differential Equations

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***Abstract*—Two point fully implicit block method of variable step size is developed for solving directly the second order system of Ordinary Differential Equations (ODEs). This method will estimate the solutions of Initial Value Problems (IVPs) at two points simultaneously. The method developed is suitable for the numerical integration of non stiff and mildly stiff differential systems. Numerical results are given to compare the efficiency of the developed method to the existence non block method.**

***Index Terms*—Block method, higher order odes, numerical method, two point block.**

I. Introduction

In this paper, we consider solving directly the second order non stiff and mildly stiff IVPs for system of ODEs of the form

$$y'' = f(x, y, y'),\ y(a) = y_0\,,\ y'(a) = y'_0,\ x \in [a, b]. \qquad (1)$$

Eq. (1) arises from many physical phenomena in a wide variety of applications especially in engineering such as the motion of rocket or satellite, fluid dynamic, electric circuit and other area of application. The approach for solving the system of higher order ODEs directly has been suggested by several researchers such as in [1] – [4].

The current multistep method for variable step (VS) or variable step and order (VSVO) technique for solving the systems of higher order ODEs as described by the above researchers will involve tedious computations of divided difference and the integration coefficients in the code.

A system of higher order can also be reduced to a system of first order equations and then solved using first order ODEs. This approach is very well established but it obviously will enlarge the system of first order ODEs. However, the developed method will solve the system of higher order ODEs directly.

The aim of this paper is to investigate the performance of the two point fully implicit block direct integration method presented as in the simple form of Adams Moulton method for solving (1) directly using variable step size. The method is in a simple form but we intend for efficiency and economically. The idea of the code developed is to avoid tedious and repetitive computation of the divided differences and integration coefficients that can be very costly. Hence, the code will store all the coefficients of the method. As the computational work increases the advantage of the method will be evident when the execution time is compared with the existence non block method in [4].

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II. Formulation of the Method

In Figure 1, the two values of y_{n+1} and y_{n+2} are simultaneously computed in a block using the same back values. The block has the step size h and the previous back block has the step size rh.

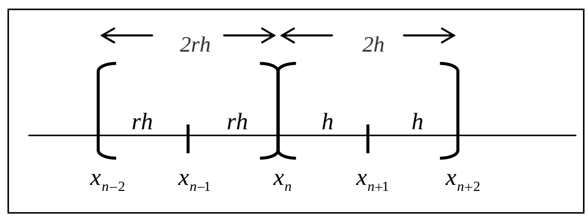

Fig. 1: Two point one block

To approximate the first point y_{n+1} at x_{n+1}, takes $x_{n+1} = x_n + h$ and integrate (1) once gives

$$\int_{x_n}^{x_{n+1}} y''(x)\,dx = \int_{x_n}^{x_{n+1}} f(x, y, y')\,dx \qquad (2)$$

which is equivalent to

$$y'(x_{n+1}) = y'(x_n) + \int_{x_n}^{x_{n+1}} f(x, y, y')\ dx \qquad (3)$$

The function $f(x, y, y')$ in (3) will be approximated using Lagrange interpolating polynomial and the interpolation points involved are $(x_{n-2}, f_{n-2}), \ldots, (x_{n+2}, f_{n+2})$. Taking $s = \dfrac{x - x_{n+2}}{h}$ and by replacing $dx = h\,ds$, the value of y_{n+1} can be obtained by integrating (3) over the interval $[x_n, x_{n+1}]$ using **MAPLE** and the following corrector formulae will be obtained,

The first point:

$$y'(x_{n+1}) = y'(x_n) + \frac{h}{240(r+1)(r+2)(2r+1)r^2} \cdot \Big[\left(-(2r+1)r^2\left(3+15r+20r^2\right)f_{n+2} + 4r^2(r+2)\left(18+75r+80r^2\right)f_{n+1} + (r+1)(r+2)(2r+1)\left(7+45r+100r^2\right)f_n - 4(2r+1)(7+30r)f_{n-1} + (r+2)(7+15r)f_{n-2}\right) \Big]. \quad (4)$$

Now, integrating (1) twice gives

$$\int_{x_n}^{x_{n+1}}\int_{x_n}^{x} y''(x)\,dxdx = \int_{x_n}^{x_{n+1}}\int_{x_n}^{x} f(x,y,y')\,dxdx \quad (5)$$

Therefore,

$$y(x_{n+1}) - y(x_n) - hy'(x_n) = \int_{x_n}^{x_{n+1}} (x_{n+1}-x) f(x,y,y')\,dx \quad (6)$$

Replacing the $f(x,y,y')$ in (6) with the same interpolation polynomial through the points $(x_{n-2}, f_{n-2}), \dots, (x_{n+2}, f_{n+2})$. Taking $s = \frac{x - x_{n+2}}{h}$ and by replacing $dx = h\,ds$ and integrate (6) over the interval $[x_n, x_{n+1}]$ using **MAPLE** and the following corrector formulae can be obtained,

The first point:

$$y(x_{n+1}) - y(x_n) - hy'(x_n) = h^2\Bigg[-\frac{\left(1+6r+10r^2\right)}{240(r+1)(r+2)} f_{n+2} + \frac{\left(4+21r+30r^2\right)}{60(2r+1)(r+1)} f_{n+1} + \frac{\left(3+24r+70r^2\right)}{240r^2} f_n - \frac{(3+16r)}{60r^2(r+1)(r+2)} f_{n-1} + \frac{(3+8r)}{240r^2(2r+1)(r+1)} f_{n-2} \Bigg] \quad (7)$$

The method is the combination of predictor of order 4 and the corrector of order 5. The predictor formulae were similarly derived where the interpolation points involved are $(x_{n-3}, f_{n-3}), \dots, (x_n, f_n)$.

Apply the same process above to find the integration coefficients of the second point y_{n+2} of the two point block direct integration method. Let $x_{n+2} = x_n + 2h$, integrating (1) once gives

$$\int_{x_n}^{x_{n+2}} y''(x)\,dx = \int_{x_n}^{x_{n+2}} f(x,y,y')\,dx\ . \quad (8)$$

Therefore,

$$y'(x_{n+2}) = y'(x_n) + \int_{x_n}^{x_{n+2}} f(x,y,y')\,. \quad (9)$$

Replacing the $f(x,y,y')$ in (9) with the same interpolation polynomial as in (3). Taking $s = \frac{x - x_{n+2}}{h}$ and by replacing $dx = h\,ds$ and integrate (9) over the interval $[x_n, x_{n+2}]$ using **MAPLE** and the following corrector formulae can be obtained,

The second point:

$$y'(x_{n+2}) = y'(x_n) + \frac{h}{15r^2(2r+1)(r+2)(r+1)} \cdot \Big[r^2(2r+1)\left(5r^2+15r+9\right)f_{n+2} + 4r^2(r+2)\left(10r^2+15r+6\right)f_{n+1} + (r+2)(r+1)(2r+1)\left(5r^2-1\right)f_n + 4(2r+1)f_{n-1} - (r+2)f_{n-2} \Big]. \quad (10)$$

Continue integrating (1) twice at the second point gives

$$\int_{x_n}^{x_{n+2}}\int_{x_n}^{x} y''(x)\,dxdx = \int_{x_n}^{x_{n+2}}\int_{x_n}^{x} f(x,y,y')\,dxdx \quad (11)$$

Therefore,

$$y(x_{n+2}) - y(x_n) - 2hy'(x_n) = \int_{x_n}^{x_{n+2}} (x_{n+2}-x) f(x,y,y')\,dx \quad (12)$$

Replacing the $f(x,y,y')$ in (12) with the same interpolation polynomial through the points $(x_{n-2}, f_{n-2}), \dots, (x_{n+2}, f_{n+2})$, taking $s = \frac{x - x_{n+2}}{h}$ and by replacing $dx = h\,ds$ and integrate (12) over the interval $[x_n, x_{n+2}]$ using **MAPLE** and the following corrector formulae can be obtained,

The second point:

$$y(x_{n+2}) - y(x_n) - 2hy'(x_n) = h^2\Bigg[-\frac{\left(1+6r+10r^2\right)}{240(r+1)(r+2)} f_{n+2} + \frac{\left(4+21r+30r^2\right)}{60(2r+1)(r+1)} f_{n+1} + \frac{\left(3+24r+70r^2\right)}{240r^2} f_n - \frac{(3+16r)}{60r^2(r+1)(r+2)} f_{n-1} + \frac{(3+8r)}{240r^2(2r+1)(r+1)} f_{n-2} \Bigg] \quad (13)$$

III. VARIABLE STEP STRATEGY

During the implementation of the method, the choices of the next step size will be restricted to half, double or the same as the previous step size and the successful step size will remain constant for at least two blocks before considered it to be doubled. This step size strategy helps to minimize the choices of the ratio *r*. In the code developed, when the next successful step size is doubled, the ratio *r* is 0.5 and if the next successful step size remain constant, *r* is 1.0. In case of step size failure, *r* is 2.0. Substituting the ratios of *r* will give the corrector formulae for the two point one block direct integration method.

Substituting the common ratios of r in (4), (7), (10) and (13) will give the corrector formulae for the two point fully implicit block direct integration method. For example, the corrector formulae when *r* = 1 in (4), (7), (10) and (13) are as follows:-

First integrating:

$$y'_{n+1} = y'_n - \frac{h}{720}\left(19f_{n+2} - 346f_{n+1} - 456f_n + 74f_{n-1} - 11f_{n-2}\right)$$

$$y'_{n+2} = y'_n + \frac{h}{90}\left(29f_{n+2} + 124f_{n+1} + 24f_n + 4f_{n-1} - f_{n-2}\right) \qquad (14)$$

Second integrating:

$$y_{n+1} = y_n + hy'_n - \frac{h^2}{1440}\left(17f_{n+2} - 220f_{n+1} - 582f_n + 76f_{n-1} - 11f_{n-2}\right). \qquad (15)$$

$$y_{n+2} = y_n + 2hy'_n + \frac{h^2}{90}\left(5f_{n+2} + 104f_{n+1} + 78f_n - 8f_{n-1} + f_{n-2}\right)$$

IV. RESULTS AND DISCUSSION

In order to study the efficiency of the developed method, we present some numerical experiments for the following three problems.

The 2PFDIR and 1PVSO were applied to the following test problems:

Problem 1:

$$y_1'' = -y_2' + \cos x, \quad y_1(0) = -1, \quad y_1'(0) = -1,$$

$$y_2'' = y_1 + \sin x, \quad y_2(0) = 1, \quad y_2'(0) = 0, \; x \in [0, 4\pi]$$

Solution: $y_1(x) = -\cos x - \sin x,\; y_2(x) = \cos x.$

Source: [5]

Problem 2:

$$y_1'' = \frac{-y_1}{r}, \quad y_1(0) = 1, \quad y_1'(0) = 0,$$

$$y_2'' = \frac{-y_2}{r}, \quad y_2(0) = 0, \quad y_2'(0) = 1,$$

$$r = \sqrt{{y_1}^2 + {y_2}^2} \qquad x \in [0, 15\pi]$$

Solution: $y_1(x) = \cos x,\; y_2(x) = \sin x.$

Source: [6]

Problem 3:

$$y_1'' = -y_2 + \sin \pi x,$$

$$y_2'' = -y_1 + 1 - \pi^2 \sin \pi x,$$

$$y_1(0) = 0, \quad y_1'(0) = -1,$$

$$y_2(0) = 1, \quad y_2'(0) = 1 + \pi, \quad [0, 10]$$

Solution: $y_1(x) = 1 - e^x,\; y_2(x) = e^x + \sin \pi x.$

Source: [5]

The following notations are used in the tables:

TOL	Tolerance
MTD	Method employed
TS	Total steps taken
FS	Total failure step
MAXE	Magnitude of the maximum error of the computed solution
TIME	The execution time taken in microseconds
2PFDIR	Implementation of the two point fully implicit block direct integration method of variable step size
1PVSO	Implementation of the one point implicit direct integration method (non block) using variable step and order in [4]
Rstep	The ratio steps of 2PFDIR compared to 1PVSO
Rtime	The ratio times of 2PFDIR compared to 1PVSO

The errors calculated are defined as

$$(e_i)_t = \left| \frac{(y_i)_t - (y(x_i))_t}{A + B(y(x_i))_t} \right| \qquad (16)$$

where $(y)_t$ is the *t*- th component of the approximate *y* and for this case we let *t*=1. The absolute error test corresponds to *A*=1, *B*=0, the mixed test corresponds to *A*=1, *B*=1 and finally *A*=0, *B*=1 corresponds to the relative error test. The mixed error tests were used for all the problems. The maximum error and average error are defined as follows:-

$$\text{MAXE} = \max_{1 \le i \le SSTEP}\left(\max_{1 \le i \le N}(e_i)_t\right) \text{ and} \qquad (17)$$

$$\text{AVE} = \frac{\sum_{i=1}^{SSTEP}\sum_{t=1}^{N}(e_i)_t}{(N)(SSTEP)} \qquad (18)$$

where N is the number of equations in the system, *SSTEP* is the number of successful steps. In the code, we iterate the corrector to convergence. The convergence test employed were

$$abs\left({y^{(s+1)}}_{n+2} - {y^{(s)}}_{n+2}\right) < 0.1 \times \text{TOL}, \quad s = 0,1,2\ldots \qquad (19)$$

where *s* is the number of iteration. After the successful convergence test of (19), local errors estimate at the point x_{n+2} will be performed to control the error for the block. The local errors estimates will be obtain by comparing the absolute difference of the corrector formula derived of order *k* and a similar corrector formula of order *k*-1.

The code was written in C language and executed on DYNIX/ptx operating system. The numerical results for the three problems are presented in Tables 1 – 3.

Table 1: Comparison between 2PFDIR and 1PVSO for solving problem 1

TOL	MTD	TS	FS	MAXE	TIME
10^{-2}	2PFDIR	33	0	1.603-2	4367
	1PVSO	31	0	5.578-1	6487
10^{-4}	2PFDIR	55	0	4.210-4	5818
	1PVSO	53	0	2.194-2	7125
10^{-6}	2PFDIR	74	0	9.058-5	8637
	1PVSO	146	0	5.408-5	17172
10^{-8}	2PFDIR	130	0	3.322-6	15233
	1PVSO	274	0	1.912-6	32195
10^{-10}	2PFDIR	278	0	3.752-8	29402
	1PVSO	700	0	1.402-8	82409

Table 2: Comparison between 2PFDIR and 1PVSO for solving problem 2

TOL	MTD	TS	FS	MAXE	TIME
10^{-2}	2PFDIR	67	0	7.982e-2	7908
	1PVSO	72	0	5.391e-1	12324
10^{-4}	2PFDIR	140	0	6.931e-4	15737
	1PVSO	178	0	1.028e-2	22692
10^{-6}	2PFDIR	316	0	7.460e-6	35048
	1PVSO	384	0	2.124e-4	49056
10^{-8}	2PFDIR	394	0	2.457e-6	43604
	1PVSO	907	0	2.744e-6	115755
10^{-10}	2PFDIR	938	0	2.539e-8	104312
	1PVSO	2250	0	2.821e-8	287955

Table 3: Comparison between 2PFDIR and 1PVSO for solving problem 3

TOL	MTD	TS	FS	MAXE	TIME
10^{-2}	2PFDIR	33	0	3.435-4	6108
	1PVSO	35	0	3.626-3	7494
10^{-4}	2PFDIR	55	0	1.084-5	6238
	1PVSO	61	0	2.875-5	8780
10^{-6}	2PFDIR	103	0	2.572-7	11672
	1PVSO	171	0	4.280-7	21652
10^{-8}	2PFDIR	152	0	2.283-8	17363
	1PVSO	451	0	1.894-8	53755
10^{-10}	2PFDIR	299	0	5.688-10	34875
	1PVSO	1133	0	6.655-11	135140

Table 4: The ratios of total steps and execution times for 2PFDIR compared to 1PVSO for solving problem 1 to 3

TOL	PROB 1		PROB 2		PROB 3	
	Rstep	Rtime	Rstep	Rtime	Rstep	Rtime
10^{-2}	0.94	1.49	1.07	1.56	1.06	1.22
10^{-4}	0.96	1.22	1.27	1.44	1.11	1.41
10^{-6}	1.97	1.98	1.22	1.40	1.66	1.86
10^{-8}	2.11	2.11	2.30	2.65	2.97	3.10
10^{-10}	2.52	2.80	2.40	2.76	3.79	3.87

In Table 1 – 3, it is observed that 2PFDIR required less number of steps compared to 1PVSO when solving the same given problems except in problem 1 at larger tolerances. However, 2PFDIR is better in terms of execution times even though the total steps taken are slightly more than 1PVSO. Therefore, we can conclude that the cost of computing the divided differences and integration coefficients in the 1PVSO is the major disadvantage when permitting random variations in the choices of step sizes and the computational cost increases when the method were implemented in variable step and order.

Most of the ratios of steps (Rstep) and all ratios of times (Rtime) in Table 4 are greater than one and these shows that 2PFDIR is more efficient compared to 1PVSO. We also could observe that the ratios are greater than two at smaller tolerances and these indicates a clear advantage of method 2PFDIR over 1PVSO. These results are expected since the two point block method would approximate the solutions at two points simultaneously. In terms of maximum error, method 2PFDIR is comparable or better compared to 1PVSO.

V. CONCLUSION

In this paper, we have shown the efficiency of the developed two point fully implicit block method presented as in the simple form of Adams Moulton Method using variable step size is suitable for solving second order ODEs directly.

REFERENCES

[1] C. W. Gear, *Numerical Initial Value Problems in Ordinary Differential Equations*. New Jersey: Prentice Hall, Inc. 1971

[2] M.B.Suleiman, Generalised Multistep Adams and Backward Differentiation Methods for the Solution of Stiff and Non-Stiff Ordinary Differential Equations. Ph.D. Thesis. University of Manchester. 1979

[3] J.D. Lambert, *Numerical Methods For Ordinary Differential Systems. The Initial Value Problem*. New York: John Wiley & Sons, Inc. 1993, ch 4

[4] Z. Omar, Developing Parallel Block Methods For Solving Higher Order ODEs Directly, Ph.D. Thesis, University Putra Malaysia, Malaysia. 1999

[5] R. Bronson, *Modern Introductory Differential Equation: Schaum's Outline Series.* USA: McGraw-Hill Book Company. 1973

[6] M.B.Suleiman, "Solving Higher Order ODEs Directly by the Direct Integration Method," *Applied Mathematics and Computation* 33, 1989. pp 197-219.