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Further results on the Craig-Sakamoto Equation

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Abstract—**In this paper necessary and sufficient conditions are stated for the Craig-Sakamoto equation** $det(I - sA - tB) = det(I - sA)\, det(I - tB)$, **for all scalars** s, t. **Moreover, spectral properties for** A **and** B **are investigated.**

Keywords: determinants, factorization of polynomials, bilinear forms, perturbation theory

1 Introduction

Let $M_n(\mathbf{C})$ be the set of $n \times n$ matrices with elements in $\mathbf{C}$. For A and $B \in M_n(\mathbf{C})$, the well known in Statisticts Craig-Sakamoto (CS) equation

$$det(I - sA - tB) = det(I - sA)\, det(I - tB) \qquad (1)$$

for all scalars s, t has occupied several researchers. In particular, in [4] O. Trussky has presented that the CS equation is equivalent to $AB = O$, when A, B are normal and most recently in [3] Olkin and in [1] Li have proved the same result in a different way. The author, together with M. Tsatsomero and P. Psarrako in [2], have investigated the CS equation involving the eigenspaces of A, B and $sA + tB$. Being more specific, if $\sigma(X)$ denotes the spectrum for a matrix X, $m_X(\lambda)$ the algebraic multiplicity of $\lambda \in \sigma(X)$, and $E_X(\lambda)$ the generalized eigenspace corresponding to λ, we have shown:

Proposition 1 For the $n \times n$ matrices A, B the following are equivalent :

I. The CS equation holds

II. for every $s, t \in \mathbf{C}$,

$$\sigma(sA \oplus tB) = \sigma\left((sa + tB) \oplus O_n\right),$$

where O_n denotes the zero matrix

III. $\sigma(sA+tB) = \{\, s\mu_i + t\nu_i \,:\, \mu_i \in \sigma(A),\ \ \nu_i \in \sigma(B)\,\}$, where the pairing of eigenvalues requires either $\mu_i = 0$ or $\nu_i = 0$.

Proposition 2 Let the $n \times n$ matrices A, B satisfy the CS equation. Then,

I. $m_A(0) + m_B(0) \geq n$.

II. If A is nonsingular, then B must be nilpotent.

III. If $\lambda = 0$ is semisimple eigenvalue of A and B, then $rank(A) + rank(B) \leq n$.

Proposition 3 Let $\lambda = 0$ be semisimple eigenvalue of $n \times n$ matrices A and B such that $BE_A(0) \subset E_A(0)$. Then the following are equivalent.

I. Condition CS holds.

II. $\mathbf{C}^n = E_A(0) + E_B(0)$.

III. $AB = O$.

The remaining results in [2] are based on the basic assumption that $\lambda = 0$ is a semisimple eigenvalue of A and B. Relaxing this restriction, we shall attempt here to look at the CS equation focused on the factorization of polynomial of two variables $f(s, t) = det(I - sA - tB)$. Also, considering the determinants in (1), new conditions necessary and sufficient on CS property are stated.

2 Criteria for CS equation

The first statement on the CS property is obtained investigating the determinantal equation through the Theory of Polynomials. By Proposition 2 II, it is clear that the CS equation is worth valuable when the $n \times n$ matrices A and B are singular. Especially, we define that " *A and B are called r-**complementary**, if and only if* at most, *r rows (columns), $a_{i_1}, a_{i_2}, \cdots, a_{i_r}$ of A are shifted and substituted by the corresponding $b_{i_1}, b_{i_2}, \cdots, b_{i_r}$ rows (columns) of B, such that the structured matrix $N(i_1, i_2, \cdots, i_r)$ of a' s and b' s rows is nonsingular.*" Then we have:

Proposition 4 Let the $n \times n$ singular matrices A and B be $[n - m_B(0)]$-complementary with $\theta = \sum_{i_1,\ldots,i_{n-m_B(0)}} detN(i_1, i_2, \ldots, i_{n-m_B(0)}) \neq 0$, where the sum is over all possible combinations $i_1, \ldots, i_{n-m_B(0)}$

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of $n-m_B(0)$ of the indices $1, 2, \ldots, n$. If they satisfy the CS equation, then

$$m_A(0) + m_B(0) \; = \; n.$$

Proposition 5 Let $\lambda = 0$ be semisimple eigenvalue of $n \times n$ matrices A and B such that $E_A(0) + E_B(0) = \mathbf{C}^n$. If for any $\lambda \in \sigma(A)\backslash\{0\}$, (or, $\mu \in \sigma(B)\backslash\{0\}$), the corresponding generalized eigenspaces $E_A(\lambda)$, ($E_B(\mu)$) satisfy $E_A(\lambda) \subseteq E_B(0)$, (or, $E_B(\mu) \subseteq E_A(0)$), then

I. A, B have the CS property.

II. $E_A(\lambda) = E_{I-sA-tB}(1 - s\lambda)$, and $E_B(\mu) = E_{I-sA-tB}(1 - t\mu)$.

3 Criteria for CS equation

Also, a new necessary and sufficient condition on CS property following is stated. Let

$$f(s,t) = det(I - sA - tB) = \sum_{p,q=0}^{n} m_{pq} s^p t^q, \qquad (2)$$

for $p + q \leq n$. Denoting by

$$x = \begin{bmatrix} 1 & s & s^2 & \cdots & s^n \end{bmatrix}^T, \; y = \begin{bmatrix} 1 & t & t^2 & \cdots & t^n \end{bmatrix}^T,$$

then (2) is written obviously

$$f(s,t) = x^T M y,$$

where $M = [m_{pq}]_{p,q=0}^{n}$, with $m_{00} = 1$.

Proposition 6 Let $A, B \in M_n(\mathbf{C})$. The CS equation holds for the pair of matrices A and B if and only if $rankM = 1$.

Following we note by $M\begin{pmatrix} a_{i_1,\ldots,i_p} \\ b_{j_1,\ldots,j_q} \end{pmatrix}$ the leading principal minor of order $p + q \, (\leq n)$, which is defined by the $i_1, \ldots, i_p$ rows of A and $j_1, \ldots, j_q$ rows of B, i.e., $M\begin{pmatrix} a_{i_1,\ldots,i_p} \\ b_{j_1,\ldots,j_q} \end{pmatrix}$ is equal to

$$\begin{vmatrix} a_{i_1 i_1} & a_{i_1 i_2} & a_{i_1 j_1} & a_{i_1 i_3} & \cdots & a_{i_1 j_q} & \cdots & a_{i_1 i_p} \\ a_{i_2 i_1} & a_{i_2 i_2} & a_{i_2 j_1} & a_{i_2 i_3} & \cdots & a_{i_2 j_q} & \cdots & a_{i_2 i_p} \\ b_{j_1 i_1} & b_{j_1 i_2} & b_{j_1 j_1} & b_{j_1 i_3} & \cdots & b_{j_1 j_q} & \cdots & b_{j_1 i_p} \\ a_{i_3 i_1} & a_{i_3 i_2} & a_{i_3 j_1} & a_{i_3 i_3} & & & & \vdots \\ \vdots & \vdots & & & \ddots & & & \\ b_{j_q i_1} & b_{j_q i_2} & & & & b_{j_q j_q} & & \\ \vdots & \vdots & & & & & \ddots & \\ a_{i_p i_1} & a_{i_p i_2} & & \cdots & & & & a_{i_p i_p} \end{vmatrix}$$

for $i_1 < i_2 < j_1 < i_3 < \cdots < j_q < \cdots < i_p$. Thus, we clarify a determinental expression of coefficients m_{pq} in (2):

$$m_{pq} = (-1)^{p+q} \sum_{1 \leq i_1 < j_1 < \cdots < j_q < i_p \leq n} M\begin{pmatrix} a_{i_1,\ldots,i_p} \\ b_{j_1,\ldots,j_q} \end{pmatrix},$$

where $m_{00} = 1$. Using the criterion in Proposition 6, the next necessary and sufficient conditions arise.

Proposition 7 The $n \times n$ matrices A and B have the CS property if and only if

$$\sum M(a_{i_1,\ldots,i_p}) \sum M(b_{j_1,\ldots,j_q}) = \sum M\begin{pmatrix} a_{i_1,\ldots,i_p} \\ b_{j_1,\ldots,j_q} \end{pmatrix},$$

for $p + q \leq n$, and

$$\sum M(a_{i_1,\ldots,i_p}) \sum M(b_{j_1,\ldots,j_q}) = 0,$$

for $p + q > n$.

The equations in Proposition 7 give also an answer to the problem "*For the $n \times n$ matrix A, clarify the set*

$$CS(A) = \{ B \; : \; A \text{ and } B \text{ follow the CS property} \}.$$

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