Avant-Garde Matrix Splitting for the Solution of Sparse Non-symmetric Linear Systems

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Abstract--A non-symmetric matrix splitting is presented for the solution of certain sparse linear systems. The author reports the comparison and the convergence performance of the previous and the contemporary methods and includes explicit comments. It is studied that some recent methods may work efficiently with a symmetric matrix but show insufficiencies like numerical instability and non-scaling invariant with non-symmetric matrix. The proposed matrix splitting can overcome these deficiencies.

Keywords: Sparse matrix, M-matrices, orthogonalization, Krylov Space, Iterative Methods

1-Introduction.

Recently some orthogonalization iterative methods e.g. **GMRES** methods or generalized minimal residual methods have been used for the solution of sparse symmetric linear systems,

$\mathbf{A\underline{x} = \underline{b}}$ **…….(1.1)**

Where A is a sparse matrix. In most of the studies these methods are matched up to and compared in many studies (see, for example **[2], [3], [15], [16], [17] and [19]**). These methods in fact minimize $\mathbf{r_n = b - Ax_n}$ **= nth residual** in a Krylov Space $\mathbf{K^n(A;r_0)}$**.** With an initial presumption $\mathbf{x_0}$ an inimitable sequence $\mathbf{\{x_n\}}$ is created with:

$\mathbf{x_n \in x_0 + < Ar_0, A^2r_0, A^3r_0, A^4r_0 \ldots\ldots, A^{n-1}r_0 >}$ **……..(1.2)**

satisfying:

$\mathbf{r_n}$ **= minimum, ……(1.3)**

which is equivalent to the orthogonality condition:

$\mathbf{r_n \perp < Ar_0, A^2r_0, A^3r_0, A^4r_0 \ldots\ldots, A^{n-1}r_0 >}$

……..(1.4)

The vigorous execution of **(1.2) – (1.4)** is **GMRES** iteration method. The execution uses Arnoldi Method (see [1]) to create an orthogonal basis for the Krylov Space $\mathbf{K^n(A;r_0)}$ which leads to an (n + 1) x n Hessenberg Least-Square problem (see [16]). At each step we have: $\mathbf{e_n = p_n(A)e_0}$, $\mathbf{r_n = p_n(A)r_0}$ where $\mathbf{p_n(Z)}$ is a polynomial of degree **n** with and $\mathbf{p_n(0) = 1.}$ Convergence will take place if and only if $\mathbf{p_n}$ exists for which $\|\mathbf{p_n(A)r_0}\|$ decreases rapidly, and a sufficient condition for this is that $\|\mathbf{p_n(A)}\|$ should decrease rapidly. **GMRES** is in fact well suited to normal matrices unfortunately nonsymmetric matrices are rarely normal. There is always storage requirement problem and to keep storage requirement under control, **GMRES** is often restarted after each **k- steps** (see [15]). It is also reported that **GMRES** algorithm is unreliable as there are instances where the residual norms produced by the algorithm, although non-increasing, do not converge to zero (see [16]). As far as the **GMRES-Like** methods, for example **BCG** method and a sufficient condition for this is that $\|\mathbf{p_n(A)}\|$ should decrease rapidly. **GMRES** is in fact well suited to normal matrices and unfortunately nonsymmetric matrices are rarely normal. There is always storage requirement problem and to keep storage requirement under control, **GMRES** is often restarted after each **k-steps** (see [15]). It is also reported that **GMRES** algorithm is unreliable as there are instances where the residual norms produced by the algorithm, although non-increasing, do not converge to zero (see **[16]**). As far as the **GMRES-Like** methods, for example **BCG** method and **CGS** methods, are concerned it is reported that they are susceptible to the possibility of breakdown-division by zero (see **[15]**). **Gutknecht ([9])** has presented a **BICGStab** method for matrices with complex spectrum.

The slightly older methods for the solution of (1.1) rely on variants of point block **ICCG**, where the conjugate gradient algorithm is preconditioned by a partial factorization of the coefficient matrix. However, these highly refined and efficient techniques do not extend to the more general case of unsymmetric linear systems (see **[4]). Kershaw [12]** generalizes the **ICCG** method to arbitrary non-singular sparse matrices arising from the partial differential equations and obtains the modified equation:

$$\underline{x}^{(k)} = \underline{x}^{(0)} + \sum_{i=1}^{i=k} \alpha_i (A^T A)^{i-1} A^T (A\underline{x}^{(0)} - \underline{y})$$

where α is chosen to minimize $\left| \underline{\mathbf{x}}^{(k)} - \underline{\mathbf{x}} \right|$.

The main problem with this approach is that the amount of work per iteration is almost doubled because it is necessary to multiply by $\mathbf{A}^T$ and $\mathbf{A}$ at each iteration in order to avoid the actual production of $\mathbf{A}^T\mathbf{A}$ in the sparse case. It can be noted that when $\mathbf{A}$ is not symmetric $(\mathbf{A}^T\mathbf{A})^{\frac{1}{2}}$ may be a poor approximation to $\mathbf{A}^T$ or $\mathbf{A}$ and, therefore, $(\mathbf{A}^T\mathbf{A})^{\frac{1}{2}}$ cannot be used in place of $\mathbf{A}$.

In other studies more importance is given to the splitting of the coefficient matrix, which draws to a close to the storage problem and provides reliable methods for the solution of non-symmetric systems (see [**17] and [13]**). We present the results and findings of one of these studies for a general reader. The computer codes of the method are published. We also present some new theorems which are proved recently.

The method is designed for the solution of non-symmetric systems **(1.1)** but it is capable of using all the robust techniques, developed for the solution of symmetric linear systems. The method is found to be more suitable for the iterative solution of **(1.1)**, where A is a sparse unsymmetric **M**-matrix or nearly symmetric structured **M**-matrix. We propose a particular class of regular splitting and call this class of regular splitting the generalized regular splitting or **GRS** and the corresponding iterative method, the generalized regular splitting method or **GRSI** method. A simple technique is designed to produce a symmetric and positive definite matrix or **SPD** splitting matrix. This allows for the stable efficient incomplete Cholesky factorization of the splitting matrix as the preliminary steps to an iterative solution. Because of the presence of a SPD matrix any other efficient technique developed for symmetric system can be used.

Definition1.1:

Given an M-matrix A, the splitting

$\mathbf{A} = (\mathbf{S} + \Delta) - (\mathbf{H} + \Delta)$……………………(1.2)

Is called the generalized regular splitting or **GRS** if $\mathbf{S} = [\mathbf{s}_{i,j}]$ is a symmetric M-matrix such that:

$$s_{i,j} = \begin{cases} \max\{a_{i,j}, a_{j,i}\} & \text{if } \mathbf{i \neq j} \\ a_{i,j} & \text{if } \mathbf{i = j} \end{cases}$$

The matrix $\mathbf{H} = [\mathbf{h}_{i,j}]$ is such that

$\mathbf{h}_{i,j} = \mathbf{s}_{i,j} - \mathbf{a}_{i,j} \geq \mathbf{0}$ for all i,j.

The non-negative diagonal matrix $\boldsymbol{\Delta} = [\boldsymbol{\delta}_{i,j}]$ is such that:

$$\delta_{i,j} = \begin{cases} 0 & \text{if } \mathbf{i \neq j} \\ \delta & \text{if } \mathbf{i = j} \end{cases}$$

2- Choice of δ.

If matrix S is not a diagonally dominant matrix, a non-negative diagonal matrix $\boldsymbol{\Delta} = [\boldsymbol{\delta}_{i,j}]$ is added to make the matrix S a diagonally dominant matrix to avoid the breakdown, where $\boldsymbol{\delta}$ is small and is chosen so that $(\mathbf{S} + \Delta)$ is a symmetric and diagonally dominant M-matrix and $(\mathbf{H} + \Delta) \geq \mathbf{0}$. A simple choice of $\boldsymbol{\delta}$ could be as follows:

$$\delta = \max_{i=1,\dots,n} \left(\sum_{j=1}^{n} |a_{ij}| - |a_{ii}| \right) + 10^7, \ \forall j \neq i$$

Such matrices $(\mathbf{S} + \Delta)$ can be easily factorized in a stable manner into incomplete **Cholesky** factors (see, for example, **[14]**). It is noted that the smaller the Δ the faster the convergence of the GRSI method.

3- Description of the algorithm.

Let us consider the system $\mathbf{A}\underline{\mathbf{x}} = \underline{\mathbf{b}}$, where $\mathbf{A} = [\mathbf{a}_{i,j}]$ is a sparse nonsymmetric, **N x N, M-**matrix . Let

$\mathbf{A} = (\mathbf{S} + \Delta) - (\mathbf{H}` + \Delta)$……………………(3.1)

be the **GRS** of **A**.

As $(\mathbf{S} + \Delta)$ is a symmetric and diagonally dominant M-matrix it can be decomposed into triangular factors or incomplete Cholesky factors i.e. $(\mathbf{S} + \Delta) = \mathbf{LDL}^T - \mathbf{E}$ is a regular splitting (see **[14]**), where **E** is computed explicitly.

It then follows that $\mathbf{A} = \mathbf{LDL}^T - (\mathbf{H}` + \mathbf{E} + \Delta)$ is a regular splitting. Substituting **H** for **H\` + E** the iterative system can be written:

$$LDL^T \underline{\mathbf{x}}^{(k)} = \underline{\mathbf{b}} + (\mathbf{H} + \Delta)\underline{\mathbf{x}}^{(k-1)}, \qquad \mathbf{k} \in \langle \mathbf{n} \rangle,$$

where $\underline{\mathbf{x}}^{(0)}$ is arbitrary. Substituting the values of $\underline{\mathbf{x}}^{(k-1)}$ in turn we obtain,

$\underline{\mathbf{x}}^{(k)} = [\mathbf{I} + \{(\mathbf{LDL}^T)^{-1}(\mathbf{H} + \Delta)\} + \{(\mathbf{LDL}^T)^{-1}(\mathbf{H} + \Delta)\}^2 + \{(\mathbf{LDL}^T)^{-1}(\mathbf{H} + \Delta)\}^3 + \dots\dots + \{(\mathbf{LDL}^T)^{-1}(\mathbf{H} + \Delta)\}^{(k-1)}](\mathbf{LDL}^T)^{-1}\underline{\mathbf{b}} + \{(\mathbf{LDL}^T)^{-1}(\mathbf{H} + \Delta)\}^k \underline{\mathbf{x}}^{(0)}$ ………………………………(3.2)

If $\mathbf{G = \{(LDL^T)^{-1}(H + \Delta)\}}$ then, $\mathbf{\underline{x}^{(k)} = [I + G + G^2 + G^3 + G^4 \ldots\ldots + G^{(k-1)}](LDL^T)^{-1}\underline{b} + + G^k\underline{x}^{(0)}}$**…….(3.3)**
Moreover $\mathbf{\underline{x}^{(k)} \rightarrow A^{-1}\underline{b}}$,
i.e. the solution of the linear system because the spectral radius **ρ(G)** is less than one, so that $\mathbf{G^{(k)} \rightarrow 0}$ and as $\mathbf{k \rightarrow \infty}$, $\mathbf{[\sum_{p=0}^{k-1} G^p] \rightarrow (I - G)^{-1}}$.

A number of authors like **[7]** and **[11]** have devised similar splittings for the case when A is a large sparse singular and irreducible M-matrix. They also worked with nearly symmetric matrices, which they define as having a symmetric, zero structure. The splitting matrix is constructed to be symmetric by dropping terms from A and, as a result, the direct part of the direct iterative method can take advantage of (i) symmetric pivoting (ii) a standard symmetric ordering scheme and (iii) a static storage scheme for the factor L. The author reports that the direct-iterative method is faster than straightforward iteration with the Gauss-Seidel method. The splitting we propose has similar advantages but no terms from A are dropped.

4-Convergence
The GRSI method associated with the GRS splitting is given by

$$(S+\Delta)\underline{x}^{(k)} = (H+\Delta)\underline{x}^{(k-1)} + \underline{b}$$

or $\underline{x}^{(k)} = (S+\Delta)^{-1}(H+\Delta)\underline{x}^{(k-1)} + (S+\Delta)^{-1}\underline{b}$ or

$$\underline{x}^{(k)} = G\underline{x}^{(k-1)} + \underline{\eta}$$

where the iteration matrix is given by $G = (S+\Delta)^{-1}(H+\Delta)$ and $\underline{\eta} = (S+\Delta)^{-1}\underline{b}$.

Theorem 4-1: ([18], p.89).
If $\mathbf{A = P - Q}$ is a regular splitting of matrix A and $A^{-1} \geq 0$, then

$$\rho(P^{-1}Q) = \rho(A^{-1}Q)/(1+\rho(A^{-1}Q) < 1 \qquad (4.1)$$

i.e. the matrix $P^{-1}Q$ is convergent, and the iterative method $\underline{x}^{(k)} = (P^{-1}Q)\underline{x}^{(k-1)} + P^{-1}\underline{b}$ converges for any initial value $\underline{x}^{(0)} of \underline{x}$ if (4.1) is satisfied.

Theorem 4-2:
The spectral radius of the iteration matrix of GRSI method is less than 1, and hence the method converges for any initial value $\underline{x}^{(0)} of \underline{x}$ when the coefficient matrix A is an M-matrix.
Proof: Follows directly from theorem 4-1 since GRS is a regular splitting and $A^{-1} \geq 0$.

Lemma 4-1:
The iteration matrix G of the GRSI method is a non-negative matrix when A is an M-matrix.
Proof: $A = (S+\Delta) - (H+\Delta)$
Now $(S+\Delta)^{-1} \geq 0$ as $(S+\Delta)$ is an M-matrix
And $(H+\Delta) \geq 0$ therefore,

$$G = (S+\Delta)^{-1}(H+\Delta) \geq 0.$$

Theorem 4-3: ([20], p 125)
Let A be a monotone matrix and let $A = Q_1 - R_1$ and $A = Q_2 - R_2$ be two regular splittings of A. If $R_2 \leq R_1$, then $S(Q_2^{-1}R_2) \leq S(Q_1^{-1}R_1)$.

Theorem 4-4. The smaller is the δ the faster is the convergence of the GRSI method.
Proof: Obvious from theorem 4-3.

Theorem 4-5. Let $A = [a_{i,j}]$ be an N x N non-symmetric M-matrix and $\underline{x} \geq 0$ be any non-zero vector such that $r_x = \min\left\{\sum_{j=1}^{n} a_{i,j}x_j / x_i\right\}$.
If $A = (S+\Delta) - (H+\Delta)$ is a GRS of A, then $\beta \geq \alpha$, where β is any eigenvalue of $(S+\Delta)$ and $\alpha = \sup_{\underline{x} \geq 0, \underline{x} \neq 0}\{r_x\}$.
Proof:
$A = (S+\Delta) - (H+\Delta) \Rightarrow$

$$(S+\Delta)^{-1}A = I - (S+\Delta)^{-1}(H+\Delta)$$

$= I - G.$
i.e. $G = I - (S+\Delta)^{-1}A$ and by Lemma 4-1 $\mathbf{G \geq 0}$.
i.e. $[I - (S+\Delta)^{-1}A]A^{-1} \geq 0 \Rightarrow A^{-1} \geq 0.$
i.e. $A^{-1} - (S+\Delta)^{-1} \geq 0$, by definition 1.1 $A^{-1} \geq (S+\Delta)^{-1}$.
By theorem 4-3, $\frac{1}{\alpha} \geq \frac{1}{\beta} \Rightarrow \beta \geq \alpha.$

Theorem 4-6. Let $G = [g_{i,j}] \geq 0$ be an N x N GRSI matrix, and P^* be the hyperoctant of vectors $\underline{x} \geq 0$. Then, for any $\underline{x} \in P^*$ either

$$\min_{1 \le i \le n} \left[\frac{\sum_{j=1}^{n} g_{i,j} x_j}{x_i} \right] < \rho(G)$$

or

$$< \max_{1 \le i \le n} \left[\frac{\sum_{j=1}^{n} g_{i,j} x_j}{x_i} \right]$$

$$\left[\frac{\sum_{j=1}^{n} g_{i,j} x_j}{x_i} \right] = \rho(G) \quad .$$

Moreover,

$$\sup_{x \in P^*} \left\{ \min_{1 \le i \le n} \left[\frac{\sum_{j=1}^{n} g_{i,j} x_j}{x_i} \right] \right\} = \rho(G)$$

$$= \inf_{x \in P^*} \left\{ \max_{1 \le i \le n} \left[\frac{\sum_{j=1}^{n} g_{i,j} x_j}{x_i} \right] \right\}$$

Proof: Follows the steps of theorem 2.2 of (Varga [1962], p.32).

Each upper and lower bound for the spectral radius of the iteration matrix of the GRSI method can thus be obtained by simple arithmetic steps.

5-Comparison with the standard methods.

There are economical and efficient techniques for the solution of linear symmetric systems. Furthermore, the situation regarding the availability of software implementing iterative schemes for symmetric systems is also satisfactory. As mentioned earlier these techniques for the solution of symmetric systems cannot be extended to the case of nonsymmetric linear systems. As far as the availability of the software is concerned apart from the codes of **Paige and Saunders** most are experimental. The packages like **SPARSPAK** (see **Geogge and Liu, 1980**) and the **Yale Sparse Matrix Package** (see **Duff and Reid, 1983**) are numerically sound if the coefficient matrix in a system of linear equations is **SPD**. If it is not, the packages may still be used but may be numerically unsound (see **[13]**). The **GRSIM** subroutine is designed to use the ease and comfort of all the efficient methods currently solving symmetric systems of linear equations, for the solution of nonsymmetric systems. Any technique, which can solve a symmetric system easily and economically, can be incorporated to the **GRSI** method to form a pristine version of the **GRSI** method e.g. the **GRSI** Cholesky factorization or **GRSI-CF** version and the **GRSI** Incomplete Cholesky factorization or **GRSI-ICF** version. The method uses a regular splitting. Therefore the convergence is guaranteed. For an unsymmetric M-matrix it can be easily proved numerically that the **GRSI** method converges faster than the **Jacobi** and **Gauss-Seidel** methods. The **GRSI** method is not compared with the direct solution method of the unsymmetric system because of the fact that the direct solution method is only about twice as expensive as a Cholesky factorization (see **[6]**).

6-Numerical Testing.

We illustrate the numerical behaviour of the **GRSI** method by the following simple examples. In most of the experiments we consider the linear elliptic equation:

$$a u_{xx} + c u_{yy} + d u_x + e u_y + f u = g(x,y) \qquad \text{.....(6.1)}$$

in the rectangular region $R: 0 \le x \le \alpha, 0 \le y \le \beta$, with Dirichlet boundary conditions. We suppose, for definiteness, that $a > 0$, $b > 0$ and $f \le 0$ and all are bounded in the region **R** and on its boundary **B**. Upon employing second-order central difference procedures, the finite-difference approximation for the above equation becomes: $\beta_1 U_{i+1,j} + \beta_2 U_{i-1,j} + \beta_3 U_{i,j+1} + \beta_4 U_{i,j-1} - \beta_0 U_{i,j} = h^2 g_{i,j}$

where β_i are functions of $x_i = ih$, $y_i = jh$, given by $\beta_0 = 2(a_{i,j} + c_{i,j} - \frac{1}{2}h^2 f_{i,j})$, $\beta_1 = a_{i,j} + \frac{1}{2}hd_{i,j}$, $\beta_2 = a_{i,j} - \frac{1}{2}hd_{i,j}$, $\beta_3 = c_{i,j} + \frac{1}{2}he_{i,j}$, $\beta_4 = c_{i,j} - \frac{1}{2}he_{i,j}$. The notation $a_{i,j}$ refer to $a(ih, jh)$, evaluated at the point where the computational module is centered. The coefficient matrix **A**, so obtained is an irreducible M-matrix.

Example-1.

We now give the approximate number of multiplication operations needed for the solution of **(1.1)**, obtained by the discretization of **(6.1)**, by different iterative methods (see **[14]**). Suppose **N** denotes the order of the coefficient matrix **A**. The initial work, such as the work

necessary for the estimation of the iteration-parameter for the SOR method and the work for the decomposition of the **GRSI** matrix into Incomplete Cholesky factors has been neglected. This work will in general be small compared to the computational work needed to carry out the actual iterations. Accurate determination of the SOR parameter may be difficult in some circumstances.

TABLE -1
Approximate Number of Multiplication Operation needed for one iteration in the solution of $A\underline{x} = \underline{b}$ (A is an N X N matrix)

Method	No. of Operations
SOR	6N
Gauss-Seidel	5N
Jacobi	5N
GRSI	7N

A number of nonsymmetric linear systems of equations were obtained by using the natural mesh ordering and discretizing **(6.1)** with arbitrary values of the coefficients **a, c, d, e, f** and the mesh size **h**. A comparison on the basis of asymptotic rate of convergence of the **GRSI-CF,** the **GRSI-ICF,** the **Jacobi,** the **Gauss-Seidel** and the **SOR** methods was made. We report a few of these results. The matrix Δ is considered to be a zero matrix and the optimum parameter for **SOR** is used through out.

Example-2. We considered a BST matrix with known sparsity and eigenvalues. The result is given in the following table.

TABLE II
Number of Iteration Needed for Convergence with Tolerance $\Im = 10^{-7}$.

Method	Spectral Radius S(G)	η = rate of convergence
GRSI-ICF	0.0781830	2.5487025
SOR	0.01528233	1.8784726
Gauss-Seidel	0.3003972	1.2026496
Jacobi	0.5433333	0.6100323

Example-3. (cf. [10], p.354)
This example illustrate that in some cases the GRSI method converges even though the **RF method** with **Chebyshev acceleration (RF-SI method)** fails to converge. Consider the partial differential equation:

$u_{xx} + u_{yy} + \beta u_x = 0$**……..……(6.2)**

Assuming Dirichlet boundary conditions in the unit square $0 \le x \le 1$, $0 \le y \le 1$, and using the standard five-point finite-difference discretization we obtain the difference equation:

$h^{-2}\{u(x+h, y) + u(x-h, y) + u(x, y+h) + u(x, y-h) - 4u(x,y)\} + (\tfrac{1}{2})\beta h^{-1}\{u(x+h, y) - u(x-h, y)\} = 0.$

Let us apply the **RF-SI method** and the **GCW method** for the case $\beta = -3$ and $h = \frac{1}{3}$. There are four interior points and the β_k for this special case are $\beta_1 = 0.5$, $\beta_2 = 1.5$, $\beta_3 = 1.0$, $\beta_4 = 1.0$, and $\beta_0 = 4.0$. Since $(\tfrac{1}{2})h^3|\beta| \le 1$, the eigenvalues of the **RF method** are given by

$$\lambda_{p,q} = (\tfrac{1}{2})\cos(p\pi h) + (\tfrac{1}{2})\cos(p\pi h)\sqrt{1 - \{(\tfrac{1}{2})h\beta\}^2}$$

$p,q = 1, \ldots h^{-1} - 1.$

The spectral radius of the **RF-SI method** is **1.4542307,** which clearly shows that the **RF-SI method** fails to converge. The spectral radius of the iteration matrix of the **GCW method** is given by:

$$S(G) = \frac{h|\beta|\cos\pi h}{4\sqrt{2}\sin\left(\frac{\pi h}{2}\right)} = 0.1767767$$

The coefficient matrix is an irreducibly diagonally dominant M-matrix.

The asymptotic average rate of convergence of the **GRSI-ICF, GCW-SI, SOR** (with opt. relax. fact.), **Gauss-Seidel**, and **Jacobi methods** are noted respectively as **2.9, 1.7, 1.6, 1.5, 0.76**. Which shows that **GRSIM** is four times faster than the **Jacobi method**, two times faster than the **SOR method** and the **Gauss-Seidel method.**

REFERENCES

[1] **W. E. Arnoldi,** The principle of minimized iteration in the solution of the matrix eigenvalue problem. Quart. Appl. Math. 9, 1951. pp. 17-29.

[2] **S. L. Campbell, I. C. F. Ipsen, C. T. Kelly, C. D. Meyer and Z. Q. Xue,** Convergence estimates for solution of integral equations with GMRES. J. of integ. equ. and appl. 8,1996, pp. 19-34.

[3] **L. Collatz,** Aufgaben monotoner Art. Arch Math. 3, 1952, pp. 366-376.

[4] **I. S. Duff,** The solution of nearly symmetric sparse linear equations. CSS150, Computer Science and System Division, A.E. R. E. Harwell, Oxon, 1983.

[5] **I. S. Duff, and J. K. Reid,** The multifrontal solution of indefinite sparse symmetric linear systems. ACM Trans. Math. Softw. 9, 1983, pp. 302-325.

[6] **I. S. Duff, and J. K. Reid,** Some design feature of a sparse matrix code. ACM Trans. Math. Softw. 5, 1979a, pp. 18-35.

[7] **R. E. Funderlic, and R. J. Plemmons,** A combined direct-iterative ethod for certain M-matrix Linear systems. SIAM J. ALG. DISC. METH., 5, 1984, pp. 33-42.

[8] **A. George, J. Liu, and E. Ng,** User Guide for SPARSPAK: Waterloo Sparse Linear Package. Report CS-78-30.Department of Computer Science, University of Waterloo, 1980.

[9] **M. H. Gutknecht,** Variants of BICGtab for matrices with complex spectrum. SIAM J. Sci. Comput., 14, 1993, pp. 1020- 1033.

[10] **L. A. Hageman, D. M. Young,** Applied iterative methods. Academic press, New York, 1981.

[11] **W. J. Harrod, and R. J. Plemmons,** Comparison of some direct methods for computing stationary distributions of Markov Chains. SIAM. J. SCI. STAT. COMPUT., 5, 1984, pp. 453-469.

[12] **D. S. Kershaw,** The incomplete Cholesky-conjugate gradient method for the iterative solution of systems of linear equations. J. Com. Phys. 26, 1978, pp. 43-65.

[13] **G. J. Makinson, and A. A. Shah,** OAHM: a FORTRAN subroutine for solving a class of unsymmetric linear systems of equations. CANMER 2, 1986, pp. 499-508.

[14] **J. A. Meijerink, and H. A. van der Vorst,** An iterative solution method for linear systems of which the coefficient matrix is symmetric M-matrix. Maths. Comp. 31, 1977, pp. 148-162.

[15] **N. M. Nachtigal, S. C. Reddy, and L. N. Trefethen,** How fast are nonsymmetric iterations? SIAM J. Matrix Anal. Appl.13, 1992, pp. 778-795.

[16] **Y. Saad, and M. H. Schultz, GMRES:** A generalized minimum residual algorithm for solving nonsymmetric linear systems. SIAM J. Sci. Statist. Com.7,1986, pp. 856-869.

[17] **A. A. Shah, GRSIM-**A FORTRAN Subroutine for the Solution of non-symmetric Linear Systems, Commun. Numer. Meth. Engng, 18, 2002, pp. 803-815.

[18] **R. S. Varga,** Matrix iterative analysis, Prentice Hall, Englewood Cliffs, New Jersey, 1962.

[19] **C. Vuik, and H. A. van der Vorst,** A comparison of some GMRES-like methods. Lin. Alg. and its Appl. 160, 1992, pp. 131-162.

[20] **D. M. Young,** Iterative Solution of large linear systems. Academic Press, New York, 1971.