Gauss-Radau Quadrature Rule Using Special Class of Polynomials

M. A. Bokhari and Asghar Qadir

Abstract— **A form of Gauss-Quadrature rule over $[0,1]$ has been investigated that involves the derivative of the integrand at the pre-assigned left or right end node. This situation arises when the underlying polynomials are orthogonal with respect to the weight function $\omega(x):=1-x$ over $[0,1]$. Along the lines of Golub's work, the nodes and weights of the quadrature rule are computed from a Jacobi-type matrix with entries related to simple rational sequences. The structure of these sequences is based on some characteristics of the identity-type polynomials recently developed by one of the authors. The devised rule has a slight advantage over that subject to the weight function $\omega(x):=1$.**

***Index Terms*— Gauss-Radau quadrature rule, Jacobi-matrix, Hypergeometric series, Identity-type polynomials, 3-term recurrence relation.**

I. INTRODUCTION

For any function $f:[0,1]\to\Re$ and a positive weight function $\omega:[0,1]\to\Re$, set $f_\omega=\dfrac{f}{\omega}$. Let π_k denote the class of all polynomials of degree up to *k*. If $\int_0^1 x^j\omega(x)dx<\infty,\ j=0,1,2,\dots$, then there exists a quadrature rule

$$\int_0^1 f(x)dx=\int_0^1 f_\omega(x)\omega(x)dx = \sum_{i=1}^{n}\nu_i f_\omega(t_i)+\nu_{n+1}f_\omega(1)+R_n \quad , \qquad (1)$$

which is exact for $f_\omega\in\pi_{2n}$. It is called the (*n*+1)-point right hand Gauss-Radau formula [4] for the weight function ω. Let $p_{n,\omega}$ denote the polynomial of degree *n* orthogonal with respect to ω. Then the nodes t_i in (1) are known to be the zeros of $p_{n,\omega}$ whereas the weights ν_i, $i=1,2,\dots n+1$, can be computed by interpolation at all the interior nodes t_i and the pre-assigned node 1. It is interesting to note that the nodes (including 1) and the weights in (1) can be determined by an elegant result due to Golub [3] which we state as:

Lemma A: *The n + 1 nodes* $t_1,t_2,\dots,t_n$ and 1 *are precisely the eigenvalues of a modified Jacobian matrix*

$$\boldsymbol{J}_{n+1,w}=\begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & 0 & 0 & . & 0\\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & 0 & . & 0\\ 0 & \sqrt{\beta_2} & \alpha_2 & \ddots & . & .\\ 0 & 0 & \ddots & \ddots & \ddots & .\\ . & . & . & \ddots & \alpha_{n-1} & \sqrt{\beta_n}\\ 0 & 0 & . & . & \sqrt{\beta_n} & \alpha_n^{R,1}\end{bmatrix} \qquad (2)$$

where α_k,β_k *are the coefficients in the 3-term recurrence relation*

$$p_{k+1}(t)=(t-\alpha_k)p_k(t)-\beta_k p_{k-1}(t) \qquad (3)$$
$$p_{-1}(t)=0,\ p_0(t)=1$$

satisfied by the monic orthogonal polynomials $p_{k,\omega}$ *and*

$$\alpha_n^{R,1}:=1-\beta_n\frac{p_{n-1}(1)}{p_n(1)}. \qquad (4)$$

In addition, the weights are given by $\nu_i:=\beta_0 u_{i,1}^2$ *with* $\beta_0:=\int_0^1\omega(x)dx$ and $u_{i,1}$ *as the first components of the associated normalized eigenvectors.*

Remark 1: An explicit representation of the recursion coefficients α_k,β_k in (3) is given by

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$$\alpha_k = \frac{\langle tp_k, p_k\rangle_w}{\langle p_k, p_k\rangle_w}, \quad \beta_{k+1} = \frac{\langle p_{k+1}, p_{k+1}\rangle_w}{\langle p_k, p_k\rangle_w}, \tag{5}$$

for $k = 0,1,2,...$

II. Gauss-Radau Formula with $\omega(x):1-x$

We noticed that the formula (1) is capable of utilizing $f'(1)$ in its structure if f is differentiable at $t = 1$ and $\omega(x) := 1-x$. In such cases, an alternative form of the quadrature rule (1) is given in

Theorem 1: *Let $f:[0,1]\to\Re$ be differentiable at $t=1$ and let $\omega(x):=1-x$. Then the quadrature rule in (1) may be expressed as*

$$\int_0^1 f(t)dt = f(1) - \sum_{i=1}^{n} \nu_i \frac{f(t_i)-1}{t_i-1} - \nu_{n+1} f'(1) + R_n, \tag{6}$$

where the nodes t_i and the weights ν_i *are subject to the weight function* $\omega(x):=1-x$.

Formula (6) may be justified by rewriting the left side as

$$\int_0^1 f(t)dt = f(1) - \int_0^1 \frac{f(t)-1}{t-1}(1-t)dt$$

and setting

$$\tilde{f}(t) := \begin{cases} \dfrac{f(t)-f(1)}{t-1} & \text{if } t \neq 1 \\ f'(1) & \text{if } t = 1 \end{cases}$$

and then using (1) for $f_\omega = \tilde{f}$ and $\omega(x) = 1-x$.

Remark 2. The quadrature formula (6) is also applicable to an integral over any finite interval $[a,b]$ subject to the differentiability of the integrand at $x = b$. For this purpose it is enough to note that

$$\int_a^b g(x)dx = \int_0^1 f(t)dt$$

with $f(t) := \dfrac{g((b-a)t+a)}{b-a}$, $t \in [0,1]$. In addition, (6) can be utilized if the pre-assigned node is 0 instead of 1. In this case, we have to consider

$$\int_a^b g(x)dx = \int_0^1 f(t)dt \text{ with } f(t) := g(1-t),\ t \in [0,1].$$

We are interested in the computational aspects of the proposed rule (6). Note that the efficacy of (6) relies on appropriate representation of the orthogonal polynomials $p_{n,\omega}$, $\omega(x) := 1-x$, and the corresponding recursion coefficients α_k, β_k (cf (5)). This feature is discussed in the next sections.

III. Identity-type Polynomials

Recently, it has been established in [1] that the identity-type function

$$\hat{e}(t;c) := \frac{\Gamma(c)}{\Gamma(c-1)} \sum_{m=0}^{\infty} \frac{(c)_m(-c)_m}{(m!)^2} t^m, \tag{7}$$

where $(c)_0 = 1$ and $(c)_n = c(c-1)(c-2)....(c-n+1)$, satisfies the second order differential equation

$$t(1-t)\frac{d^2y}{dt^2} + (1-t)\frac{dy}{dt} + c^2 y = 0, \qquad c > 0. \tag{8}$$

Note that the hypergeometric series $\sum_{m=0}^{\infty} \frac{(c)_m(-c)_m}{(m!)^2} t^m$ in (7) is defined for all integral values of *c*. Here, our interest lies with the associated polynomials

$$\begin{aligned} e_n^*(t) &:= F[n,-n;1;t] \\ &= \sum_{m=0}^{\infty} \frac{(n)_m(-n)_m}{(m!)^2} t^m, \quad n = 1,2,3,..... \end{aligned} \tag{9}$$

which indeed provide solution of the 2nd order differential equation (8) when *c* is replaced by *n*. Some properties of these polynomials are listed below [1]:

(a) For $n = 1,2,3,.....$, $e_n^*(t) \in \pi_n$ and

$$e_n^*(t) = (1-t)F[1+n, 1-n; 1; t] \tag{10}$$

since $F[a,b;c;z] = (1-z)^{c-a-b} F[c-a, c-b; c; z]$ ([5], p.60). In particular,

$$e_1^*(t) = 1-t$$
$$e_2^*(t) = (1-t)(1-3t)$$
$$e_3^*(t) = (1-t)(1-8t+10t^2)$$
$$\vdots$$

(b) With the notation $\langle h,k\rangle_{\omega} := \int_0^1 h(t)k(t)\omega(t)dt$,

$$\langle e_n^*, e_m^* \rangle_{w^*} = \begin{cases} 0 & \text{if } n \neq m \\ \frac{1}{2n} & \text{if } n = m \end{cases}, \tag{11}$$

where $w^*(t) = 1/(1-t)$, i.e., the polynomials $e_n^*(t)$, $n = 1,2,....$, are orthogonal with respect to w^* over $[0,1]$.

(c) The leading coefficient of $e_n^*(t)$ in (4), say κ_n, is given by

$$\kappa_n = (-1)^n \frac{(2n-1)!}{n!(n-1)!}. \tag{12}$$

IV. Orthogonality of Factor Polynomials $F[1+n,1-n;1;t]$

We set (cf (10), (12))

$$e_{n-1}(t) := \frac{-1}{\kappa_n} F[1+n,1-n;1;t], \tag{13}$$

and note that these are monic polynomials of degree $n-1$, $n = 1,2,...,$ and orthogonal with respect to the weight function

$$w(t) := 1-t \tag{14}$$

over the interval [0,1]. The polynomials $e_n(t)$, $n = 0,1,2,...$ indeed satisfy the 3-term recurrence relation

$$e_{n+1}(t) = (t-\alpha_n)e_n(t) - \beta_n e_{n-1}(t), \tag{15}$$

where $\beta_0 = 1/2$ and for $n = 0,1,2,...$

$$\alpha_n = \frac{\langle te_n, e_n \rangle_w}{\langle e_n, e_n \rangle_w}, \quad \beta_{n+1} = \frac{\langle e_{n+1}, e_{n+1} \rangle_w}{\langle e_n, e_n \rangle_w}. \tag{16}$$

Remark 3: The recursion coefficients in (16) may be expressed as

$$\left.\begin{aligned} \alpha_n &= \frac{\langle te_{n+1}^*, e_{n+1}^* \rangle_{w^*}}{\langle e_{n+1}^*, e_{n+1}^* \rangle_{w^*}} \\ \beta_{n+1} &= \left(\frac{\kappa_{n+1}}{\kappa_{n+2}}\right)^2 \frac{\langle e_{n+2}^*, e_{n+2}^* \rangle_{w^*}}{\langle e_{n+1}^*, e_{n+1}^* \rangle_{w^*}} \end{aligned}\right\}, \; n = 0,1,2,... \tag{17}$$

A simple manipulation based on (9), (10) and (13) leads to

$$\left.\begin{aligned} F[1+n,1-n;1;0] &= 1 \\ F[1+n,1-n;1;1] &= (-1)^{n+1} n \end{aligned}\right\}. \tag{18}$$

and

$$\kappa_{n-1}\kappa_n = -\kappa_{n+1}(\kappa_{n-1}\alpha_n + \kappa_n\beta_n). \tag{19}$$

Based on the relations (15)-(19), we have

Theorem 2: Let $\omega(t) := 1-t$ be the weight function (cf (14)) in Lemma A. Then

(1) $\alpha_n = \dfrac{2(n+1)^2 - 1}{4(n+1)^2 - 1}$, $n = 0,1,2,\ldots$

(2) $\beta_n = \dfrac{n(n+1)}{4(2n+1)^2}$, $n = 1,2,3,\ldots$

(3) $\alpha_n^{R,1} = 1 - \beta_n \dfrac{e_{n-1}(1)}{e_n(1)} = \dfrac{3n^2+6n+2}{4n^2+6n+2}$, $n = 0,1,2,\ldots.$

Remark 4: The sequences $\{\alpha_n\}$ and $\{\beta_n\}$ as determined in the above theorem provide an explicit representation of the Jacobian-type matrix (cf (2)) in terms of *n*. Thus the nodes and weights required in the proposed quadrature formula (6) easily obtainable by an application of Lemma A.

V. Numerical Examples

We have applied the 6-point quadrature rule (6) subject to $\omega(t) := 1-t$ to different type of functions. The results thus obtained are compared with those given in ([2], p.81) for ordinary right hand 6-point Gauss-Radau rule as described in (5) with $\omega(t) := 1$. The outcomes of our simulation results are given in the following table:

Table 1

Integrals	$R_{6,1}$	$R_{6,(1-t)}$	Exact Value
$\int_0^1 \sqrt{x}dx$	0.6671 5566	0.6669 1977	0.6666 6667
$\int_0^1 x^{\frac{3}{2}}dx$	0.3999 8857	0.3999 9623	0.4000 0000
$\int_0^1 \frac{dx}{1+x}$	0.6931 4718	0.6931 4718	0.69314718
$\int_0^1 \frac{dx}{(1+x^4)}$	0.8669 7059	0.8669 7291	0.8669 72987
$\int_0^1 \frac{dx}{1+e^x}.$	0.3798 8549	0.3798 8549	0.3798 8549
$\int_0^1 \frac{xdx}{e^x-1}.$	0.7775 0463	0.7775 0463	0.7775 0463
$\int_0^1 \frac{2dx}{2+\sin 10\pi x}$	0.8793 0050	1.1535 1517	1.1547 0054

Note: $R_{6,1=}$ ordinary right hand 6-point Gauss-Radau rule with $\omega(t):=1$

$R_{6,(1-t)}$ = Right hand 6-point Gauss-Radau rule subject to $\omega(t):=1-t$

VI. Conclusion

We have introduced a form of Gauss-Radau quadrature rule (cf (6)) which utilizes the derivative of the integrand at the pre-assigned right end node. This rule also preserves the exactness and convergence properties like the standard Gauss-Radau quadrature rule (cf (1)) [4]. As indicated in Table 1, it produces slightly better results, when compared with the Gauss-Radau quadrature rule subject to weight function $\omega(t):=1$. In fact, for the last example there is a dramatic improvement of accuracy. The proposed rule can be easily used as the left hand Gauss-Radau rule by a trivial modification of the integrand, and also for any finite interval $[a,b]$ instead of $[0,1]$ (See Remark 2).

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