A Comparison of Mann and Ishikawa iterations of quasi-contraction operators

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Abstract—**It is generally conjectured that the Mann iteration converges faster than the Ishikawa iteration for any operator defined on an arbitrary closed convex subset of a Banach space. The recent result of Babu et al [1] shows that this conjecture can be proved for a class of quasi-contractive operators called the Zamfirescu operators[10]. In this paper it is shown that the proof can indeed be generalised to that of quasi-contraction maps.**

Keywords: fixed point, quasi-contraction, Mann iteration, Ishikawa iteration. 2000 AMS Mathematics Classification: 47H10, 46A16

1 Introduction

Let X be a real Banach space, and C a nonempty convex subset of X. Let T is a self map of C, and let $x_o \in C$. The Mann iteration (see [8]) is defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n \tag{1}$$

The Ishikawa iteration (see [5], [8]) is defined by

$$y_{n+1} = (1 - \alpha_n)y_n + \alpha_n Tz_n \tag{2}$$

$$z_n = (1 - \beta_n)y_n + \beta_n Ty_n \tag{3}$$

where $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset [0, 1)$.

The following result was proved in [1].

Theorem 1. *Let X be a Banach space, C a closed convex subset of X, and $T : C \to C$ be a Zamfirescu operator [10], i.e. an operator for which there exist the real numbers a, b, c satisfying $0 < a < 1$, $0 < b$, $c < 1/2$ such that for each pair $x, y \in C$, at least one of the following is true:*
(i) $\|Tx - Ty\| \le a\|x - y\|$;
(ii) $\|Tx - Ty\| \le b[\|x - Tx\| + \|y - Ty\|]$;
(iii) $\|Tx - Ty\| \le c[\|x - Ty\| + \|y - Tx\|]$.
If (a) $\sum \alpha_n = \infty$ and (b) $0 \le \alpha_n, \beta_n \le 1$, then the Mann iteration defined by (1) and the Ishikawa iteration defined by (2)-(3) converge strongly to the unique fixed point of T. Moreover, the Mann iteration converges faster than the Ishikawa iteration to the fixed point of T.

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We want to show that this Theorem is true for a more general class of operators, called quasi-contraction operators.

Definition 1 [4]. Let $T : M \to M$ be a mapping of a metric space (M, d) into itself. A mapping T will be called $quasi - contraction$ if for some $0 \le k < 1$ and all $x, y \in M$,

$$\begin{aligned} d(Tx, Ty) \le\ & k.max\{d(x, y); d(x, Tx); d(y, Ty); \\ & d(x, Ty); d(y, Tx)\} \end{aligned}$$

Berinde [3] showed that when X is a Banach space, the Picard iteration converges faster to the unique fixed point of Zamfirescu operator T than the Mann iteration. This was confirmed to be true emperically on some numerical tests performed with the aid of the software package fixed point (see [3]).
The open problem posed by Berinde [3]: Does the Picard iteration converges faster than the Mann iteration for the larger class of quasi-contraction maps was answered answered in the affirmative in [7].

When a Zamfirescu operator, (see Theorem 1 or [10]), is defined on a closed convex subset of a Banach space, the Ishikawa iteration converges [2]. It was shown in [6] that the Mann iteration converges for these same class of operators defined on a closed convex subset of a complete metrisable locally convex space. It was shown in [9] that the Mann and the Ishikawa iterations are equivalent when dealing with Zamfirescu operators. Babu and Prasad [1] further showed that the Mann iteration converges faster than the Ishikawa iteration for the class of Zamfirescu operators. Consequently, in view of [3], the Picard iteration converges faster than the Ishikawa iteration for these same class of operators.
For a more general class of operators, called the quasi-contraction maps (Definition 1), it was shown in [5] that the Ishikawa iteration for quasi-contraction maps

ISBN:978-988-98671-2-6

converges to the unique fixed point of T. Rhoades and Soltuz [8] showed that the Mann iteration (1) and the Ishikawa iteration (2)-(3) are equivalent when applied to quasi-contractions.
It is therefore natural and important to ask which of the two iterations, i.e. the Mann and the Ishikawa iterations, converges faster when applied to quasi-contractions. This paper shows that the Mann iterations converges faster.

We need the following definitions for our results.

Definition 2 [3]. Let $\{a_n\}_{n=0}^{n=\infty}$ and $\{b_n\}_{n=0}^{n=\infty}$ be two sequences of real numbers that converge to a and b respectively, and assume there exists

$$l = lim_{n\to\infty}\frac{|a_n - a|}{|b_n - b|}$$

If $l = 0$, then we say that $\{a_n\}_{n=0}^{n=\infty}$ *converges faster* to a than $\{b_n\}_{n=0}^{n=\infty}$ to b.

Definition 3 [3]. Let $\{u_n\}_{n=0}^{n=\infty}$ and $\{v_n\}_{n=0}^{n=\infty}$ be two fixed point iteration procedures that converge to the same fixed point p on a normed space X such that the error estimates

$$\|u_n - p\| \le a_n, \quad n = 0, 1, 2, ... \tag{4}$$

and

$$\|v_n - p\| \le b_n, \quad n = 0, 1, 2, ... \tag{5}$$

are available, where $\{a_n\}_{n=0}^{n=\infty}$ and $\{b_n\}_{n=0}^{n=\infty}$ are two sequences of positive numbers (converging to zero). If $\{a_n\}_{n=0}^{n=\infty}$ converges faster than $\{b_n\}_{n=0}^{n=\infty}$, then we say that $\{u_n\}_{n=0}^{n=\infty}$ converges faster to p than $\{v_n\}_{n=0}^{n=\infty}$.

2 The Main Result

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Theorem 2. *Let C be a nonempty closed convex subset of a Banach space X and let $T : C \to C$ be a quasi-contraction map. Let $\{x_n\}$ and $\{y_n\}$ be the Mann and Ishikawa iterations respectively defined by (1) and (2)-(3) for $x_o,\ y_o \in C$ with $\{\alpha_n\}$ and $\{\beta_n\}$ real sequences such that (a) $0 \le \alpha_n,\ \beta_n \le 1$ and $\sum \alpha_n = \infty$. Then $\{x_n\}$ and $\{y_n\}$ converge strongly to the unique fixed point of T, and moreover, the Mann iteration converges faster, than the Ishikawa iteration, to the fixed point of T.*

Proof. We follow the technique in [1] which was originally used in [3]. The Ishikawa iteration defined by (2)-(3) converges strongly to the unique fixed point of T (e.g. see [5]). Also, the Mann iteration converges strongly to the unique fixed point of T (e.g. see [8]). Since the fixed point of T is unique [4], then both iterations must converge to the same fixed point which we denote by p.
It is not difficult to see that the quasi-contraction map satisfies the following inequalities

$$\|Tx - Ty\| \le \delta\{\|x - y\| + \|x - Tx\|\} \tag{6}$$

$$\|Tx - Ty\| \le \delta\{\|x - y\| + \|y - Tx\|\} \tag{7}$$

for all $x, y \in K$ where $\delta = \max\{k, \frac{k}{1-k}\}$ Let $\{x_n\}$ be the Mann iteration associated with T, then, in view of (1), we have

$$\|x_{n+1} - p\| \le (1 - \alpha_n)\|x_n - p\| + \alpha_n\|Tx_n - p\|. \tag{8}$$

Suppose $x = p$ and $y = x_n$, (6) becomes

$$\|Tx_n - p\| \le \delta\|x_n - p\| \tag{9}$$

In view of (8) and (9), we have

$$\begin{array}{rcl}\|x_{n+1} - p\| & \le & (1 - \alpha_n)\|x_n - p\| + \alpha_n\delta\|x_n - p\| \\ & = & [1 - \alpha_n(1 - \delta)]\|x_n - p\|\end{array}$$

Hence

$$\|x_{n+1} - p\| \le \prod_{k=1}^{n}[1 - \alpha_k(1 - \delta)].\|x_1 - p\|,\ n = 0,\ 1,\ 2, ... \tag{10}$$

It is clear that

$$1 - \alpha_k(1 - \delta) > 0\ \forall\ k = 0,\ 1,\ 2, ... \tag{11}$$

Similarly, let $\{y_n\}$ be the Ishikawa iteration defined in (2)-(3), then, we have

$$\|y_{n+1} - p\| \le (1 - \alpha_n)\|y_n - p\| + \alpha_n\|Tz_n - p\|. \tag{12}$$

If $x = p$ and $y = z_n$ in (7), we have

$$\|Tz_n - p\| \le \delta\|z_n - p\| + \delta\|z_n - p\| = 2\delta\|z_n - p\| \tag{13}$$

If $x = p$ and $y = y_n$ in (7), we have

$$\|Ty_n - p\| \le \delta\|y_n - p\| + \delta\|y_n - p\| = 2\delta\|y_n - p\|. \tag{14}$$

We know that

$$\|z_n - p\| \le (1 - \beta_n)\|y_n - p\| + \beta_n\|Ty_n - p\|. \tag{15}$$

In view of (12)-(15), we have

$$\begin{array}{rcl}\|y_{n+1} - p\| & \le & (1 - \alpha_n)\|y_n - p\| + 2\delta\alpha_n\|z_n - p\| \\ & \le & (1 - \alpha_n)\|y_n - p\| + 2\delta\alpha_n[(1 - \beta_n) \\ & & \|y_n - p\| + \beta_n\|Ty_n - p\|] \\ & = & (1 - \alpha_n)\|y_n - p\| + 2\delta\alpha_n(1 - \beta_n) \\ & & \|y_n - p\| + 2\delta\alpha_n\beta_n\|Ty_n - p\| \\ & \le & (1 - \alpha_n)\|y_n - p\| + 2\delta\alpha_n(1 - \beta_n) \\ & & \|y_n - p\| + 2\delta\alpha_n\beta_n 2\delta\|y_n - p\| \\ & = & [(1 - \alpha_n) + 2\delta\alpha_n(1 - \beta_n) \\ & + & 4\alpha_n\beta_n\delta^2]\|y_n - p\| \\ & = & [1 - \alpha_n(1 - 2\delta + 2\beta_n\delta - 4\beta_n\delta^2)] \\ & & \|y_n - p\| \\ & = & [1 - \alpha_n(1 - 2\delta)(1 + 2\beta_n\delta)]\|y_n - p\|....(**)\end{array}$$

Since $(1-2\delta)(1+2\beta_n\delta) < 1-4\delta^2 \leq 1$, it is clear that

$$1-\alpha_n(1-2\delta)(1+2\beta_n\delta) > 0 \ \forall\ n = 0, 1, 2, ... \tag{16}$$

We consider the following two cases.
Case (1). Let $\delta \in (0, 1/2]$. Hence

$$1-\alpha_n(1-2\delta)(1+2\beta_n\delta) \leq 1 \ \forall\ n = 0, 1, 2, ... \tag{17}$$

(**) then becomes

$$\|y_{n+1} - p\| \leq \|y_n - p\| \ \forall\ n \tag{18}$$

and hence

$$\|y_{n+1} - p\| \leq \|y_1 - p\| \ \forall\ n \tag{19}$$

If we compare the coefficients of (10) and (19), and using Definition 3 so that

$$a_n = \prod_{k=1}^{n}[1-\alpha_k(1-\delta)] \ and \ b_n = 1, \tag{20}$$

we have $lim_{n\to\infty}(a_n/b_n) = 0$
Case (ii). Let $\delta > 1/2$. In this case we have

$$1-\alpha_n(1-2\delta)(1+2\beta_n\delta) \leq 1-\alpha_n(1-4\delta^2) \tag{21}$$

and so (**) becomes

$$\|y_{n+1} - p\| \leq [1-\alpha_n(1-4\delta^2)]\|y_n - p\| \ \forall\ n \tag{22}$$

Hence

$$\|y_{n+1} - p\| \leq \prod_{k=1}^{n}[1-\alpha_k(1-4\delta^2)]\|y_1 - p\| \tag{23}$$

Comparing (10) and (23) and using Definition 3, we have

$$a_n = \prod_{k=1}^{n}[1-\alpha_k(1-\delta)] \ and \ b_n = \prod_{k=1}^{n}[1-\alpha_k(1-4\delta^2)] \tag{24}$$

Clearly, $a_n \geq 0$ and $b_n \geq 0 \ \forall\ n$ and $\frac{a_n}{b_n} = \prod_{k=1}^{n}\frac{1-\alpha_k(1-\delta)}{1-\alpha_k(1-4\delta^2)}$. Also, since $1-\alpha_k(1-\delta) < 1-\alpha_k(1-4\delta^2)$ for each k, then

$$\frac{min[1-\alpha_k(1-\delta), k = 1, 2..n]}{max[1-\alpha_k(1-4\delta), k = 1.2..n]} < 1$$

Clearly $\prod_{k=1}^{n}\frac{1-\alpha_k(1-\delta)}{1-\alpha_k(1-4\delta^2)} < (\frac{min[1-\alpha_k(1-\delta),\ k=1,2....n]}{max[1-\alpha_k(1-4\delta),\ k=1,2....n]})^n$ and $lim_{n\to\infty}\frac{a_n}{b_n} = 0$.
Therefore in all the two cases $\{a_n\}$ converges faster than $\{b_n\}$ and hence the Mann iteration converges faster than the Ishikawa iteration to the fixed point p of T.

In [4] it was shown that the Picard iteration defined by $x_{n+1} = Tx_n,\ n = 0, 1, 2....$ converges to the unique fixed point of a quasi-contraction map T defined in Definition 1 and converges faster than the Mann iteration. Furthermore, in [8], it was shown that the Ishikawa iteration and the Mann iteration are equivalent for a quasi-contraction map. In view of these results and Theorem 2, we have the following result which is a generalisation of the equvalence result in [1].

Corollary 1. *The Picard iteration converges faster than the Ishikawa iteration (2)-(3), to the fixed point of a quasi-contraction operator defined in Definition 1.*

Corollary 2 [1]. *The Picard iteration converges faster than the Ishikawa iteration (2)-(3), to the fixed point of a Zamfirescu operator defined in Theorem 1.*

Remark. It should be observed, just like in the case of Zamfirescu operators, that for a quasi-contraction operator T, the Picard iteration converges faster than the Mann iteration [7], and the Mann iteration converges faster than the Ishikawa iteration to the fixed point of T (Theorem 2). Ishikawa has two parameters, $\{\alpha_n\}$ and $\{\beta_n\}$, the Mann iteration has only one parameters $\{\alpha_n\}$ while the Picard iteration has none. It appears that the more the parameters for an iteration process, the slower the rate of convergence. At least this is true in the case of Picard, Mann and the Ishikawa iterations when applied to quasi-contraction maps. It is therefore an open problem whether this conjecture is true for other known iteration procedures and for a more general class of operators.

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