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Using Wavelet for Numerical Solution of Fredholm Integral Equations

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Abstract— **In this paper, we bring three theorems that enable us to approximate the solution of Ferdholm integral equations of the second kind. Then we use the Coifman wavelets or Coiflets as scaling functions for projection that satisfied the conditions of theorems for approximation. Also we use this projection to convert the integral equation to a Galerkin system, which is the most important of the expansion methods for solving linear integral equations. Finally, by using numerical examples we show that our estimation have a good degree of accuracy**

Keywords: Integral Equation, Wavelet, Galerkin System.

1 Introduction

This section provide an overview of the topics that we need in this paper. A **wave** is usually defined as an oscillating function of time or space, such as a sinusoid. Fourier analysis is wave analysis, it expands a signal or function in term of sinusoid. A **wavelet** is a "small wave", which has its energy concentrated in time to give a tool for the analysis of transient, nonstationary, or time-varying phenomena([1, 2]).

We will take wavelets and use them in series expansion of signals or functions as the same way a Fourier series uses the wave or sinusiod to represent a signal or function, ([3, 4]).

1.1 Wavelet and Wavelets Expansion Systems

A signal or function $f(t)$ can be often better analyzed, described, or processed if expressed as a linear decomposition by

$$f(t) = \sum_l a_l \psi_l(t) \tag{1}$$

where l is an integer index for the finite or infinite sum, a_l are the real valued expansion coefficients, and $\psi_l(t)$ are a set of real valued functions of t called expansion set. if the expansion (1) is unique, the set is called a **basis** for the class of function that can be so expressed. If the basis is orthogonal, meaning

$$< \psi_k(t), \psi_l(t) > = \int \psi_k(t)\psi_l(t)\, dt = 0, \quad k \neq l, \tag{2}$$

then the coefficients can be calculated by the inner product

$$a_k = < f(t), \psi_k(t) > = \int f(t)\psi_k(t)\, dt. \tag{3}$$

For the **Wavelet expansion**, a two parameter system is constructed such that (1) becomes

$$f(t) = \sum_k \sum_j a_{j,k}\psi_{j,k}(t) \tag{4}$$

where both j and k are integer indices and the $\psi_{j,k}(t)$are the wavelet expansion functions that usually form an orthogonal basis. The set of expansion coefficients $a_{j,k}$ are called the **discrete wavelet transform (DWT)** of $f(t)$ and (4) is the inverse transform ([2]).

1.2 The Discrete Wavelet Transform

Our goal is to generate a set of expansion functions such that any signal in $\mathbf{L}^2(\mathbb{R})$ (the space of square integral functions) can be represented by the series

$$f(t) = \sum_{j,k} a_{j,k} 2^{j/2} \psi(2^j t - k), \tag{5}$$

now let $\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k), \quad j, k \in \mathbb{Z}$, then

$$f(t) = \sum_{j,k} a_{j,k}\psi_{j,k}(t), \tag{6}$$

A more specific form indicating how the $a_{j,k}$ are calculated can be written using inner products if the $\psi_{j,k}(t)$ form an orthonormal basis for the space of signals of interest, we can write

$$f(t) = \sum_{j,k} < \psi_{j,k}(t), f(t) > \psi_{j,k}(t), \tag{7}$$

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1.3 The Scaling Function

In order to use the idea of multiresolution, we will start by defining the scaling function and then define the wavelet in terms of it, ([3]). As we described for the wavelet in previous section, we define a set of scaling functions in terms of integer translates of the basic scaling function by

$$\phi_k(t) = \phi(t-k), \quad k \in \mathbb{Z}, \quad \phi \in \mathbf{L}^2. \tag{8}$$

The subspace of $\mathbf{L}^2(\mathbb{R})$ spanned by these functions is defined as

$$V_0 = \overline{Span_{k\in\mathbb{Z}}\{\phi_k(t)\}}. \tag{9}$$

The over-bar denotes closure. This means that

$$f(t) = \sum_k a_k\, \phi_k(t), \quad \forall f(t) \in V_0. \tag{10}$$

One can generally increase the size of the subspace by changing the time scale of the scaling functions. A two dimentsional family of function is generated from the basic scaling function by scaling and translation by

$$\phi_{j,k} = 2^{j/2}\phi(2^j t - k) \tag{11}$$

whose span over k is

$$V_j = \overline{Span_k\{\phi_k(2^j t)\}} = \overline{Span_k\{\phi_{j,k}(t)\}}, \quad \forall k \in \mathbb{Z}. \tag{12}$$

This means that if $f(t) \in V_j$, then it can be expressed as

$$f(t) = \sum_k a_k \phi(2^j t + k). \tag{13}$$

2 Approximation of Signals by Scaling Function Projection

We define the k^{th} moments of $\phi(t)$ and $\psi(t)$ as

$$m(k) = \int t^k \phi(t)\, dt, \qquad m_1(k) = \int t^k \psi(t)\, dt, \tag{14}$$

and the discrete k^{th} moments of $h(n)$ and $h_1(n)$ as

$$\mu(k) = \sum_n n^k h(n), \qquad \mu_1(k) = \sum_n n^k h_1(n). \tag{15}$$

The orthogonal projection of a signal $f(t)$ on the scaling function subspace V_j is given and denoted by

$$P^j\{f(t)\} = \sum_k < f(t), \phi_{j,k}(t) > \phi_{j,k}(t), \tag{16}$$

which gives the component of $f(t)$ which is in V_j and which is the best least squares approximation to $f(t)$ in V_j. We now state an important relation of the projection (16) as an approximation to $f(t)$ in terms of the number of zero wavelet moments and the scale, ([1]).

Theorem 1. . *If $m_1(l) = 0$ for $l = 0, 1, \ldots, L$ then the $\mathbf{L}^2$ error is*

$$\varepsilon_1 = \|f(t) - P^j\{f(t)\}\|_2 \leq C_1 2^{-j(L+1)}, \tag{17}$$

where C_1 is a constant independent of j and L, but depend on $f(t)$ and the wavelet system, ([1]).

This theorem states, ([1, 2]): A second approximation involves using the samples of $f(t)$ as the inner product coefficients in the wavelet expansion of $f(t)$ in (16). We denotes this sampling approximation by

$$S^j\{f(t)\} = \sum_k 2^{-j/2} f(k/2^j)\phi_{j,k}(t). \tag{18}$$

Theorem 2. . *If $m(l) = 0$ for $l = 1, \ldots, L$ then the $\mathbf{L}^2$ error is*

$$\varepsilon_2 = \|f(t) - S^j\{f(t)\}\|_2 \leq C_2 2^{-j(L+1)}, \tag{19}$$

where C_2 is a constant independent of j and L, but depend on $f(t)$ and the wavelet system, ([1]).

This is a similar approximation or converges to the previous theorem results, but relates the projection of $f(t)$ on j-scale subspace to the sampling approximation in that same subspace. This "vector space" shows the nature and relationship of the two types of approximations. The use of samples as inner products is an approximation within the expansion subspace V_j. The use of a finite expansion to represent a signal $f(t)$ is an approximation from $\mathbf{L}^2$ onto subspace V_j. Theorems (1) and (2) shows the nature of those approximations, which is very good for wavelet.

If we consider a wavelet system where the number of scaling functions and wavelets are set zero and this number is as large as possible, then the following results is true ([1]):

Theorem 3. . *If $m_1(l) = m(l) = 0$ for $l = 1, \ldots, L$, and $m_1(0) = 0$, then the $\mathbf{L}^2$ error is*

$$\varepsilon_3 = \|f(t) - S^j\{f(t)\}\|_2 \leq C_3 2^{-j(L+1)}, \tag{20}$$

where C_3 is a constant independent of j and L, but depend on $f(t)$ and the wavelet system.

3 Coifman Wavelet-Galerkin Method by Scaling Function Projection

In this section, we make Galerkin system, that its projection uses Coifman wavelet as basis. We first describe general scaling function projection, and then, discuss about special case, where say it Coifman wavelet.

Consider a linear Fredholm integral equation of second kind:

$$x(s) = y(s) + \int_a^b k(s,t)x(t)\,dt, \tag{21}$$

where $k \in \mathbf{L}^2([a,b]\times[a,b])$, $x \in \mathbf{L}^2([a,b])$, and $s \in [a,b]$. Set operator K as below, ([6]):

$$(Kx)(s) = \int_a^b k(s,t)x(t)\,dt.$$

By this notation, we can write (21), as

$$(I-K)x = y. \tag{22}$$

3.1 Wavelet-Galerkin Method

Now we approximate solution (signal) $x(s)$, and kernel $k(s,t)$ by wavelet in m scale,([7, 8]):

$$x(s) = \sum_{i\in\mathbb{Z}} \tilde{x}_i 2^{m/2}\phi(2^m s - i) \tag{23}$$

$$k(s,t) = \sum_{i\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} \tilde{k}_{i,l} 2^{m/2}\phi(2^m s - i)2^{m/2}\phi(2^m t - l) \tag{24}$$

and

$$y(s) = \sum_{i\in\mathbb{Z}} \tilde{y}_i 2^{m/2}\phi(2^m s - i) \tag{25}$$

where $\tilde{x}_i, \tilde{y}_i$ and $\tilde{k}_{i,l}$ are the wavelet coefficient of $x(s)$, $y(s)$ and $k(s,t)$, respectively. By setting $u = 2^m s$, $v = 2^m t$, $x_i = \tilde{x}_i 2^{m/2}$, $y_i = \tilde{y}_i 2^{m/2}$ and $k_{i,l} = \tilde{k}_{i,l}2^m$, from (23), (24) and (25), we have ([7, 8, 9]):

$$x(s) = \sum_{i\in\mathbb{Z}} x_i\phi(u-i) \tag{26}$$

$$k(s,t) = \sum_{i\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} k_{i,l}\phi(u-i)\phi(v-l) \tag{27}$$

and

$$y(s) = \sum_{i\in\mathbb{Z}} y_i\phi(u-i). \tag{28}$$

By applying equations (26), (27), and (28) in integral equation (21),we have ([9]):

$$\sum_{i\in\mathbb{Z}} x_i\phi(u-i) = \sum_{i\in\mathbb{Z}} y_i\phi(u-i) + \tag{29}$$

$$\int_a^b \sum_{i\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} k_{i,l}\phi(u-i)\phi(v-l)\sum_{i\in\mathbb{Z}} x_i\phi(v-i)\,dv \tag{30}$$

Now by orthonormality of bases, we have ([2, 4]):

$$\int_a^b \phi(u-i)\phi(u-l)\,du = \delta_{i,l}, \tag{31}$$

and therefore by taking the inner product of both sides of equation (30) with $\phi(u-i)$, we have, ([2]):

$$x_i - \sum_{l\in\mathbb{Z}} k_{i,l}x_l = y_i, \quad i \in \mathbb{Z}. \tag{32}$$

We can write this system in compact form as bellow:

$$(\mathbf{I}-\mathbf{K})\mathbf{x} = \mathbf{y} \tag{33}$$

where

$$\mathbf{I} = [\delta_{i,l}], \quad \mathbf{K} = [k_{i,l}], \quad \mathbf{x} = [x_i], \quad \textit{and}\ \mathbf{y} = [y_i].$$

4 Numerical performances

For showing efficiency of numerical method, we consider the following examples. We note that, ([6]):

$$\| e_N \| = \left(\int_{-1}^1 e_N^2(t)\,dt\right)^{\frac{1}{2}} \approx \left(\frac{1}{N}\sum_{i=0}^N e_N^2(x_i)\right)^{\frac{1}{2}},$$

where

$$e(s_i) = x(s_i) - x_N(s_i), \quad i = 0,1,\ldots,N.$$

Such that $x_N(s_i)$ and $x(s_i)$ are, respectively the approximate and exact solutions of the integral equations.

Notation 1. *In the following examples, we consider Coiflet scaling function and coefficient by $N = 6$, ([1, 5]).*

4.1 Examples

Example 1 : Consider $x(s) = \sin s - s + \int_0^{\pi/2} st\,x(t)\,dt$ with exact solution $x(s) = \sin s$.

Example 2 : Consider $x(s) = e^s - \frac{e^{s+1}-1}{s+1} + \int_0^1 e^{st}\,x(t)\,dt$ with exact solution $x(s) = e^s$.

Example 3 : Consider $x(s) = s + \int_0^1 K(s,t)\,x(t)\,dt, K(s,t) = \begin{cases} s, & s \le t \\ t, & s \ge t \end{cases}$ with exact solution $x(s) = \sec 1 \sin s$.
The following table shows the computed error $\| e_N \|$ for the before examples.

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Table 1 : Errors $\|e_N\|$ at scale m=4

N	Example 1	Example 2	Example 3
2	3.2×10^{-2}	4.1×10^{-2}	5.7×10^{-2}
3	5.2×10^{-3}	3.7×10^{-3}	3.7×10^{-3}
4	4.7×10^{-6}	6.3×10^{-5}	5.9×10^{-4}
5	1.3×10^{-9}	9.4×10^{-7}	4.3×10^{-7}
6	2.1×10^{-12}	2.7×10^{-10}	8.1×10^{-9}

5 Conclusion

We know that, the choice of basis is always important in determining the conditioning of the linear equations and hence the stability of the calculation against both quadrature and round-off errors ([6]).

Therefore we use scaling function projection with Coifman wavelet to obtain orthonormal basis, which is very useful for projection methods, since an orthonormal basis has the advantage that it guarantees the stability of the matrix equations in Galerkin Method ([4, 5, 7, 8]).

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