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Local Absorbing Boundary Conditions for a Finite Element Discretization of the Cubic Nonlinear Schrödinger Equation

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Abstract—**We consider in this work the initial value problem for the one dimensional cubic nonlinear Schrödinger equation. In order to integrate it numerically, one option frequently used, is to impose local absorbing boundary conditions. A finite element discretization in space of the cubic nonlinear Schrödinger equation is considered along with the absorbing boundary conditions obtained for an analogous discretization of the linear equation. For the implementation of these boundary conditions, an adaptive strategy is proposed, so that the boundary conditions change at each time step, depending on the numerical solution that is arriving to the boundary at that moment. The numerical experiments are satisfactory, obtaining a good absorption at the boundary.**

Keywords: absorbing boundary conditions, Schrödinger equation.

1 Introduction

Let us consider the initial value problem for the cubic nonlinear Schrödinger equation,

$$\left\{ \begin{array}{rcl} \partial_t u(x,t) & = & i(\partial_{xx}u(x,t) + \nu |u(x,t)|^2 u(x,t)), \\ & & x \in \mathbb{R}, \quad t \geq 0, \\ u(x,0) & = & u_0(x), \quad x \in \mathbb{R}, \end{array} \right. \tag{1}$$

with ν a real constant. In order to integrate numerically this problem, that is defined in an unbounded domain, it is necessary to consider a finite subdomain $[x_l, x_r]$ (where the support of $u_0(x)$ is included) and to impose artificial boundary conditions. When the solution of the new problem, defined for $x \in [x_l, x_r]$, is just the restriction of the original solution to the bounded subdomain, the boundary conditions are called transparent (TBCs). Nevertheless, the TBCs have the disadvantage of being usually nonlocal. That is why in many cases, for a practical purpose, local absorbing boundary conditions (ABCs) are preferred. The ABCs are constructed as an approximation to the TBCs and they only allow small reflections of the numerical solution at the boundary. There are many works about this subject. We remark the pioneer paper of Engquist and Majda [6] for the wave equation.

There are also works where ABCs are developed for the linear Schrödinger equation [1, 2, 3, 7]. In the previous references, finite differences discretizations in space are considered. Moreover, in [2, 3] the ABCs are specific for the finite differences discretization. On the other hand, in [4] a linear finite element discretization in space is considered and ABCs are obtained which are specific for it. These ABCs have the advantage of being suitable with a higher order finite element discretization in the interior domain (see [4] for details).

For the nonlinear Schrödinger equation there are fewer works in the literature. One option (see for example [8]) is to modify the original equation in (1), including an absorbing potential. Nevertheless this technique requires an artificial change in the equation. Another option is to use the ABCs obtained for the linear Schrödinger equation. This is the idea followed by [5] and [9]. However, the results in [9] are not very optimistic.

In this work we have considered a linear finite element discretization in space for the cubic nonlinear Schrödinger equation and we have fitted, in an adaptive way, the ABCs obtained in [4] for the linear equation. Moreover the implementation of the ABCs is also adaptive in a similar way to [3]. Let us see these questions with more detail in Section 2.

2 Discretization of the problem and absorbing boundary conditions

Let us take a positive parameter $h = (x_r - x_l)/N > 0$, and consider the grid of $[x_l, x_r]$ given by $x^j = x_l + jh$, $j = 0, \ldots, N$. As we have already said, we are going to use linear finite elements to discretize in space the one dimensional cubic nonlinear Schrödinger equation. In this way, we obtain the following discretization for the

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interior domain,

$$\frac{1}{6}\left(\frac{d}{dt}u^{j-1}+4\frac{d}{dt}u^{j}+\frac{d}{dt}u^{j+1}\right)= \qquad (2)$$
$$\frac{i}{h^2}\left(u^{j-1}-2u^{j}+u^{j+1}\right)+\varphi_j(u_h),$$

for $j=1,\ldots,N-1$, where

$$u_h(t)=\sum_{i=1}^{N}u^{i}(t)\rho_i$$

with ρ_i the shape functions and $\varphi_j(u_h)$ denotes the nonlinear term.

The ABC for the analogous discretization of the linear equation is given by

$$\delta_0 u^{N-1}+\delta_1\frac{d}{dt}u^{N-1}=\delta_2 u^{N}+\delta_3 v^{N}+\delta_4\frac{d}{dt}v^{N}, \qquad (3)$$

for the right boundary (a similar expression is obtained for the left boundary), where v^N is such that

$$\frac{d}{dt}u^{N}=v^{N}. \qquad (4)$$

The coefficients δ_j depend on the constant potential V of the linear equation. In order to use these ABCs for the nonlinear equation, at each time step we are going to choose $V=\nu|u^J|^2$, for a fixed J closed to N. On the other hand, the coefficients δ_j also depend on certain nodes which are chosen in an adaptive way, similarly to [3] for the linear case. In this way, we can obtain good absorption results without using a priori information of the solution.

Finally, the result of (2), (3) and (4) is a system of ordinary differential equations

$$\begin{array}{rcl} R\frac{d}{dt}\mathbf{u}(t) & = & M\mathbf{u}(t)+\phi(\mathbf{u}(t)) \\ \mathbf{u}(0) & = & \mathbf{u}_0 \end{array} \qquad (5)$$

where

$$\begin{array}{rcl} \mathbf{u}(t) & = & [v^0(t),u^0(t),u^1(t),\ldots,u^N(t),v^N(t)]^T, \\ \mathbf{u_0} & = & [0,u_0^0,u_0^1,\ldots,u_0^N,0]^T, \end{array}$$

which we integrate with the implicit mid point rule.

Next, we are going to see how the numerical results are satisfactory and we obtain a good absorption of the numerical solution at the boundary. Let us see, for example, the result when we consider the following initial condition

$$u_0(x)=\sqrt{\frac{2\alpha}{\nu}}\exp(iUx/2)\,\mathrm{sech}(\sqrt{\alpha}x), \qquad x\in[-L,L],$$

where, we will take $\alpha=1$, $\nu=1$ and $U=5$. We will consider $L=35$ so that the computational domain $[-L,L]$ is big enough to contain the support of the initial condition. In Figures 1 and 2 we can observe the modulus of the numerical solution at four different fixed times. It travels to the right and when it arrives to the boundary, it is absorbed by the ABCs. Notice the good results of absorption that we obtain with the adaptive strategy we propose.

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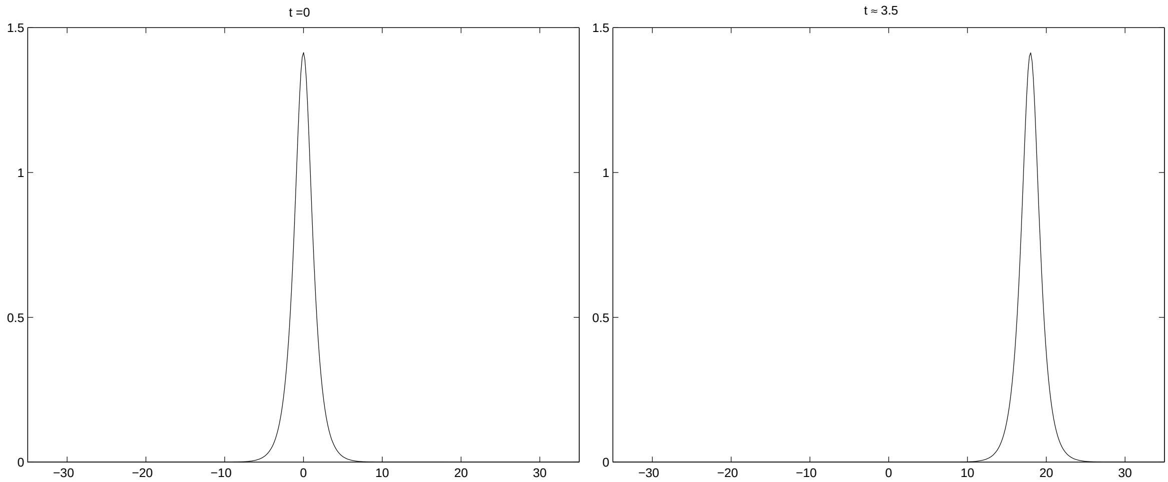

Figure 1: Modulus of the numerical solution for $t = 0$ and $t \approx 3.5$.

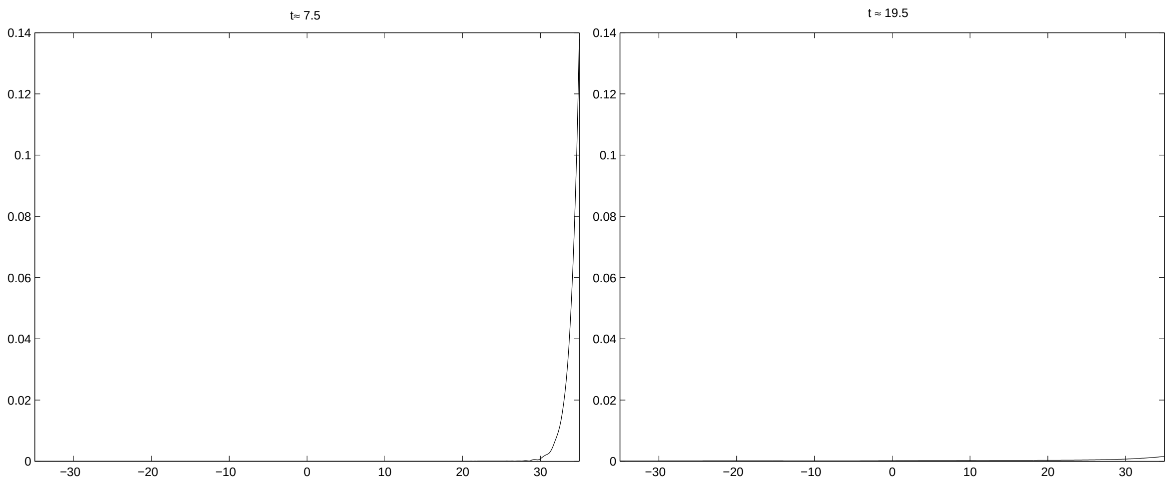

Figure 2: Modulus of the numerical solution for $t \approx 7.5$ and $t \approx 19.5$.