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No Classic Boundary Conditions

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Abstract–**We consider the boundary value problem:**

$$\begin{cases} x^{(m)}(t) = f(t, \overline{x}(t)), \quad a \le t \le b, \quad m > 1 \\ x(a) = \beta_0 \\ \Delta x^{(k)} \equiv x^{(k)}(b) - x^{(k)}(a) = \beta_{k+1}, \quad k = 0, ..., m-2 \end{cases}$$

where $\overline{x}(t) = (x(t), x'(t), ..., x^{(m-1)}(t))$**,** $\beta_i \in \mathbf{R}$, $i = 0, ..., m-1$**, and** f **is continuous at least in the interior of the domain of interest. We prove the existence and uniqueness of the solution under certain conditions.**

Keywords: Bernoulli polynomials, Green's function, Differential Equation.

1 Introduction

In this paper we consider the following boundary problem:

$$\begin{cases} (1a) \quad x^{(m)}(t) = f(t, \overline{x}(t)), \quad a \le t \le b, \quad m > 1 \\ (1b) \quad x(a) = \beta_0, \quad \Delta x_a^{(k)} \equiv x^{(k)}(b) - x^{(k)}(a) = \beta_{k+1} \\ \qquad k = 0, ..., m-2 \end{cases} \tag{1}$$

where $\overline{x}(t) = (x(t), x'(t), ..., x^{(m-1)}(t))$, f is defined and continuous at least in the domain of interest included in $[a, b] \times \mathbf{R}^m$; $[a, b] \subset \mathbf{R}$, and $\beta_i \in \mathbf{R}$, $i = 0, ..., m-1$.
The equation $(1a)$ is very frequent in mathematical applications, as example for *m=3,4* it is related to beam's analysis. The boundary conditions in $(1b)$ aren't classic and we don't find them in literature because it is easy to give them physical interpretations; this is the motivation of our investigation. The outline of the paper is the following: in section 2 we give the preliminaries, in section 3 we investigate the existence and uniqueness of the solution.

2 Definition and preliminaries

If $B_n(x)$ is the Bernoulli polynomial of degree n defined by [3]

$$\begin{cases} B_0(x) = 1 \\ B_n'(x) = nB_{n-1}(x) \qquad n \ge 1 \\ \int_0^1 B_n(x)dx = 0 \qquad n \ge 1 \end{cases} \tag{2}$$

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in a recent paper Costabile [2] proved the following theorems.
Theorem 1. Let $f \in \mathbf{C}^{(\nu)}[a, b]$ we have

$$f(x) = f(a) + \sum_{k=1}^{\nu} S_k\left(\frac{x-a}{h}\right)\frac{h^{(k-1)}}{k!}\Delta f_a^{(k-1)} - R_\nu[f](x) \tag{3}$$

where
$h = b - a, \quad S_k(t) = B_k(t) - B_k(0),$
$f_a = f(a), \quad \Delta f_a^{(k)} = f^{(k)}(b) - f^{(k)}(a)$

$$R_\nu[f](x) = \frac{h^{(\nu-1)}}{\nu!}\cdot$$

$$\cdot \int_a^b \left(f^{(\nu)}(t)\left(B_\nu^*\left(\frac{x-t}{h}\right) + (-1)^{\nu+1}B_\nu\left(\frac{t-a}{h}\right)\right)\right)dt \tag{4}$$

and

$$B_m^*(t) = B_m(t) \quad 0 \le t \le 1, \quad B_m^*(t+1) = B_m^*(t) \tag{5}$$

Theorem 2. Putting

$$P_\nu[f](x) = f_a + \sum_{k=1}^{\nu} S_k\left(\frac{x-a}{h}\right)\frac{h^{(k-1)}}{k!}\Delta f_a^{(k-1)} \tag{6}$$

the following equalities are true

$$\begin{cases} P_\nu[f](a) = f_a = f(a) \\ P_\nu[f](b) = f_b = f(b) \\ \Delta P_\nu^{(k)} \equiv P_\nu^{(k)}(b) - P_\nu^{(k)}(a) = \Delta f_a^{(k)} \equiv f^{(k)}(b) - f^{(k)}(a), \\ k = 1, ..., \nu - 1 \end{cases} \tag{7}$$

The conditions (7) in the previous equalities are called *Bernoulli interpolatory conditions* analogously to Lidstone interpolatory conditions [1].
Theorem 3. If $f \in C^{(\nu+1)}[a, b]$ we have

$$R_\nu[f](x) = \int_a^b G(x, t) f^{(\nu+1)}(t)dt \tag{8}$$

where

$$G(x, t) = \frac{1}{\nu!}\left[(x-t)_+^\nu - \sum_{k=1}^{\nu} S_k\left(\frac{x-a}{h}\right)\cdot\right.$$

$$\left.\cdot\frac{h^{(k-1)}}{k!}\binom{\nu}{k-1}(b-t)^{\nu-k+1}\right] \tag{9}$$

with

$$(x)_+^k = \begin{cases} x^k & if \quad x \ge 0 \\ 0 & if \quad x < 0 \end{cases} \tag{10}$$

Theorem 4. For $f \in C^{(\nu)}[a,b]$ we have

$$|R_\nu[f](x)| \leq \frac{h^{\nu-1}}{6(2\pi)^{\nu-2}} \int_a^b \left| f^{(\nu)}(t) \right| dt \qquad (11)$$

For the following, we need
Lemma 1. If $f \in C^{(\nu)}[a,b]$ and satisfies the homogeneous Bernoulli interpolatory conditions i.e:

$$\begin{cases} f(a) = 0 \\ f^{(k)}(b) - f^{(k)}(a) = 0 \qquad k = 0, ..., \nu - 2 \end{cases} \qquad (12)$$

putting

$$M_\nu = \max_{a \leq t \leq b} \left| f^{(\nu)}(t) \right| \qquad (13)$$

the following inequalities hold

$$\left| f^{(k)}(t) \right| \leq C_{\nu,k} \cdot M_\nu \cdot (b-a)^{\nu-k} \quad 0 \leq k \leq \nu - 1 \quad (14)$$

where

$$\begin{cases} C_{\nu,0} = \dfrac{1}{3(2\pi)^{\nu-2}} \\ C_{\nu,k} = \dfrac{1}{6(2\pi)^{\nu-k-2}} \qquad k = 1, 2, .., \nu - 1 \end{cases} \qquad (15)$$

Proof. From (12) the expansion (3) becomes

$$f(t) = \frac{h^{\nu-1}}{\nu!} \left[B_\nu \left(\frac{t-a}{h} \right) - B_\nu \right] \Delta f_a^{(\nu-1)} - R_\nu[f](t) \quad (16)$$

We also have

$$f^{(\nu-1)}(t) = f^{(\nu-1)}(a) + \int_a^t f^{(\nu)}(s) ds$$

from which

$$\left| \Delta f_a^{(\nu-1)} \right| \equiv \left| f^{(\nu-1)}(b) - f^{(\nu-1)}(a) \right| \leq M_\nu (b-a) \quad (17)$$

Using the known inequalities in [3]

$$|B_l(x)| \leq \frac{l!}{12(2\pi)^{l-2}} \quad l \in N, \quad l \geq 0, \quad 0 \leq x \leq 1$$

and (11), (17) we have from (16)

$$|f(t)| \leq \frac{h^\nu \cdot M_\nu}{3(2\pi)^{\nu-2}} \qquad (18)$$

that is (14) for $k = 0$.
With a successive derivation of (16) and by applying (12) we have

$$f^{(k)}(t) = \frac{h^{\nu-(k+1)}}{(\nu-k)!} \Delta f_a^{(\nu-1)} B_{\nu-k} \left(\frac{t-a}{h} \right) - \frac{h^{\nu-(k+1)}}{(\nu-k)!} \cdot$$

$$\cdot \int_a^b f^{(\nu)}(t) B_{\nu-k}^* \left(\frac{t-s}{h} \right) ds \qquad k = 1, 2, ..., \nu - 1 \qquad (19)$$

and applying the previous inequalities we give

$$\left| f^{(k)}(t) \right| \leq \frac{h^{\nu-k} \cdot M_\nu}{6(2\pi)^{\nu-k-2}} \qquad k = 1, 2, .., \nu - 1$$

that is (14). Furthermore,
Lemma 2. If $f, g \in C^{(\nu)}[a,b]$ which satisfies (12) then

$$\left(f^{(\nu)}(t) = g^{(\nu)}(t) \quad \forall t \in [a,b] \right) \Rightarrow f(t) = g(t) \quad \forall t \in [a,b]$$

Proof. The result follows by the application of (3) and (12).

3 Existence and uniqueness

To the boundary value problem (1a)-(1b), which after (7) is called the *Bernoulli boundary value problem*, we associate the homogeneous boundary value problem

$$\begin{cases} x^{(m)}(t) = f(t, \overline{x}(t)), \qquad a \leq t \leq b, \qquad m > 1 \\ x(a) = x(b) = 0 \\ \Delta x^{(k)} \equiv x^{(k)}(b) - x^{(k)}(a) = 0 \qquad k = 1, ..., m-2 \end{cases} \qquad (20)$$

From Theorem 3, the solution of the boundary value problem (20) is

$$x(t) = \int_a^b G(t,s) f(s, \overline{x}(s)) \, ds \qquad (21)$$

where $G(t,s)$ is the *Green* function [4] defined by (9), with $\nu = m - 1$.
The polynomial $P_{m-1}[x](t)$ defined by (6) with $x(a) = \beta_0$, $x^{(k)}(b) - x^{(k)}(a) = \beta_{k+1}, \quad k = 0, ..., m-2$, satisfies the boundary value problem:

$$\begin{cases} P_{m-1}^{(m)}[x](t) = 0 \\ P_{m-1}[x](a) = \beta_0 \\ \Delta P_{m-1}^{(k)} \equiv P_{m-1}^{(k)}(b) - P_{m-1}^{(k)}(a) = \beta_{k+1}, \; k = 0, ..., m-2 \end{cases}$$

Therefore, the boundary value problem $(1a)$-$(1b)$ is equivalent to the following nonlinear *Fredholm* integral equation:

$$x(t) = P_{m-1}[x](t) + \int_a^b G(t,s) f(s, \overline{x}(s)) \, ds \qquad (21a)$$

Now we use a well-known tecnique to prove the existence of a solution for problem $(1a)$-$(1b)$, [1], but different proofs are also possible.
Theorem 5. Let us suppose that

(i) $k_i > 0 \quad 0 \leq i \leq m-1$ are given real numbers and let Q be the maximum of $|f(t, x_0, ..., x_{m-1})|$ on the compact set $[a,b] \times D_0$, where $D_0 = \{(x_0, ..., x_{m-1}) : |x_i| \leq 2k_i, \quad 0 \leq i \leq m-1\}$;

(ii) $\max \left| P_{m-1}^{(i)}[x](t) \right| \leq k_i \quad 0 \leq i \leq m-1$, where $P_{m-1}[x](t)$ is the polynomial relative to x as in (6);

(iii) $(b-a) \leq \left(\dfrac{k_i}{Q \cdot C_{m,i}} \right)^{\frac{1}{(m-i)}} \qquad 0 \leq i \leq m-1.$

Then, the Bernoulli boundary value problem has a solution in D_0.
Proof. The set

$$B[a,b]=\left\{x(t)\in C^{(m-1)}[a,b]:\left\|x^{(i)}\right\|_\infty\leq 2k_i,\ 0\leq i\leq m-1\right\}$$

is a closed convex subset of the Banach space $C^{(m-1)}[a,b]$. Now we define an operator $T: C^{(m-1)}[a,b]\to C^{(m)}[a,b]$ as follows:

$$(T[x](t)) = P_{m-1}[x](t) + \int_a^b G(t,s)f(s,\overline{x}(s))ds \quad (22)$$

It is clear, after (21a), that any fixed point of (22) is a solution of the boundary value problem (1a) and (1b). Let $x(t)\in B[a,b]$, then from (22), lemma 1, hypothesis **(i)**,**(ii)**,**(iii)** we find:

- **(a)** $TB[a,b]\subseteq B[a,b]$;
- **(b)** the sets $\left\{T[x]^{(i)}(t) : x(t)\in B[a,b]\right\}$, $0\leq i\leq m-1$ are uniformly bounded and equicontinuous in $[a,b]$;
- **(c)** $\overline{TB[a,b]}$ is compact from the *Ascoli - Arzela theorem*;
- **(d)** from the *Schauder fixed point theorem* a fixed point of T exists in D_0.

Corollary 1. Suppose that the function $f(t,x_0,x_1...,x_{m-1})$ on $[a,b]\times\mathbf{R}^m$ satisfies the following condition

$$|f(t,x_0,x_1...,x_{m-1})|\leq L+\sum_{i=0}^{m-1}L_i\,|x_i|^{\alpha_i}$$

where $L, L_i\quad 0\leq i\leq m-1$ are non negative constants, and $0\leq\alpha_i\leq 1$.
Then the boundary value problem (1a) and (1b) has a solution.
Lemma 3. For the *Green* function defined by (9), for $\nu=m-1$ the following inequalities hold:

$$|G(t,s)|\leq g \qquad (23)$$

with

$$g=\frac{1}{\nu!}(b-a)^m\left(1+\frac{2\pi^2 m!}{3(2\pi-1)}\right).$$

Proof.
The proof follows from the known inequalities of Bernoulli polynomials and from simple calculations.
Theorem 6. Suppose that the function $f(t,x_0,x_1...,x_{m-1})$ on $[a,b]\times D_1$ satisfies the following condition

$$|f(t,x_0,x_1...,x_{m-1})|\leq L+\sum_{i=0}^{m-1}L_i\,|x_i| \qquad (24a)$$

where

$$D_1=\{(x_0,x_1...,x_{m-1}) : |x_i|\leq\max_{a\leq t\leq b}\left|P_{m-1}^{(i)}[x](t)\right|+$$
$$+C_{m,i}(b-a)^m gh\left(\frac{L+C}{1-\theta}\right),\quad 0\leq i\leq m-1\}$$

$$C=\max_{a\leq t\leq b}\sum_{i=0}^{m-1}L_i\left|P_{m-1}^{(i)}[x](t)\right|$$

$$\vartheta=h\cdot g\cdot\left(\sum_{i=0}^{m-1}C_{m,i}L_i(b-a)^{m-i}\right)<1,\quad h=b-a \quad (24b)$$

Then, the boundary value problem (1a) and (1b) has a solution in D_1.
Proof. Let $y(t)=x(t)-P_{m-1}[x](t)$, so that ($1a$) and (1b) is the same as

$$\begin{cases} y^{(m)}(t)=f(t,\overline{y}(t)) \\ y(a)=y(b)=0 \\ \Delta y_a^{(k)}=0 \qquad 1\leq k\leq m-2 \end{cases} \qquad (25)$$

where

$$\overline{y}(t)=y(t)+P_{m-1}[x](t),$$
$$y'(t)+P'_{m-1}[x](t),....,y^{(m-1)}(t)+P_{m-1}^{(m-1)}[x](t).$$

Define $M[a,b]$ as the space of m times continuously differentiable functions satisfying the boundary conditions of (25). If we introduce in $M[a,b]$ the norm:

$$\|y(t)\|_\infty=\max_{a\leq t\leq b}\left|y^{(m)}(t)\right|$$

then it becomes a Banach space. As in theorem 5, it suffices to show that the operator $T: M[a,b]\to M[a,b]$ defined by

$$T[y](t)=\int_a^b G(t,s)f(s,\overline{y}(s))ds$$

maps the set

$$S=\left\{y(t)\in M[a,b]:\|y\|_\infty\leq hg\left(\frac{L+C}{1-\theta}\right)\right\}$$

into itself. In order to demonstrate this, it is sufficient to utilise the conditions (24a), lemma 1 and lemma 3.
The thesis follows from the application of the *Schauder fixed point* theorem to the operator T.
Theorem 7 Suppose that $(t,x_0,x_1,...,x_{m-1})$, $(t,y_0,y_1,...,y_{m-1})\in[a,b]\times D_1$

- **(i)** the function $f(t,x_0,x_1...,x_{m-1})$ satisfies the following Lipschitz condition $|f(t,x_0,x_1...,x_{m-1})-f(t,y_0,y_1...,y_{m-1})|\leq\sum_{i=0}^{m-1}L_i\,|x_i-y_i|$
- **(ii)** $\theta_1=\sum_{i=1}^{m-1}C_{m,i}L_i(b-a)^{m-i}<1$

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(iii) $L = \max_{a \leq t \leq b} |f(t, 0, ..., 0)|$

Then the boundary value problem $(1a)$ and $(1b)$ has a unique solution in D_1.

Proof. The existence of a solution follows from theorem (6). Let, now, $x_1(t), x_2(t)$ two solutions on D_1 and $y_1(t), y_2(t)$ defined as theorem 6, then after calculation, we have

$$\left| y_1^{(m)}(t) - y_2^{(m)}(t) \right| \leq \theta_1 \max \left| y_1^{(m)}(t) - y_2^{(m)}(t) \right|$$

and because $\theta_1 < 1$ follows $y_1^{(m)}(t) = y_2^{(m)}(t) \quad \forall t \in [a, b]$; thus from lemma 2 we have $y_1(t) = y_2(t)$.

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