Identification of Optimum Statistical Models for Time Series Analysis and Forecasting using Akaike's information Criterion and Akram Test Statistic: A Comparative Study

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Abstract—For the analysis and forecasting of time series, we always search for a statistical model capable of understanding the underlying processes of data and filtering the unwanted noise. This noise may either be white or coloured, having some pattern of autoregressive moving average i.e., ARMA(p,q) processes. This search is carried out by using various forecast accuracy criteria and tools such as, Akaike's information criterion (AIC) and Akram test statistic (ATS).

As compared to others, the AIC and ATS are noted to be more effective for identification of good models. However, AIC, being parametric in nature, is found to be comparatively more sensitive to noise volatilities and cumbersome to use; whereas, ATS, the base of which is distribution free is observed to be quite robust to noise variations, parsimonious in nature and relatively more easy to use. In this paper both the AIC and ATS are reviewed, practical implication discussed and their role in identifying optimum models from a class of candidate statistical models, especially, the linear dynamic system models is examined. For better insight, into these gadgets an example on analysis and forecasting of daily copper prices is given.

Index Terms—Akaike's information criterion, Akram test statistic, optimum forecast, statistical models, ASL, Coloured noise.

1 Introduction

Model selection is a decision theoretic approach. The main purpose is to identify the model that shows the best balance between data fitting and model complexity. Information criteria offer various procedures to choose the best model amongst a set of many possible models through certain guiding principles in particular situations but not across a variety of situations. Akaike [1] found the unbiased estimator of (relative) Kullback-Leibler (K-L) information or distance which is simply the distance between the true density and estimated density for each model.

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The basic objective of this paper is not only to explain and compare Akaike's information criterion (AIC) and Akram test statistic (ATS), but also to highlight the computational simplicity and effective performance of the latter. The paper is organized as follows: ATS is introduced through stepwise identification procedure in Section 2. A real life example is presented in Section 3. The concluding remarks are put in Section 4. A table of theoretical values of ASL corresponding to various values of coefficient of AR(1) coloured noise process is also presented as appendix at the end of the paper.

2 Akram Test Statistic

A model selection tool cum identifier of the colour of one step ahead forecast errors, Akram test statistic (ATS), was proposed by Akram [2]. It is applied to check the suitability and capability of a model to generate optimum forecasts. To search a suitable model by applying ATS, one has to proceed as per the following stepwise identification procedure.

Step 1: Analyze discrete time series using a candidate model with a white noise component and capable of accommodating p-th order autoregressive i.e., AR(p) noise processes, estimate the parameters of the model using an optimum estimation technique and then generate one-step-ahead forecasts and residuals. Standardize the residuals and compute Average String Length (ASL) as under

$$\mathbf{ASL = (n_e - 1) / (n_{-\to} n_+ + n_{+\to} n_-)}$$

Here n_e is the number of residuals, $n_{-\to} n_+$ represents the number of shifts from negative to positive signs of the standardized residulas, while $n_{+\to} n_-$ means the reverse meaning.

Step 2: The three different scenarios for formulating the null hypothesis (H_0) and the alternative hypothesis (H_1) are described as follows.

Scenario#1	**Scenario#2**	**Scenario#3**
H_0: $\mu_{ASL} \geq 2$	H_0: $\mu_{ASL} = 2$	H_0: $\mu_{ASL} \leq 2$
H_1: $\mu_{ASL} < 2$	H_1: $\mu_{ASL} \neq 2$	H_1: $\mu_{ASL} > 2$

ISBN:978-988-98671-2-6

where μ_{ASL} is the parent population's Average String Length of residuals or error terms. One of these three scenarios is selected in the light of the sampled information and purpose of study.

Step 3: For the standardized residuals and notations having the usual meanings, compute ATS with the constraint $n \geq 30$

$$\textbf{ATS}: \tau = \{ 2 (n_e - 1) / (n_{-\rightarrow} n_{+} + n_{+\rightarrow} n_{-}) \}$$

Step 4: For making decision on the fate of null hypothesis, define the acceptance region (Ar) and the rejection regions (Rr) at some level of significance α are defined e.g. for a two tailed test of significance

Ar: $2n / (n-1) + Z_{\alpha/2}\sqrt{(n-1)} \leq \text{Region} \leq 2n / (n-1) - Z_{\alpha/2} \sqrt{(n-1)}$

Rr: $\text{Region} < 2n / (n-1) + Z_{\alpha/2} \sqrt{(n-1)}$

or $\text{Region} > \{2n / (n-1) - Z_{\alpha/2} \sqrt{(n-1)}\}$

Step 5: Accept H_0 if the value of test statistic τ lies in the acceptance region AR; otherwise, reject H_0 and accept H_1. The acceptance of H_0 implies that the residuals are white; whereas rejection of H_0 leads us to the conclusion that the residuals are not white, but coloured.

Step 6: In case of coloured noise process, the question is, of what type this noise is? AR type, MA or ARMA type and what is the order of the noise process? p, q or (p, q). Here, discussion is confined to AR(p) processes as an ARMA process can be approximated and adequately represented by an AR process [3]. For the empirical value of ASL computed from the data, a corresponding value Φ of AR(1) coefficient is determined. Using this Φ value the initially used candidate model is restructured or updated and again applied to the data. Then, generate one-step-ahead forecasts and residuals and move through the above testing procedure again. If H_0 is accepted, it depicts that the model with AR(1) noise process is suitable. Its one-step-ahead forecast, therefore would be optimum. If H_0 is rejected again, it means that the model with AR(1) coloured noise component, has failed to filter the noise and therefore is not suitable for analysis of data. In this case, determine Φ again from the table of theoretical values of ASL given at the end as Appendix. The candidate model with AR(1) process would therefore be considered inappropriate and model with AR(1) process will be reconstructed considering AR(2) noise process, using Φ_1 (the first Φ) and Φ_2 (the second Φ) coefficients.

The above procedure is repeated until the colour is filtered out i.e., H_0 is accepted. To highlight the practical aspects of ATS, a real life example is presented.

3 Example

Consider a local model at time t, the observation and state equations are as under.

$$y_t = f\,\theta_t + v_t$$
$$\theta_t = G\,\theta_{t-1} + w_t$$

where at time t, for an observation y_t, f = (1 x n) is the vector of some known functions of independent variables or constants, θ = (n x 1) is the vector of unknown stochastic parameters, G = $\text{diag}\{G_i\}_{i=1,2,\ldots,r}$ is a (n x n) state or transition matrix having n number of non–zero eigenvalues $\{\lambda_i\}$ such that $i = 1, \ldots, n$, v = white noise (having independently and identically and normally distributed terms) with mean zero and some constant variance V and w = (n x 1) is the parameter noise vector.

For a known prior of θ at time t-1

$$(\theta_{i\,t-1}, | D_{i\,t-1}) \sim N[m_{i\,t-1} ; C_{i\,t-1}]$$

and the posterior of θ at time t

$$(\theta_{i\,t} | D_{i\,t}) \sim N[m_{i\,t} ; C_{i\,t}]$$

the updating mechanism of m_t, the estimate of θ_t is given as,

$$R_t = G\,C_{t-1}\,G' + W_t$$
$$A_t = R_t F\,[I + F R_t F']^{-1}$$
$$C_t = [I - A_t F]\,R_t$$
$$m_t = G\,m_{t-1} + A_t\,[y_t - F\,G\,m_{t-1}]$$

where at time t, R is a system matrix, W = $\text{diag}\{W_{it}\}$ for i =1,…,n. A is an updating or gain vector and I is an identity matrix. All vectors and matrices are assumed compatible in dimensions with their associated vectors and matrices of the system. Analogous to linear control theory these stochastic difference equations cluster themselves into an ensemble of a closed loop of linear system. On the basis of updating, the one-step-ahead forecasts are given by y^t+1 = f G mt-1. one-step-ahead forecast residuals are obtained as under.

$$e_t = y_t - f\,G\,m_{t-1}$$

Data consisting of 158 observations on closing day copper prices per 1000 kilograms, in Pak Rupees (PKR) during the period from August 01, 2005 to March 06, 2006 are noted from the London Metal Exchange (LME) and plotted to observe their pattern.

These data are analysed by employing 14 statistical models, one-step-ahead forecasts generated, residuals computed and standardized. Based on these standardized residuals various statistics of forecast accuracy measuring criteria are computed and given in Table 1. It is noticed that AIC indicates moving average model of third order, i.e., MA(3) as the best model. However, both Durbin-Watson test statistic and ATS confirm the presence of coloured noise process. It is therefore concluded that the selected model MA(3) failed to generate optimum forecasts. This situation demands construction and application of dynamic linear models (DLMs) capable of filtering the colour of noise processes. Resultantly, generalized exponentially weighted regression (GEWR) type dynamic linear models are applied to get optimal forecast after due filtration of the coloured noise processes through linear dynamic system models with AR type coloured noise components, as is evident from Table 2. Both ATS and D-W test statistic indicate presence of coloured noise processes in AR(0) white noise model and demand restructuring of DLM by redefining f, θ, G and W values. Incorporation of a value of 0.52 of the AR coefficient indicated by ASL = 3.0784 and smoothing coefficient equal to 0.35 leads to the dynamic optimal model with ASL equal to 2.492, generates better forecasts as may be visualized from the graph of the optimum one step ahead forecasts. The components of this model, i.e. the f-vector, G matrix, W matrix plus the prior values of m_0 vector and C_0 matrix for the linear growth white noise AR(0) model as well as the AR(1) coloured noise model, in diagonal form, are given as follows.

For AR(0) Model

$$f = \begin{bmatrix} 1 & 1 \end{bmatrix} \qquad G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$W = \begin{bmatrix} 1.6671 & 01 \\ 0 & 0.3474 \end{bmatrix}$$

$$C_0 = \begin{bmatrix} 24082 & -23308.9 \\ -23308.9 & 23412 \end{bmatrix}$$

$$m_0 = \begin{bmatrix} 3925.88 \\ -153.07 \end{bmatrix}$$

For AR(1) Model

$$f \quad \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

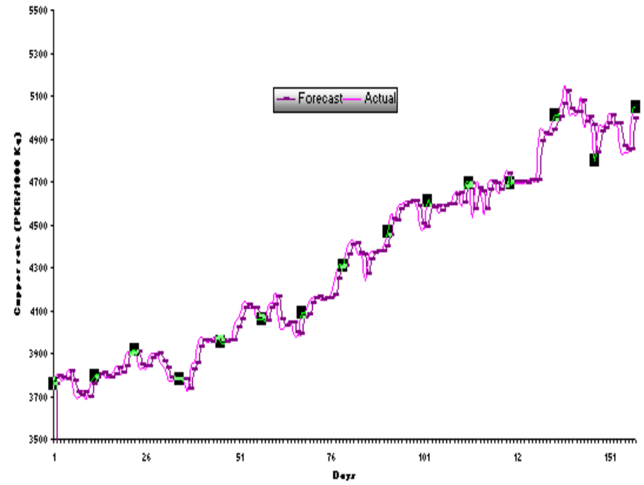
$$G = \begin{bmatrix} 0.3076 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$W = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4.8286 & 0 \\ 0 & 0 & 2.9144 \end{bmatrix}$$

$$m_0 = \begin{bmatrix} -19.55 \\ 3192.46 \\ 618.55 \end{bmatrix}$$

A similar setting of the model may be made in canonical form, if desired, following Akram [3, 4].

$$C_0 = \begin{bmatrix} 325.17 & -203.76 & -45.20 \\ -203.83 & 1.84e+06 & -1.84e+06 \\ -45.13 & -1.86e+06 & 1.84e+6 \end{bmatrix}$$



Forecast for AR(2) coloured noise process

4 Conclusion

Model selection is not simply hypothesis testing. Rather, it ranks various alternative rival models from best to worst. In real life situations, the assumption of whiteness of residuals is found to be rarely true and the data are usually polluted with coloured noise processes. In such situations, it becomes imperative to identify the noise processes prior to going for any formal analysis so that the models having capability of filtering the colour of noise may be constructed to generate results with residuals having whiteness in their structures. For such purposes, ATS is found to be very simple, parsimonious and straight forward test statistic which performs at least equally well as AIC.

References

[1] Akaike H., Information theory as an extension of maximum likelihood principle. Proceedings of second international symposium on information theory. B.N.Petrov and Csaki (editors). Akademiai Kiado, Budapest. pp. 267-281, 1973.

[2] Akram M., A test statistic for identification of noise processes. Pakistan Journal of Statistics.Vol.17, No. 2; pp. 103-115, 2001.

[3] Akram M., Computational aspects of state space models for time series forecasting. Proc. 11th Symposium on Computational Statistics, Austria. Pp.116-117, 1994.

[4] Akram M., Recursive transformation matrices for linear dynamic system models. J. Computational Statistics & Data Analysis. Vol.6, pp.119-127.

[5] Anděl J., An Autoregressive Representation of ARMA processes. Proc. 2nd Symposium on Mathematical Statistics, Austria, pp. 13-21, 1981.

Theoretical Values of ASL
{AR(1) Colored Noise Process}

Φ	ASL	Φ	ASL
-0.99	1.047380	-0.49	1.508438
-0.98	1.068312	-0.48	1.516762
-0.97	1.084994	-0.47	1.525126
-0.96	1.099504	-0.46	1.533532
-0.95	1.112648	-0.45	1.541981
-0.94	1.124836	-0.44	1.550476
-0.93	1.136311	-0.43	1.559018
-0.92	1.147234	-0.42	1.567608
-0.91	1.157713	-0.41	1.576250
-0.90	1.167828	-0.40	1.584944
-0.89	1.177641	-0.39	1.593692
-0.88	1.187198	-0.38	1.602497
-0.87	1.196536	-0.37	1.611360
-0.86	1.205685	-0.36	1.620283
-0.85	1.214671	-0.35	1.629268
-0.84	1.223515	-0.34	1.638317
-0.83	1.232234	-0.33	1.647431
-0.82	1.240843	-0.32	1.656613
-0.81	1.249357	-0.31	1.665865
-0.80	1.257785	-0.30	1.675189
-0.79	1.266139	-0.29	1.684586
-0.78	1.274427	-0.28	1.694059
-0.77	1.282658	-0.27	1.703610
-0.76	1.290838	-0.26	1.713242
-0.75	1.298975	-0.25	1.722955
-0.74	1.307074	-0.24	1.732753
-0.73	1.315140	-0.23	1.742637
-0.72	1.323180	-0.22	1.752611
-0.71	1.331196	-0.21	1.762676
-0.70	1.339195	-0.20	1.772835
-0.69	1.347179	-0.19	1.783090
-0.68	1.355152	-0.18	1.793443
-0.67	1.363119	-0.17	1.803899
-0.66	1.371081	-0.16	1.814458
-0.65	1.379043	-0.15	1.825124
-0.64	1.387007	-0.14	1.835899
-0.63	1.394977	-0.13	1.846787
-0.62	1.402954	-0.12	1.857790
-0.61	1.410942	-0.11	1.868911
-0.60	1.418942	-0.10	1.880154
-0.59	1.426958	-0.09	1.891521
-0.58	1.434991	-0.08	1.903015
-0.57	1.443045	-0.07	1.914641
-0.56	1.451120	-0.06	1.926402
-0.55	1.459220	-0.05	1.938301
-0.54	1.467346	-0.04	1.950342
-0.53	1.475500	-0.03	1.962528
-0.52	1.483684	-0.02	1.974864
-0.51	1.491900	-0.01	1.987353
-0.50	1.500151	0.00	2.000000

Φ	ASL	Φ	ASL
0.01	2.012809	0.51	3.032932
0.02	2.025785	0.52	3.067466
0.03	2.038931	0.53	3.103051
0.04	2.052253	0.54	3.139744
0.05	2.065756	0.55	3.177606
0.06	2.079445	0.56	3.216704
0.07	2.093324	0.57	3.257108
0.08	2.107401	0.58	3.298896
0.09	2.121679	0.59	3.342151
0.10	2.136166	0.60	3.386964
0.11	2.150866	0.61	3.433436
0.12	2.165787	0.62	3.481673
0.13	2.180935	0.63	3.531795
0.14	2.196317	0.64	3.583930
0.15	2.211940	0.65	3.638223
0.16	2.227811	0.66	3.694828
0.17	2.243938	0.67	3.753921
0.18	2.260329	0.68	3.815694
0.19	2.276993	0.69	3.880361
0.20	2.293938	0.70	3.948159
0.21	2.311173	0.71	4.019357
0.22	2.328708	0.72	4.094255
0.23	2.346552	0.73	4.173190
0.24	2.364717	0.74	4.256549
0.25	2.383212	0.75	4.344765
0.26	2.402050	0.76	4.438340
0.27	2.421241	0.77	4.537850
0.28	2.440799	0.78	4.643957
0.29	2.460737	0.79	4.757439
0.30	2.481068	0.80	4.879203
0.31	2.501806	0.81	5.010323
0.32	2.522967	0.82	5.152078
0.33	2.544566	0.83	5.306005
0.34	2.566621	0.84	5.473975
0.35	2.589149	0.85	5.658286
0.36	2.612167	0.86	5.861803
0.37	2.635697	0.87	6.088140
0.38	2.659759	0.88	6.341950
0.39	2.684374	0.89	6.629334
0.40	2.709566	0.90	6.958474
0.41	2.735359	0.91	7.340647
0.42	2.761778	0.92	7.791923
0.43	2.788852	0.93	8.336141
0.44	2.816610	0.94	9.010532
0.45	2.845082	0.95	9.877212
0.46	2.874301	0.96	11.04981
0.47	2.904304	0.97	12.76558
0.48	2.935127	0.98	15.63866
0.49	2.966810	0.99	22.10603
0.50	2.999397	0.999	69.29176

Model	Std RSS	AIC	ASL	D-W	Bias	MAE	MAPE	MSE	RMSE	TS
Table 1: Forecast accuracy measures for LME copper rate forecast										
LR	155.00	2.97	4.62	0.39	2.20e-08	0.79	0.02	1.00	1.00	4.41e-06
MA (1)	155.00	1.98	9.24	-	1.62e-07	0.87	0.02	1.00	1.00	2.97e-05
MA (2)	150.36	-3.83	2.86	1.54	9.05e-09	0.73	0.02	0.96	0.98	1.95e-06
MA (3)	**147.44**	**-6.93**	**3.34**	**1.15**	**-1.65e-08**	**0.76**	**0.02**	**0.95**	**0.97**	**-3.47e-06**
MAT (3)	154.98	2.95	2.01	2.36	3.39e-09	0.70	0.02	1.00	1.00	7.62e-07
SES	153.10	-0.98	2.09	2.34	7.54e-09	0.67	0.02	0.98	0.99	1.78e-06
ARIMA (1,1,0)	155.00	2.97	2.42	-	8.29e-09	0.70	0.02	1.00	1.00	1.88e-06
ARIMA (1,1,1)	154.00	4.21	2.38	-	9.24e-09	0.72	0.02	1.00	1.00	2.02e-06
ARIMA (2,1,0)	154.00	4.21	2.49	-	-2.26e-09	0.70	0.02	1.00	1.00	-5.11e-07
ARIMA (2,1,1)	152.98	5.30	2.45	-	2.16e-08	0.72	0.02	1.00	1.00	4.78e-06
ARIMA (0,1,1)	155.00	2.97	2.49	-	4.14e-09	0.71	0.02	1.00	1.00	9.29e-07
ARIMA (2,1,2)	152.00	6.44	2.45	-	9.57e-09	0.71	0.02	1.00	1.00	2.11e-06
Table 2: Accuracy measures for dynamic forecasts for LME copper rate										
AR (0)	148.12	-4.21	3.08	1.65	1.05e-08	0.72	0.02	0.96	0.98	2.32e-06
AR (1)	149.20	-0.80	2.49	1.95	6.03e-09	0.70	0.02	0.97	0.98	1.36e-06

Key:
LR=Linear regression,
MA=Moving average,
SES= Single exponential smoothing,
MAT=Moving average with trend,
ARIMA=Autoregressive integrated moving average, Std.
RSS = Standardized residual sum of squares,
MAE = Mean absolute error,
MAPE = Mean absolute percentage error
MSE = Mean square error,
RMSE = Root mean square error,
TS = Tracking signal

ISBN:978-988-98671-2-6