Copula Marginal Expected Tail Loss Efficient Frontiers for CDOs of Bespoke Portfolios

Diresh Jewan, Renkuan Guo, and Gareth Witten

***Abstract*—The behaviour of the efficient frontier for CDOs of bespoke portfolios is investigated under one-factor copula marginal distributional assumptions. This approach has been thoroughly used in statistical literature. The main feature of these models is that default events, conditional on some latent state variable, are independent. This eases the computation of the aggregate loss distribution, a crucial element in credit portfolio optimisation. Both Gaussian and Clayton copula models are applied to the default dependence structure. The Clayton copula model demonstrates superiority in capturing the default dependence inherent in credit portfolios. The portfolio optimisation problem set-up under the newly defined Copula Marginal Expected Tail Loss (abbreviated as CMETL) risk measure is convex and can be easily solved in terms of linear programming algorithms. Numerical analysis is conducted by creating a Bespoke CDO collateral portfolio using the iTraxx Europe IG Series 5 index constituents as an illustrative example**.

***Index Terms*—Bespoke CDOs, Copula Marginal Expected Tail Loss, Factor Copula Models, Heavy-tail distribution.**

I. Introduction

The global structured credit landscape has been irrevocably changed with the innovation of Collateralized Debt Obligations (abbreviated as CDOs). As of 2006, the volume of synthetic CDO structures outstanding grew to over $1.9 trillion, making it the fastest growing investment vehicle in the financial markets, with Bespoke CDO deals making up 21% of the total volume [1].

Understanding the risk/return trade-off dynamics of Bespoke CDO collateral portfolios is crucial when maximising the utility provided by these instruments. Credit losses are characterized by small likelihoods of large losses coupled with large likelihoods of small losses and thus credit loss distributions are heavily skewed with long heavy tails.

From an investor's perspective, optimising the collateral portfolio should result in a maximum return for a given level of credit risk whereas, from a structurer's perspective the risk/return analysis is a starting point in an efficient capital allocation process.

According to [2], there exist two types of risk measures: relevant and tractable. Relevant measures capture key properties of a credit loss distribution, while tractable measures can be optimised using computationally efficient methods. The following analysis uses a tractable, quantile based risk measure that has more attractive properties than that of the unexpected loss measure for investigating the behaviour of the efficient frontier. Efficient frontiers are defined by a collection of optimal risk/return portfolios.

In this paper, we study the behaviour of the efficient frontier under Expected Tail Loss (abbreviated as ETL) as the risk measure for a collateral portfolio with the heavy-tailed distribution assumptions via copulas.

ETL was initially introduced by [3], in the portfolio optimisation context. ETL has proved a more consistent risk measure, since it is sub-additive and convex [4]. Reference [3] has shown that the portfolio optimisation under this risk measure results in a linear programming problem. This measure has been shown in numerous studies across the different asset classes, to be a superior measure to derive empirical efficient frontiers that act as an useful approximation to the true efficient frontier [5]-[6].

The loss distribution, a key element in credit portfolio optimisation procedures is generated through one-factor Copula Marginal models. Copulas represent a convenient way of describing the joint distribution, a crucial element in credit portfolio optimisation. De Finetti's theorem for exchangeable sequence of binary random variables provides a theoretical background for the use of factor copula models in credit risk applications [7].

The factor copula approach has been widely used in credit portfolio modelling [8]-[10]. The main feature of these models is that default events, conditional on some latent state variable are independent, simplifying the computation of the aggregate loss distribution [7]. This approach is well suited for large dimensional problems. Reference [11] relates the factor and copula approaches.

For a comparative analysis, both the Gaussian and Clayton copula models were used to model the default dependence structure inherent in credit portfolios.

II. Bespoke CDO Mechanics & Empirical Analysis

A Bespoke CDO is a popular second-generation credit product. This standalone single-tranche transaction is referred to as a bespoke because it allows the investor to

Manuscript received December 19, 2007.

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customise the various deal characteristics such as the collateral composition, level of subordination, tranche thickness, and rating. Other features, such as substitution rights, may also play an important role [12].

Only a specific portion of the credit risk is transferred in these transactions, in contrast with the entire capital structure as in the case of standardized synthetic CDOs. Most of the transactions involve 100-200 liquid corporate credit default swaps.

In a typical Bespoke CDO transaction, the main decision step for potential investors is to select the portfolio of credits, to which they want exposure. A short position can also be held on a subset of names to have overweight views on certain credits or sectors. The analysis will focus on the portfolio optimisation problem that exists within this step of the structuring process.

The next step is to determine the level of subordination, and tranche thickness corresponding to the risk appetite of the investor. These deal parameters determine the degree of leverage and the required premium payments [12]. This step of the process adds a further two constraints to the transaction optimisation problem. The first is related to the tranche thickness, and the second is the credit rating assigned to the tranche. At this level of the problem, from an investor's perspective, the cost of the transaction is minimised.

The schematic below outline a typical Bespoke CDO transaction. In this transaction the investor goes long the credit risk in a mezzanine tranche. The challenge of issuing Bespoke CDOs is the ongoing need and expense of the risk management of the tranche position [13].

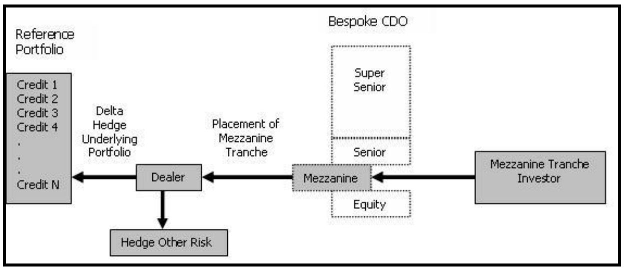

Figure 1. Placement of a Mezzanine Tranche in a Bespoke CDO transaction (Source: Lehman Brothers)

The collateral test portfolio used in this study consists of 114 constituents of the iTraxx Europe IG Series 5 Credit Default Swap (abbreviated as CDS) index. Eleven of the constituents were removed due to insufficient historical data. Five-year tenor CDS spread data for all constituents in the test portfolio were obtained from Bloomberg Financial Services for the 3-year period November-2004 to November-2007. The underlying CDS contracts are denominated in Euros.

Recovery rates for all constituents are assumed constant and equal to 40% in the event of a default. The single factor model described in Section IV drives asset correlations. The following figure shows the evolution of the iTraxx Europe IG index over the observation period.

Figure 2. Evolution of iTraxx Europe IG Series 5 Index Spread Levels

The analysis covers two very important credit market events. The first is the default correlation crises in May 2005, and the second is the recent credit crisis caused by high delinquency rates in the U.S. sub-prime sector of the credit market.

Preliminary analysis on the CDS spread data shows the following empirical evidence:

(1) The CDS spreads are intertemporally stationary. This is a crucial issue, since credit risk models implicitly assume that spreads follow stationary processes [14]. The following table displays this evidence using the Dickey-Fuller regression.

Table 1. Unit Root test results for credit spreads of the iTraxx Europe IG Index, and several portfolio constituents

Corporate entity	**t-statistic**	
	Level	**First Difference**
iTraxx Europe IG Index	-0.6243	-38.7868
BAE Systems PLC	-1.6559	-31.3635
Unilever NV	-2.2556	-32.4789
Continental AG	-2.2823	-31.6315
Peugeot SA	-2.3278	-14.6058
Commerzbank AG	-0.8791	-13.6451

Following [15], continuously compounded percentage changes in spreads between successive trading days were computed as first differences in the logarithm of spread levels.

(2) The empirical distributions of the spread changes for all portfolio constituents and the iTraxx Europe CDS index are highly leptokurtotic. Figure 3 support this fact.

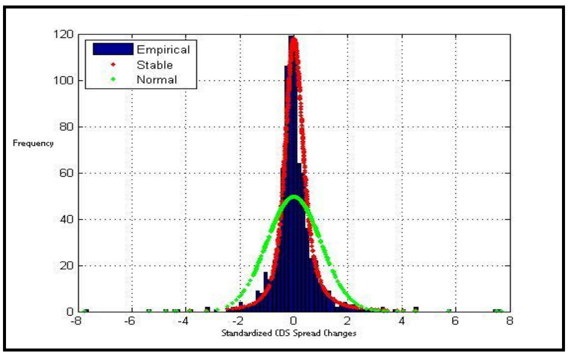

Figure 3a. Estimated probability density functions using both the Gaussian and Stable Paretian assumptions for the iTraxx Europe IG index.

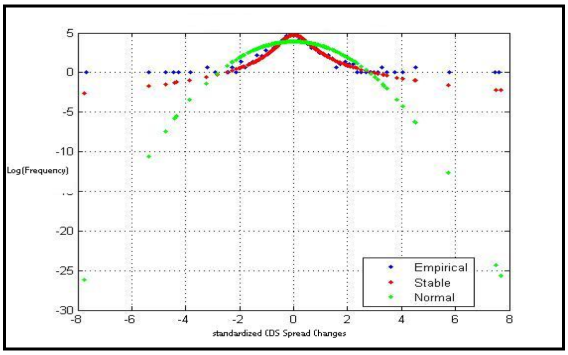

Figure 3b. The natural logarithm of the estimated probability densities

We observe that the Stable Paretian fit is able to match the frequency for both small and intermediate spread changes; however, both fits are not perfect in the tails. The Gaussian model would badly underestimate the likelihood of the tail events. Stable Paretian distributions are a better model for the credit risk application.

III. RISK MEASURES AND ETL

The focus now is placed on the optimisation of credit portfolios using the method initially proposed by [3]. Many credit risk measures were used in portfolio optimization applications; for example, standard deviation (denoted σ), semi standard deviation, Unexpected Losses (analogous to Value at Risk), Expected Tail Loss, Beta, etc. However, in recent years, Coherent risk measures like ETL have found useful applications in portfolio optimisation [2], [3], [5], and [6].

The main advantage of using coherent risk measures is that it results in the portfolio optimisation problem becoming convex, which can be easily solved using standard optimisation techniques.

A. Coherent Risk Measures

Coherent risk measures are functional on a space of bounded random variables on a probability space $\left(\Omega, \mathfrak{A}, P\right)$ with the following properties:

- If $X \geq 0$ then $\rho(X) \leq 0$.
- $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$
- For $\lambda \geq 0$, $\rho(\lambda X) = \lambda \rho(X)$.
- For every constant function a ,
- $\rho(a + X) = \rho(X) - a$.

Coherence is important in credit portfolio management because it supports the benefits of diversification. This means that the credit risk of the portfolio decreases as the number of instruments that make up the investment increases, allowing portfolio risk managers to reap the rewards of diversification.

B. Expected Tail Loss

The coherent risk measure engaged in this paper is ETL, which is essentially the expected loss conditional on losses exceeding the unexpected loss. Mathematically, ETL is defines as:

$$ETL_p(\beta) = E[L_p \mid L_p > \mathrm{UL}_p(\beta)], \qquad (1)$$

where L_p denotes the loss for portfolio p and UL_p , the unexpected losses at the confidence level β . Portfolio optimisation with *ETL* as objective function will result in a smooth, convex problem with a unique solution [3].

C. Portfolio Optimisation under ETL

ETL optimal portfolio techniques, combined with copula marginal distribution modelling of the portfolio risk factors can lead to significant improvements in risk-adjusted returns.

A Copula Marginal Expected Tail Loss (abbreviated as CMETL) is one that minimizes credit portfolio ETL subject to a constraint of achieving expected portfolio returns at least as large as an investor defined level, along with other typical constraints on weights, where both quantities are evaluated in the CMETL framework. The CMETL measure is similar to that of *Stable (Distribution) Expected Total Loss (abbreviated as SETL)* measure proposed by [5].

In order to define the above CMETL precisely we use the following quantities:

R_p : the random return of portfolio p ,

$CMER_p$ **:** the expected return of portfolio p with respect to the copula marginal distribution, and

β **:** tail loss probability.

The following assumptions are imposed by [5] for the *CMETL* investor:

- the universe of assets is $\boldsymbol{Q}$ (the set of mandate admissible portfolios);
- the investor may borrow or deposit at the risk-free rate r_f without restriction;
- the investor seeks an expected return of at least μ .

The CMETL investor's optimal portfolio is then given by the following:

$$\begin{aligned}
&X_c\left(\mu \mid \beta\right) = \arg\min_{q \in Q} \left\{CMETL_p\left(\beta\right)\right\}, \\
&\text{s.t.} \\
&\sum_r x_r = 1, \\
&l \leq x_r \leq u, \quad \forall\ \mathrm{r}, \\
&CMER_p \geq \mu.
\end{aligned} \qquad (2)$$

where X_c , denotes the resulting portfolios weights. The subscript *c* indicates that the problem is defined under the copula framework. The constraints, in the order written above, require that:

i. the sum of the portfolio weights should equal to one,
ii. the position weights should lie within the trading limits *l* and *u* to avoid unrealistic long and short positions , and
iii. the expected return on the portfolio in the absence of rating transitions should be at least equal to some investor defined level µ.

As the required return µ increases, so does the minimum amount of risk. Together, these optimal risk/return trade-offs define the efficient frontier.

IV. COPULA MARGINAL EFFICIENT FRONTIERS FOR CDOS

The primary objective of portfolio credit risk modelling is the quantification of credit risk inherent in the portfolio. Application of this tool to the CDO optimisation problem creates efficient bespoke portfolios that have minimum credit risk for an investor defined level of return.

Monte Carlo simulation is most useful in generating the portfolio loss distribution, since it provides a simple mode of tracking the defaulted assets in a portfolio under a given scenario, which allows efficient implementation of the optimisation algorithm.

The analysis is performed in a one-period, default-only mode structural framework. Consider a portfolio composed of *n* credit instruments. The exposure in each instrument is denoted by N_i. Under the binomial setting, the loss of counterparty i for scenario k is given by the following:

$$L_i^k = N_i - V_i^k, \quad for\ i = 1,\dots,n, \quad k = 1,\dots,s, \tag{3}$$

where V_i^k is the value of exposure i at the given horizon in scenario k. In particular:

$$V_i^k = \begin{cases} N_i R_i, & default\ state\ , \\ N_{i,} & otherwise, \end{cases} \tag{4}$$

where R_i is the recovery rate for reference entity i.

The portfolio loss function in scenario *k*, $L^k : \mathbb{R}^{n+1} \to \mathbb{R}$, over the chosen time horizon is given by the following:

$$L_k\left(\underline{x}\right) = \sum_{i=1}^{n} L_i^k x_i, \tag{5}$$

where $\underline{x} = (x_1, x_2, \cdots, x_n)^T$ is the vector of positions held in the portfolio. We use the *factor copula* approach in order to generate scenarios for the credits states of each reference entity. This approach can be understood as a combination of the one-parameter copula and Firm-value approach.

In the firm-value approach a company will default when its 'default-like' stochastic process, $A_i : \mathbb{R}^{n+1} \to \mathbb{R}$, falls below a barrier. This latent variable is defined by the following:

$$A_i\left(x_1, x_2, \ldots, x_n, \varepsilon_i\right) := \rho_i M + \sqrt{1-\rho_i}\,\varepsilon_{i,} \tag{6}$$

where M and ε_i are independent standard normal variants in the case of a Gaussian copula model, and $\mathbf{cov}(\varepsilon_k, \varepsilon_l) \neq 0,\ for\ all\ k \neq l.$ A_i can be interpreted as the value of the assets of the company, and M, the general state of the economy.

The default dependence come from the factor *M*. Unconditionally, the stochastic processes are correlated but conditionally they are independent. The default probability of an entity i, denoted by F_i, can be observed from market prices of credit default swaps, and defined by the following:

$$F_i\left(t\right) = 1 - \exp\left(-\int_0^t h_i(u)du\right), \tag{7}$$

where $h_i(u)$ represents the hazard rate for reference entity *i*. For simplicity, we assume that the CDS spread term structure is flat, and calibration of the hazard rates for all reference entities is straightforward.

The default barrier $B_i(t)$ is defined as: $B_i\left(t\right) = G^{-1}\left(F_i\left(t\right)\right)$, where G defines the inverse distribution function. In the case of a Gaussian copula, this would be the inverse cumulative Gaussian distribution function.

A second type of copula model considered comes from the Archimedean family. In this family we consider the Clayton copula. In this setting, the stochastic process A_i is defined by the following:

$$A_i := \varphi_{\frac{1}{\theta}}\left(-\frac{\ln(\varepsilon_i)}{M}\right) = \left(-\frac{\ln(\varepsilon_i)}{M} + 1\right)^{\frac{1}{\theta}}, \tag{8}$$

where *φ(.)* is the Laplace transform of the *Gamma(1/θ)* distribution, ε_i is a uniform random variable and *M* is a *Gamma(1/θ)* distributed random variable. Using the credit models presented above, the loss distribution for the test portfolio can easily be derived.

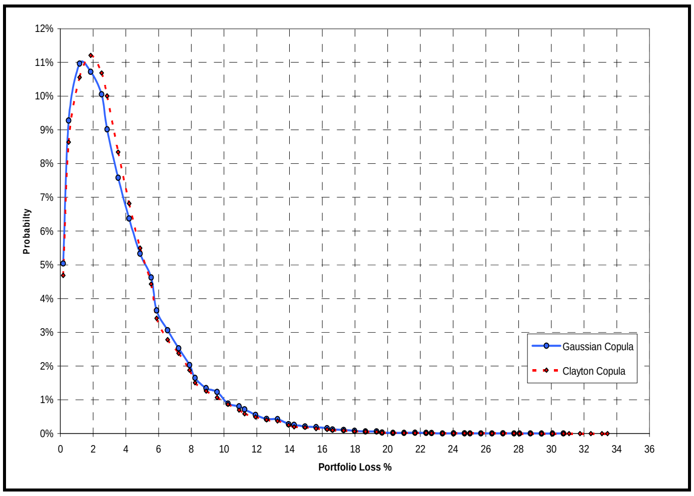

Figure 4. Credit loss distributions for homogenous test portfolio at 5-year horizon under different default dependence assumptions

The portfolio optimisation is examined under both the Gaussian and Clayton copula assumptions for the default dependence. The five-year portfolio loss distribution is generated by Monte Carlo simulation. 50,000 scenarios of joint credit states of reference entities and related losses were generated. The sensitivity study with respect to the number of scenarios indicates that 50,000 scenarios were sufficient to

ISBN:978-988-17012-3-7

estimate unexpected losses and CMETL with precision. Prior to the simulation both, the single factor Clayton and Gaussian copulas were calibrated to the market data by a statistical regression technique. The sum of quadratic deviation in the two models was minimized to obtain the least squares fit copula parameters.

The following table displays the parameters of the two models for few of the iTraxx Europe IG index constituents.

Table 2. Calibrated copula parameters

Corporate entity	Copula Parameter	
	ρ (Gaussian)	θ (Clayton)
BAE Systems PLC	0.9008	0.6861
Unilever NV	0.7130	0.4444
Continental AG	0.9123	0.9364
Peugeot SA	0.9130	0.7821
Commerzbank AG	0.7208	0.4189

The coefficients in the copula regression analysis, for all constituents in the bespoke test portfolio were significant at a 95% confidence level.

The following table summarizes the expected loss, unexpected loss, and CMETL at the 99.9% confidence level.

Table 3. Risk Measure differences for initial homogenous test portfolio under both Gaussian & Clayton copulas assumptions.

Risk Measure	Gaussian Copula	Clayton Copula
Expected Loss	3.7248%	3.6952%
Unexpected Loss @ 99.9%	20.349%	20.328%
CMETL @ 99.9%	25.534%	27.292%
Maximum Loss	30.447%	33.434%

Although the Gaussian copula predicts a slightly higher expected loss than that of the Clayton copula, there is a 1.758% absolute difference in the CMETL, and a 3% difference in the maximum loss between the Clayton and Gaussian copula models.

We optimise all positions and solve the linear programming problem represented by (2). Three scenarios are considered

iv. An examination of the efficient frontiers for a well-diversified portfolio under both Gaussian and Clayton copula assumptions. Only long positions in the underlying credit default swaps are allowed. The upper trading limit is set to 5%.

v. An investigation of the behaviour of the efficient frontiers under the Clayton copula assumption when the upper trading limit is increased, consequently introducing concentration risk.

vi. The comparison of efficient frontiers under the Clayton copula assumption when both long and short positions are permitted.

In the first case lower and upper trading limit of 0.5% and 5% are set respectively. This is to ensure that no reference asset is excluded from the portfolio. This also maintains the well-diversified features of the iTraxx Europe IG portfolio within the bespoke structure. The following figure presents the efficient frontiers under the two default dependence assumptions.

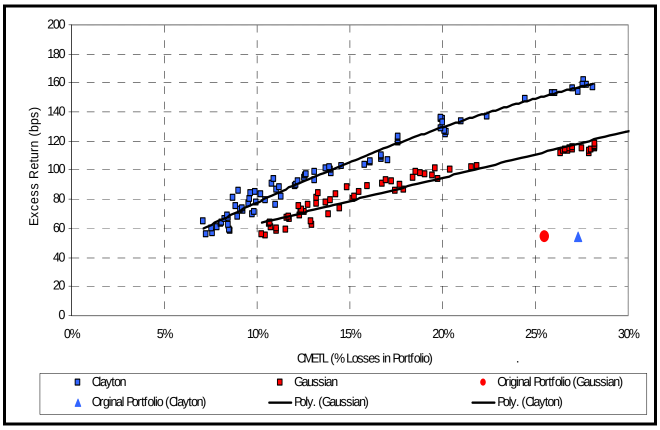

Figure 5. Comparison of efficient frontiers under Gaussian and Clayton copula assumptions

The above figure shows the difference in efficient frontiers between the Gaussian and Clayton copula assumptions. For a given level of risk, a higher return should be expected if the portfolio composition is based on the Clayton copula assumption rather then the Gaussian copula. The deviation of the expected returns between the two assumptions is an increasing function of CMETL.

The above figure also indicates the inefficient risk/return levels for the original portfolio under both the Gaussian and Clayton copula assumptions. It is interesting to compare the risk profile of the original portfolio, with that of the optimal portfolio having the same return. Under the Clayton copula assumption, the same level of return is achieved with less than one-fifth of the original risk.

In the second case, the upper trading limit is increased from 10% to 100%. The effect of concentration risk on the efficient frontiers is investigated.

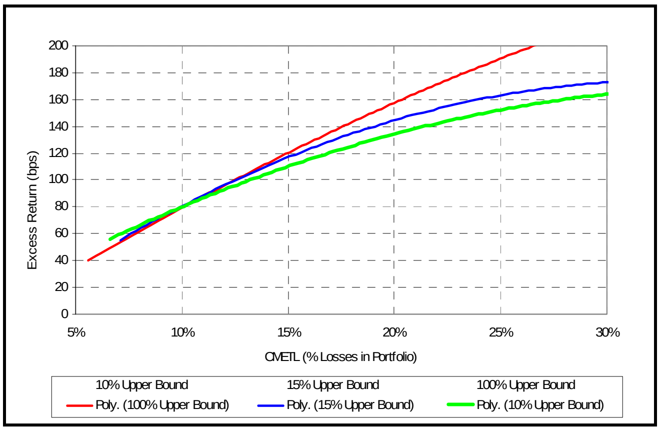

Figure 6. Behaviour of efficient frontiers under increasing concentration risk

The figure displays the expected variation in the second order polynomial fittings of the efficient frontiers when the upper trading limit is slowly relaxed. Under these circumstances, concentration risk is introduced into the portfolio. Investors demand a higher premium for taking on this risk. For a 20% level of risk, investors demand an extra 25 bps premium for holding a more concentrated portfolio. At these levels, the concentrated portfolio only holds positions in 49 of the 114 names, with the largest position size being 19.5%.

In the final case, the effect on the efficient frontier for allowing short positions is examined. Under this scenario, only the well-diversified portfolio case is considered. The lower and upper trading limits are set at -5% and 5% respectively.

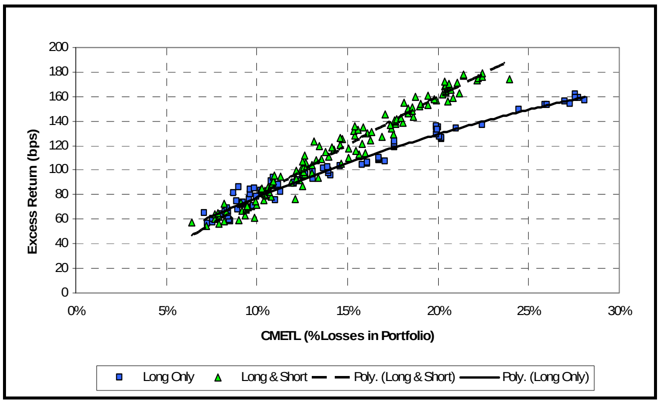

Figure 7. Effect of short positions on the efficient frontier

The figure displays an important result: allowing short positions on credits in the portfolio, provides the investor with superior returns to those in the long only case. At a 20% level of risk, investors can earn an extra 30 bps premium for taking on overweight views on certain credits in the portfolio. This indicates why hedge fund strategies involving Bespoke CDO structures have become increasingly popular.

The results for all efficient frontier second order polynomial regressions were significant at a 95% confidence level. The resulting R^2 coefficients were all above 90%.

V. Concluding Remarks

In this paper, we propose a new model, referred to as the CMETL model, for analyzing the behaviour of efficient frontier structures of CDOs of bespoke portfolios. Although our analysis only considers a specific CDS portfolio, the CMETL model extends naturally to other credit sensitive instruments and trading constraints of a more general nature.

Using the CMETL optimisation framework, we simultaneously adjusted two closely related risk measures: ETL and Unexpected Losses. The Gaussian copula asset allocation is proved sub-optimal. Clayton copula efficient frontiers provided a higher return for a given level of risk. A closer examination of the original risk profile shows that the risk can be reduced to one-fifth of the original amount, under the Clayton asset allocation method.

The permission of short positions in the bespoke CDO portfolio allows investors to increase returns beyond a specific risk tolerance level. In the case study considered, a maximum increase of 37.5% in investor-defined return is achieved by allowing overweight positions in certain credits.

The above analysis provides Bespoke CDO investors with a starting point for choosing optimal collateral portfolios. The return earned on the single tranche position will be higher than the pre-specified collateral portfolio returns.

Future work on Bespoke CDO cost minimisation is required to provide a complete depiction of the optimisation problem that lies within these credit structures. This will require an investigation into the different Bespoke CDO pricing methodologies.

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