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A numerical Solution of the Burgers Equation Using Quintic B-Splines

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Abstract— **In this paper, numerical solutions of the nonlinear Burgers_ equation are obtained by a method based on collocation of quintic B-splines over finite elements. Applying the Von-Neumann stability analysis, the proposed method is shown to be unconditionally stable. The accuracy of the presented method is demonstrated by one test problem. The numerical results are found to be in good agreement with the exact solutions.**

Index Terms— Collocation, B-Spline.

I. Introduction

This paper is concerned with applying the quintic B-spline function to develop a numerical method for approximating the analytic solution of the Burgers of the form

$$U_t + UU_x - \nu U_{xx} = 0 \qquad a \le x \le b \tag{1}$$

with the initial condition

$$U(x,0) = f(x) \tag{2}$$

and the boundary conditions

$$\begin{aligned} &U(a,t) = \alpha_1 \quad , \quad U(b,t) = \alpha_2 \\ &U_x(a,t) = U_x(b,t) = 0 \\ &U_{xx}(a,t) = U_{xx}(b,t) = 0 \end{aligned} \tag{3}$$

where α_1, α_2 are constants as the problem need, U =U(x, t) is a sufficiently-often differentiable function, and f(x) is bounded.

2. Collocation method

The interval [a,b] is partitioned into N finite elements of uniformly equal length h by the knots x_j , j = 0,1,2,. . . ,N such that $a = x_0 \le x_1 \le \cdots \le b = x_N$.

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Let the quintic B-spline function $\Phi_i(x)$ at these knots be given by: $\Phi_m(x)$ =

$$\frac{1}{h^5}\begin{cases} (x-x_{m-3})^5 & , \ [x_{m-3} \ , \ x_{m-2}] \\ (x-x_{m-3})^5 - 6\ (x-x_{m-2})^5 & , \ [x_{m-3} \ , \ x_{m-2}] \\ (x-x_{m-3})^5 - 6\ (x-x_{m-2})^5 + 15\ (x-x_{m-1})^5 & , \ [x_{m-2} \ , \ x_{m-1}] \\ (x-x_{m-3})^5 - 6\ (x-x_{m-2})^5 + 15\ (x-x_{m-1})^5 - 20\ (x-x_1)^5 & , \ [x_{m-1} \ , \ x_m] \\ (x-x_{m-3})^5 - 6\ (x-x_{m-2})^5 + 15\ (x-x_{m-1})^5 - 20\ (x-x_1)^5 & \\ \quad +15\ (x-x_{m-1})^5 & , \ [x_m \ , \ x_{m+1}] \\ (x-x_{m-3})^5 - 6\ (x-x_{m-2})^5 + 15\ (x-x_{m-1})^5 - 20\ (x-x_1)^5 & \\ \quad +15\ (x-x_{m-1})^5 - 6\ (x-x_{m-2})^5 & , \ [x_{m+1} \ , \ x_{m+2}] \\ 0 & , \ otherwise \end{cases} \tag{4}$$

where $h = x_{m+1} - x_m$ for all i, implying that all intervals $[x_{m-1}, x_m]$. This means that the values of the quintic B-spline function $\Phi_m(x)$, and all its first, second, and third derivatives vanish outside the interval $[x_{m-1}, x_m]$. The set of splines $\{\Phi_{-2}, \Phi_{-1}, \Phi_0, \cdots, \Phi_N, \Phi_{N+1}, \Phi_{N+2}\}$ forms a basis for the functions defined over [a,b]. One can easily verify that the values of $\Phi_m(x)$ and its derivatives are as shown in Table 1.

Our task is to find an approximate solution $U_N(x,t)$ to the exact the solution U(x, t) in the form:

$$U_N(x,t) = \sum_{i=-2}^{n+2} w_i(t)\Phi_i(x) \quad , \ j = 0,1,\ldots,N \tag{5}$$

Where $w_i(t)$ are time dependent quantities to be determined using the boundary conditions:

$$\begin{aligned} &U_N(a,t) = \alpha_1 \quad , \quad U_N(b,t) = \alpha_2 \\ &(U_N)_x(a,t) = (U_N)_x(b,t) = 0 \\ &(U_N)_{xx}(a,t) = (U_N)_{xx}(b,t) = 0 \end{aligned} \tag{6}$$

and the collocation condition:

$$(U_N)_t(x_j,t) + U_N(x_j,t)(U_N)_x(x_j,t) - \nu(U_N)_{xx}(x_j,t) = 0 \tag{7}$$

Substitute from Eq. (5) into Eq. (7) to get the following equation:

$$\sum_{i=-2}^{N+2} \frac{\partial w_i(t)}{\partial t}\Phi_i(x_j) + \sum_{i=-2}^{N+2} w_i(t)\Phi_i(x_j)\sum_{i=-2}^{N+2} w_i(t)\Phi_i'(x_j) - \nu\sum_{i=-2}^{N+2} w_i(t)\Phi_i''(x_j) = 0$$

where j = 0,1,2, . . . ,N. (8)

Table 1

The values of $\Phi_i(x)$ and its derivatives with knots at the shown points

x	x_{i-3}	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}	x_{i+3}
$\Phi_i(x)$	0	1	26	66	26	1	0
$\Phi'_i(x)$	0	5/h	50/h	0	-50/h	-5/h	0
$\Phi''_i(x)$	0	$20/h^2$	$40/h^2$	$120/h^2$	$40/h^2$	$20/h^2$	0

Suppose that w_i is linearly interpolated between two time levels n and n + 1 as:

$$w_i = (1-\theta)w_i^n + \theta w_i^{n+1} \tag{9}$$

$where\, 0 \le \theta \le 1$ and w_i^n are the parameters at the time $n\Delta t$

Using the finite difference method, we have

$$\frac{dw_i}{dt} = \frac{w_i^{n+1} - w_i^n}{\Delta t} \tag{10}$$

and hence Eq. (8) can be written as

$$\sum_{i=-2}^{N+2}\Phi_i(x_j)\left(\frac{w_i^{n+1}-w_i^n}{\Delta t}\right)+\sum_{i=-2}^{N+2}\Phi'_i(x_j)\left((1-\theta)w_i^n+\theta w_i^{n+1}\right)\sum_{i=-2}^{N+2}\Phi_i(x_j)w_i$$
$$-\nu\sum_{i=-2}^{N+2}\Phi''_i(x_j)\left((1-\theta)w_i^n+\theta w_i^{n+1}\right)=0 \tag{11}$$

Given the parameter θ the value 1/2 we get the Crank–Nicolson formula which implies the recurrence relation

$$\sum_{i=-1}^{N+1}\left\{\Phi_i(x_j)+\frac{\Delta t}{2}\Phi'_i(x_j)\left(\sum_{k=-1}^{N+1}\Phi_k(x_j)w_k\right)-\frac{\nu\Delta t}{2}\Phi''_i(x_j)\right\}w_i^{n+1} =$$
$$\sum_{i=-1}^{N+1}\left\{\Phi_i(x_j)-\frac{\Delta t}{2}\Phi'_i(x_j)\left(\sum_{k=-1}^{N+1}\Phi_k(x_j)w_k\right)+\frac{\nu\Delta t}{2}\Phi''_i(x_j)\right\}w_i^{n} \tag{12}$$

Using the values given in Table 1, Eq. (12) can be calculated at the knots x_j, j = 0, 1, 2, …, N so that at x = x_0, Eq. (12) reduces to:

$$\begin{aligned} &a_0 w_{-2}^{n+1} + b_0 w_{-1}^{n+1} + c_0 w_0^{n+1} + d_0 w_1^{n+1} + e_0 w_2^{n+1} = \\ &a'_0 w_{-2}^{n} + b'_0 w_{-1}^{n} + c'_0 w_0^{n} + d'_0 w_1^{n} + e'_0 w_2^{n} \end{aligned} \tag{13}$$

Where

$$\begin{aligned} a_0 &= 1 - r_1 Z_{i-2} - r_2 , & a'_0 &= 1 + r_1 Z_{-2} + r_2 \\ b_0 &= 26 - 10 r_1 Z_{-2} - 2r_2 , & b'_0 &= 26 + 10 r_1 Z_{-2} + 2r_2 \\ c_0 &= 66 + 6r_2 , & c'_0 &= 66 - 6r_2 \\ d_0 &= 26 + 10 r_1 Z_{-2} - 2r_2 , & d'_0 &= 26 - 10 r_1 Z_{-2} + 2r_2 \\ e_0 &= 1 + r_1 Z_{-2} - r_2 , & e'_0 &= 1 - r_1 Z_{-2} + r_2 \end{aligned} \tag{14}$$

$$Z_{-2} = w_{-2} + 26 w_{-1} + 66 w_0 + 26 w_1 + w_2$$

At $x = x_i$ Eq. (12) becomes

$$\begin{aligned} &a_i w_{i-2}^{n+1} + b_i w_{i-1}^{n+1} + c_i w_i^{n+1} + d_i w_{i+1}^{n+1} + e_i w_{i+2}^{n+1} = \\ &a'_i w_{i-2}^{n} + b'_i w_{i-1}^{n} + c'_i w_i^{n} + d'_i w_{i+1}^{n} + e'_i w_{i+2}^{n} \end{aligned} \tag{15}$$

Where

$$\begin{aligned} a_i &= 1 - r_1 Z_{i-2} - r_2 , & a'_i &= 1 + r_1 Z_{-2} + r_2 \\ b_i &= 26 - 10 r_1 Z_{-2} - 2r_2 , & b'_i &= 26 + 10 r_1 Z_{-2} + 2r_2 \\ c_i &= 66 + 6r_2 , & c'_i &= 66 - 6r_2 \\ d_i &= 26 + 10 r_1 Z_{-2} - 2r_2 , & d'_i &= 26 - 10 r_1 Z_{-2} + 2r_2 \\ ei &= 1 + r_1 Z_{-2} - r_2 , & e'_i &= 1 - r_1 Z_{-2} + r_2 \end{aligned} \tag{16}$$

$$with \quad r_1 = \frac{5\Delta t}{2h} , \quad r_2 = \frac{10\nu\Delta t}{h^2}$$

$$Z_{i-2} = w_{i-2} + 26 w_{i-1} + 66 w_i + 26 w_{i+1} + w_{i+2}$$

The system (15) consists of N+1 equations in the N+5 knowns $\left(w_{-2}, w_{-1}, w_0, \cdots, w_N, w_{N+1}, w_{N+2}\right)^T$. Toget a solution to this system, we need four additional constraints. These constraints are obtained from the boundary conditions (6) and can be used to eliminate from the system (15). Then, we have the matrix system equation

$$A\left(w^n\right)w^{n+1} = B\left(w^n\right)w^n + r \tag{17}$$

where the matrices $A\left(w^n\right)$ and $B\left(w^n\right)$ are septa-diagonal (N + 1)(N + 1) matrices and r is the N + 1 dimensional column vector. The penta-diagonal algorithm is then used to solve the system (17). We apply first the initial condition:

$$U_N(x,0) = \sum_{i=-2}^{N+2} w_i^0 \Phi_i(x) \tag{18}$$

Todeterminetheinitialstate $\left\{w_{-2}, w_{-1}, w_0, \cdots, w_N, w_{N+1}, w_{N+2}\right\}$.

The approximate solution $U_N(x,0)$ must satisfy the following conditions:

(a) It must agree with the initial condition u(x, 0) at the knots x_j.

(b) The first, second and the third derivatives of the approximate initial condition agree with those of the exact initial conditions at both ends of the range.

These two conditions can be expressed as:

$$\begin{aligned} &U_N(x_i,0) = U(x_i,0) \\ &(U_N)_x(x_0,0) = U_x(a,0) = 0 \ , \ (U_N)_x(x_N,0) = U_x(b,0) = 0 \\ &(U_N)_{xx}(x_0,0) = U_{xx}(a,0) = 0 \ , \ (U_N)_{xx}(x_N,0) = U_{xx}(b,0) = 0 \\ &i = 0,1,2,\cdots,N \end{aligned} \tag{19}$$

Eliminating w_{-2}^0, w_{-1}^0 and w_{N+1}^0, w_{N+2}^0 from the system (19), we have

$$Aw^0 = r \tag{20}$$

Where A is the penta-diagonal matrix given by:

$$A=\begin{bmatrix} 54 & 60 & 6 & 0 & 0 & 0 & \cdots & 0 \\ \frac{101}{4} & \frac{135}{2} & \frac{105}{4} & 1 & 0 & 0 & \cdots & 0 \\ 1 & 26 & 66 & 26 & 1 & 0 & \cdots & 0 \\ & & \cdots & & \cdots & & & \\ & & \cdots & & \cdots & & & \\ 0 & \cdots & 0 & 1 & 26 & 66 & 26 & 1 \\ 0 & \cdots & 0 & 0 & 1 & \frac{105}{4} & \frac{135}{2} & \frac{101}{4} \\ 0 & \cdots & 0 & 0 & 0 & 6 & 60 & 54 \end{bmatrix}$$

$and\ w^0=\left[w_0^0, w_1^0, w_2^0, \cdots, w_N^0\right]^T$ while the vector r has the form $r=\left[U\left(x_0,0\right), U\left(x_1,0\right), \cdots, U\left(x_N,0\right)\right]^T$.

3. Stability analysis

To apply the Von-Neumann stability for the system (15), we must first linearize this system. According to the Von-Neumann we have

$$w_j^n=\varepsilon^n \exp\left(i\beta jh\right) \quad , \ i=\sqrt{-1} \tag{21}$$

where β is the mode number and h is the element size will be determined for a linearization of numerical scheme. At $x=x_j$, Eq. (15) can be written as:

$$\begin{aligned} &a_j w_{j-2}^{n+1}+b_j w_{j-1}^{n+1}+c_j w_j^{n+1}+d_j w_{j+1}^{n+1}+e_j w_{j+2}^{n+1}= \\ &a_j' w_{j-2}^{n}+b_j' w_{j-1}^{n}+c_j' w_j^{n}+d_j' w_{j+1}^{n}+e_j' w_{j+2}^{n} \end{aligned} \tag{22}$$

Substitute from Eq. (21) into the recurrence relation (22) to get:

$$\begin{aligned} &\varepsilon^{n+1}\left\{a_j \exp(-2i\beta h)+b_j \exp(-i\beta h)+c_j+d_j \exp(i\beta h)+e_j \exp(2i\beta h)\right\}= \\ &\varepsilon^{n}\left\{a_j' \exp(-2i\beta h)+b_j' \exp(-i\beta h)+c_j'+d_j' \exp(i\beta h)+e_j' \exp(2i\beta h)\right\} \end{aligned} \tag{23}$$

Eq. (23) can be rewritten in a simple form as:

$$\varepsilon=\frac{X-iY}{X_1+iY} \tag{24}$$

$where\ X_1,\ X\ and\ Y\ are\ as\ follows:$

$$X=\left(2+2r_2\right)\cos\left(2\beta h\right)+\left(52+4r_2\right)\cos\left(\beta h\right)+\left(66-6r_2\right) \tag{25}$$
$$X_1=\left(2-2r_2\right)\cos\left(2\beta h\right)+\left(52-4r_2\right)\cos\left(\beta h\right)+\left(66+6r_2\right)$$
$$Y=2r_1 Z_{i-2}\sin\left(2\beta h\right)+20r_1 Z_{i-2}\sin\left(\beta h\right)$$

We note that $\left|X\right| \le X_1$, $so\ \left|\varepsilon\right|=\sqrt{\dfrac{X^2+Y^2}{X_1{}^2+Y^2}} \le 1$. (26)

Therefore, the linearized numerical scheme for the Burgers equation is unconditionally stable.

4. Numerical results

We now obtain the numerical solutions of Burger for one standard problem. The accuracy of the numerical method is measured by computing the difference between the analytic and numerical solutions at each mesh point, and uses these to compute the $L_2\ and\ L_\infty$ -error norms.

Example 1. Consider an analytic solution of the Burgers equation [3,4] of the form:

$$U(x,t)=\frac{\frac{x}{t}}{1+\sqrt{\frac{t}{s}}\exp\left(\frac{x^2}{4\nu t}\right)} \quad , \quad t\ge 1\ ,\ 0\le x\le 1\ ,\ s=\exp\left(\frac{1}{8\nu}\right) \tag{27}$$

Initial condition is obtained from Eq. (28) when t = 1 is used. The analytical solution represents shock-like solutions of the one-dimensional Burgers_ equation. The boundary conditions are:

$$U(0,t)=U(1,t)=0$$
$$U_x(0,t)=U_x(1,t)=0 \quad , \quad U_x(0,t)=U_x(1,t)=0 \tag{28}$$

The obtained results are summarized in Tables 2 as follows: From the above tables, we note that as the viscosity value v is increased the errors tend to increase, but for all the values of v used here, the errors are acceptable.

The following figures give the behavior of the numerical solutions for various times:

Figs. 1 illustrate the propagation of shock for $\nu=0.0015$, $\Delta t=0.01$, $\Delta x=0.005$.

The figures show that as the time increases the curve of the numerical solution decays.

5. Conclusion

In this paper a numerical treatment for the nonlinear Burgers_ equation is proposed using a collection method with the quintic B-splines. The stability analysis of the method is shown to be unconditionally stable. The obtained approximate numerical solutions maintain good accuracy compared with the exact solutions especially or small values of the viscosity parameter

Table.2
Comparison of results at different times for $\nu = 0.0015$, $\Delta t = 0.01$ *and* $\Delta x = 0.005$

x	CBCM	Exact	CBCM	Exact	CBCM	Exact
	t=1.7	t=1.7	t=2.5	T=2.5	T=3.25	T=3.25
0.1	0.05893	0.05882	0.04003	0.04000	0.03117	0.03079
0.2	0.11799	0.11764	0.08016	0.08000	0.06163	0.06163
0.3	0.17699	0.17647	0.12024	0.12000	0.09245	0.09245
0.4	0.23599	0.23529	0.16032	0.16000	0.12327	0.12327
0.5	0.29499	0.29412	0.20040	0.20000	0.15408	0.15408
0.6	0.35313	0.35216	0.24048	0.24000	0.18490	0.18490
0.7	0.00048	0.00053	0.28050	0.27995	0.21572	0.21572
0.8	0.00000	0.00000	0.05644	0.06040	0.24647	0.24647
0.9	0.00000	0.00000	0.00000	0.00000	0.10331	0.10709
L_2	2.26079e-004		3.05381e-004		3.10732e-004	
L_∞	0.00118		0.00352		0.0037	

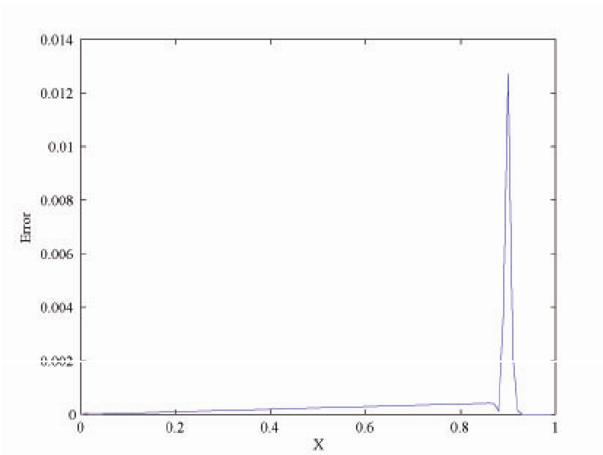

Figure 2. Errors $(|\text{Numerical - Analytical}|)$ *at time* $t = 3.25$ *with* $\nu = 0.0015$

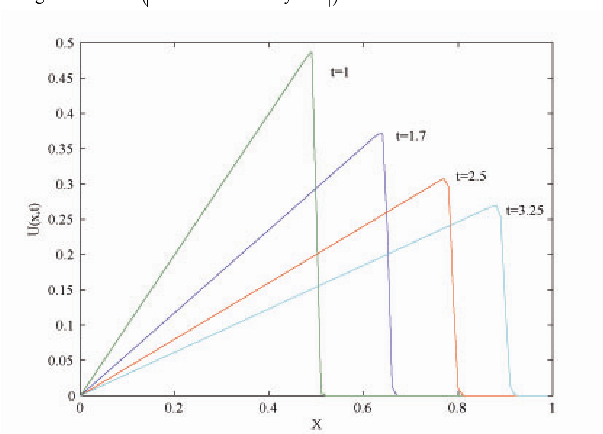

Figure 1. $\nu = 0.0015$, $\Delta t = 0.01$ *and* $\Delta x = 0.005$

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