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Orthogonal Rational Functions, Associated Rational Functions And Functions Of The Second Kind*

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Abstract— **Consider the sequence of poles $\mathcal{A} = \{\alpha_1, \alpha_2, \ldots\}$, and suppose the rational functions φ_j with poles in $\mathcal{A}$ form an orthonormal system with respect to a Hermitian positive-definite inner product. Further, assume the φ_j satisfy a three-term recurrence relation. Let the rational function $\varphi_{j\setminus 1}^{(1)}$ with poles in $\{\alpha_2, \alpha_3, \ldots\}$ represent the associated rational function of φ_j of order 1; i.e. the $\varphi_{j\setminus 1}^{(1)}$ do satisfy the same three-term recurrence relation as the φ_j. In this paper we then give a relation between φ_j and $\varphi_{j\setminus 1}^{(1)}$ in terms of the so-called rational functions of the second kind. Next, under certain conditions on the poles in $\mathcal{A}$, we prove that the $\varphi_{j\setminus 1}^{(1)}$ form an orthonormal system of rational functions with respect to a Hermitian positive-definite inner product. Finally, we give a relation between associated rational functions of different order, independent of whether they form an orthonormal system.**

Keywords: Orthogonal rational functions, associated rational functions, rational functions of the second kind, three-term recurrence relation, Favard theorem.

1 Introduction

Let ϕ_j denote the polynomial of degree j that is orthogonal with respect to a positive measure μ on a subset S of the real line. Further, suppose the orthogonal polynomials (OPs) ϕ_j are monic (i.e. they are of the form $\phi_j(x) = x^j + \ldots$) and satisfy a three-term recurrence relation given by

$$\begin{aligned}&\phi_{-1}(x) \equiv 0, \quad \phi_0(x) \equiv 1,\\ &\phi_j(x) = (x - \alpha_j)\phi_{j-1}(x) - \beta_j \phi_{j-2}(x), \quad j \geq 1.\end{aligned}$$

Let the monic polynomial $\phi_{j-k}^{(k)}$ of degree $j - k$ denote the associated polynomial (AP) of order $k \geq 0$, with $j \geq k$. By definition, these APs are the polynomials generated by the three-term recurrence relation given by

$$\begin{aligned}&\phi_{-1}^{(k)}(x) \equiv 0, \quad \phi_0^{(k)}(x) \equiv 1,\\ &\phi_{j-k}^{(k)}(x) = (x - \alpha_j)\phi_{(j-1)-k}^{(k)}(x) - \beta_j \phi_{(j-2)-k}^{(k)}(x),\\ &\qquad\qquad\qquad\qquad\qquad\qquad j \geq k + 1.\end{aligned}$$

Note that this way the monic APs of order 0 and the monic OPs are in fact the same.

The following relations exist between monic APs of different order (see e.g. [7])

$$\phi_{j-k}^{(k)}(x) = (x - \alpha_{k+1})\phi_{j-(k+1)}^{(k+1)}(x) - \beta_{k+2}\phi_{j-(k+2)}^{(k+2)}(x), \quad j \geq k + 1 \tag{1}$$

and

$$\phi_{j-k}^{(k)}(x) = \phi_{j-l}^{(l)}(x)\phi_{l-k}^{(k)}(x) - \beta_{l+1}\phi_{j-(l+1)}^{(l+1)}(x)\phi_{(l-1)-k}^{(k)}(x), \quad k + 1 \leq l \leq j - 1. \tag{2}$$

From the Favard theorem it follows that the APs of order k form an orthogonal system with respect to a positive measure $\mu^{(k)}$ on S. Therefore, another relation exists between the APs of order $k - 1$ and k in terms of polynomials of the second kind:

$$\phi_{j-k}^{(k)}(x) = \int_S \frac{\phi_{j-(k-1)}^{(k-1)}(t) - \phi_{j-(k-1)}^{(k-1)}(x)}{t - x} d\mu^{(k-1)}(t). \tag{3}$$

Orthogonal rational functions (ORFs) on a subset S of the real line (see e.g. [2, 5, 6] and [1, Chapt. 11]) are a generalisation of OPs on S in such a way that they are of increasing degree with a given sequence of complex poles, and the OPs result if all the poles are at infinity. Let φ_j denote the rational function with j poles outside S that is orthogonal with respect to a positive measure μ on S. Under certain conditions on the poles, these ORFs do satisfy a three-term recurrence relation as well. Consequently, associated rational functions (ARFs) can be defined based on this three-term recurrence relation. Furthermore, in [1, Chapt. 11.2], the rational function of the second kind $\varphi_j^{[1]}$ of φ_j is defined similarly as in (3); i.e.

$$\varphi_j^{[1]}(x) = \int_S \frac{\varphi_j(t) - \varphi_j(x)}{t - x} d\mu(t). \tag{4}$$

The aim of this paper is to generalise the relations for APs, given by (1)–(3), to the case of ARFs. But first, we start with the necessary theoretical background in the next section.

2 Preliminaries

The field of complex numbers will be denoted by $\mathbb{C}$ and the Riemann sphere by $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. For the real line we use

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the symbol $\mathbb{R}$, while the extended real line will be denoted by $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. Further, we represent the positive real line by $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$. If the value $a \in X$ is omitted in the set X, this will be represented by X_a; e.g.

$$\mathbb{C}_0 = \mathbb{C} \setminus \{0\}.$$

Let $c = a + \mathbf{i}b$, where $a, b \in \mathbb{R}$, then we represent the real part of $c \in \mathbb{C}$ by $\Re\{c\} = a$ and the imaginary part by $\Im\{c\} = b$.

Given a sequence $\mathcal{A}_j = \{\alpha_1, \alpha_2, \dots, \alpha_j\} \subset \overline{\mathbb{C}}_0$, we define the factors

$$Z_l(x) = \frac{x}{1 - x/\alpha_l}, \qquad l = 1, 2, \dots, j,$$

and products

$$b_0(x) \equiv 1, \qquad b_l(x) = Z_l(x) b_{l-1}(x), \qquad l = 1, 2, \dots, j,$$

or equivalently,

$$b_l(x) = \frac{x^l}{\pi_l(x)}, \quad \pi_l(x) = \prod_{i=1}^{l} (1 - x/\alpha_i), \quad \pi_0(x) \equiv 1.$$

The space of rational functions with poles in $\mathcal{A}_j$ is then given by

$$\mathcal{L}_j = \operatorname{span}\{b_0(x), b_1(x), \dots, b_j(x)\}.$$

We will also need the reduced sequence of poles $\mathcal{A}_{j\setminus k} = \{\alpha_{k+1}, \alpha_{k+2}, \dots, \alpha_j\}$, where $0 \leq k \leq j$, and the reduced space of rational functions with poles in $\mathcal{A}_{j\setminus k}$ given by

$$\mathcal{L}_{j\setminus k} = \operatorname{span}\{b_{k\setminus k}(x), b_{(k+1)\setminus k}(x), \dots, b_{j\setminus k}(x)\},$$

where

$$b_{l\setminus k}(x) = \frac{b_l(x)}{b_k(x)} = \frac{x^{l-k}}{\pi_{l\setminus k}(x)},$$

for $l \geq k$ and

$$\pi_{l\setminus k}(x) = \prod_{i=k+1}^{l} (1 - x/\alpha_i), \quad \pi_{l\setminus l}(x) \equiv 1.$$

In the special case in which $k = 0$ or $k = j$, we have that $\mathcal{A}_{j\setminus 0} = \mathcal{A}_j$ and $\mathcal{L}_{j\setminus 0} = \mathcal{L}_j$, respectively $\mathcal{A}_{j\setminus j} = \emptyset$ and $\mathcal{L}_{j\setminus j} = \mathcal{L}_0 = \overline{\mathbb{C}}$. We will assume that the poles in $\mathcal{A}_j$ are arbitrary complex or infinite; hence, they do not have to appear in pairs of complex conjugates.

We define the substar conjugate of a function $f(x) \in \mathcal{L}_\infty$ by

$$f_*(x) = \overline{f(\overline{x})}.$$

Consider an inner product that is defined by the linear functional M:

$$\langle f, g \rangle = M\{f g_*\}, \quad f, g \in \mathcal{L}_\infty.$$

We say that M is a Hermitian positive-definite linear functional (HPDLF) if for every $f, g \in \mathcal{L}_\infty$ it holds that

$$f \neq 0 \Leftrightarrow M\{f f_*\} > 0 \quad \text{and} \quad M\{f g_*\} = \overline{M\{f_* g\}}.$$

Further, let μ_0 be defined as $\mu_0 = M\{1\} \in \mathbb{R}_0^+$, and suppose there exists a sequence of rational functions $\{\varphi_j\}$, with $\varphi_j \in \mathcal{L}_j \setminus \mathcal{L}_{j-1}$, so that the φ_j form an orthonormal system with respect to M.

Let $\alpha_0 \in \overline{\mathbb{C}}_0$ be arbitrary but fixed in advance. Then the orthonormal rational functions (ORFs) $\varphi_j = \frac{p_j}{\pi_j}$ are said to be regular for $j \geq 1$ if $p_j(\alpha_{j-1}) \neq 0$ and $p_j(\overline{\alpha}_{j-1}) \neq 0$. A zero of p_j at ∞ means that the degree of p_j is less than j. We now have the following recurrence relation for ORFs. For the proof, we refer to [5, Sec. 2] and [3, Sec. 3].

Theorem 2.1. *Let $E_0 \in \mathbb{C}_0$, $\alpha_{-1} \in \overline{\mathbb{R}}_0$ and $\alpha_0 \in \overline{\mathbb{C}}_0$ be arbitrary but fixed in advance. Then the ORFs φ_j, $j = n-2, n-1, n$, with $n \geq 1$, are regular iff there exists a three-term recurrence relation of the form*

$$\varphi_n(x) = Z_n(x) \left\{ \left[E_n + \frac{F_n}{Z_{n-1}(x)} \right] \varphi_{n-1}(x) + \frac{C_n}{Z_{n-2*}(x)} \varphi_{n-2}(x) \right\}, \quad (5)$$

with $E_n \neq 0$ and

$$C_n = -\frac{E_n + F_n / Z_{n-1}(\overline{\alpha}_{n-1})}{\overline{E}_{n-1}} \neq 0.$$

The initial conditions are $\varphi_{-1}(x) \equiv 0$ and $\varphi_0(x) \equiv \frac{\eta}{\sqrt{\mu_0}}$, where η is a unimodular constant ($|\eta| = 1$).

Let $\varphi_{j\setminus k}^{(k)} \in \mathcal{L}_{j\setminus k}$ denote the associated rational function (ARF) of φ_j of order k; i.e. $\varphi_{j\setminus k}^{(k)}$, $j = k+1, k+2, \dots$, is generated by the same recurrence relation as φ_j with initial conditions $\varphi_{(k-1)\setminus k}^{(k)}(x) \equiv 0$ and $\varphi_{k\setminus k}^{(k)}(x) \equiv \kappa_0^{(k)}$. Note that in the special case in which $k = 0$, we have that $\varphi_{j\setminus 0}^{(0)} = \varphi_j$. In the remainder of this paper we will assume that $\varphi_{j\setminus k}^{(k)}$ is of the form

$$\varphi_{j\setminus k}^{(k)}(x) = \frac{p_{j-k}^{(k)}(x)}{\pi_{j\setminus k}(x)},$$

with

$$p_{j-k}^{(k)}(x) = \kappa_{j-k}^{(k)} x^{j-k} + \dots, \quad \kappa_{j-k}^{(k)} \in \mathbb{C}$$

and $\kappa_0^{(k)} \neq 0$. We now have the following Favard theorem. For the proof, we refer to [4].

Theorem 2.2 (Favard). *Let $\{\varphi_{j\setminus k}^{(k)}\}$ be a sequence of rational functions, and assume that*

(A1) $\alpha_{k-1} \in \overline{\mathbb{R}}_0$ *and* $\alpha_j \in \overline{\mathbb{C}}_0$, $j = k, k+1, \dots$,

(A2) $\varphi_{j\setminus k}^{(k)}$, $j = k+1, k+2, \dots$, *is generated by a three-term recurrence relation of the form given by Equation (5),*

(A3) $\varphi_{j\setminus k}^{(k)} \in \mathcal{L}_{j\setminus k} \setminus \mathcal{L}_{(j-1)\setminus k}$, $j = k+1, k+2, \dots$, *and* $\varphi_{k\setminus k}^{(k)} \in \mathbb{C}_0$,

(A4) *Let* $\hat{F}_j = F_j/E_j$, *with* $E_j = E^{(k)}_{j\backslash k}$ *and* $F_j = F^{(k)}_{j\backslash k}$. *Then* $|\hat{F}_j| < \infty$ *and*

$$\frac{\Im\{\alpha_{j-2}\}}{|\alpha_{j-2}|^2} - \frac{\Im\{\alpha_j\}}{|\alpha_j|^2} \cdot \frac{|E_{j-1}|^2}{|E_j|^2} = \left[\frac{\Im\{\alpha_{j-1}\}}{|\alpha_{j-1}|^2}|\hat{F}_j|^2 - \Im\{\hat{F}_j\}\right] \times \left[|E_{j-1}|^2 - 4\frac{\Im\{\alpha_{j-1}\}}{|\alpha_{j-1}|^2} \cdot \frac{\Im\{\alpha_{j-2}\}}{|\alpha_{j-2}|^2}\right], \quad j = k+1, k+2, \ldots,$$

(A5) *max* $\left\{0, 4\frac{\Im\{\alpha_j\}}{|\alpha_j|^2} \cdot \frac{\Im\{\alpha_{j-1}\}}{|\alpha_{j-1}|^2}\right\} < |E_j|^2 < \infty$, $j = k, k+1, \ldots$,

(A6) $C_j\overline{E}_{j-1} = -[E_j + F_j/Z_{j-1}(\overline{\alpha}_{j-1})] \neq 0$, *with* $C_j = C^{(k)}_{j\backslash k}$, $j = k+1, k+2, \ldots$.

Then there exists a HPDLF $M^{(k)}$ *so that*

$$\langle f, g\rangle = M^{(k)}\{fg_*\} = \int_S fg_* d\mu^{(k)}$$

defines a Hermitian positive-definite inner product for which the rational functions $\varphi^{(k)}_{j\backslash k}$ *form an orthonormal system.*

In the remainder we will assume that the system of ORFs φ_j satisfies every assumption in Theorem 2.2. This way, if $\alpha_{k-1} \in \overline{\mathbb{R}}_0$, it is sufficient to prove that $\varphi^{(k)}_{j\backslash k} \notin \mathcal{L}_{(j-1)\backslash k}$ for $j = k+1, k+2, \ldots$, so that the ARFs $\varphi^{(k)}_{j\backslash k}$ form an orthonormal system with respect to a HPDLF $M^{(k)}$. If $\alpha_{k-1} \in \overline{\mathbb{R}}_0$ and condition (A3) is satisfied as well, we let $M^{(k)}\{1\} = \mu^{(k)}_0 = |\kappa^{(k)}_0|^{-2}$.

3 Associated rational functions

Suppose the ARFs $\varphi^{(k-1)}_{j\backslash(k-1)}$ of order $k-1 \geq 0$ form an orthonormal system with respect to a HPDLF $M^{(k-1)}$, and let $\Phi_{j\backslash(k-1)}$ be given by

$$\Phi_{j\backslash(k-1)}(x,t) = (1 - t/\overline{\alpha}_{k-1})\varphi^{(k-1)}_{j\backslash(k-1)}(x). \quad (6)$$

Then we define the rational functions of the second kind $\psi_{j\backslash k}$ by

$$\psi_{j\backslash k}(x) = (1 - x/\alpha_k) \times \left[M^{(k-1)}_t\left\{\frac{\Phi_{j\backslash(k-1)}(t,x) - \Phi_{j\backslash(k-1)}(x,t)}{t-x}\right\} - \delta_{j,k-1}\overline{R}_{k-1}\right], \quad j \geq k-1, \quad (7)$$

where $\delta_{j,k-1}$ is the Kronecker Delta and

$$R_{k-1} = \left[\kappa^{(k-1)}_0 \alpha_{k-1}\right]^{-1}. \quad (8)$$

Note that this definition is very similar to, but not exactly the same as the one given before in (4). We will then prove that the $\psi_{j\backslash k}$ satisfy the same three-term recurrence relation as $\varphi^{(k-1)}_{j\backslash(k-1)}$ with initial conditions $\psi_{(k-1)\backslash k}(x) \equiv 0$ and

$$\psi_{k\backslash k}(x) \equiv -\overline{E}_{k-1}C_k/\overline{\kappa}^{(k-1)}_0 \neq 0.$$

First, we need the following lemma.

Lemma 3.1. *Let* $\psi_{j\backslash k}$, *with* $j \geq k-1 \geq 0$, *be defined as before in (7). Then it holds that*

$$\psi_{j\backslash k}(x) \equiv \begin{cases} 0, & j = k-1 \\ -\overline{E}_{k-1}C_k/\overline{\kappa}^{(k-1)}_0 \neq 0, & j = k, \end{cases}$$

while $\psi_{j\backslash k} \in \mathcal{L}_{j\backslash k}$ *for* $j > k$.

Proof. Define $q_{j-(k-2)}$ by

$$q_{j-(k-2)}(x) = (1 - x/\overline{\alpha}_{k-1})\pi_{j\backslash(k-1)}(x)$$

For $j \geq k$ it then follows from (6) and (7) that

$$\psi_{j\backslash k}(x) = \frac{1}{\pi_{j\backslash k}(x)} M^{(k-1)}_t\left\{\frac{1}{t-x} \times \left[\varphi^{(k-1)}_{j\backslash(k-1)}(t)q_{j-(k-2)}(x) - (1 - t/\overline{\alpha}_{k-1})p^{(k-1)}_{j-(k-1)}(x)\right]\right\} = \frac{\sum_{i=0}^{j-(k-1)} M^{(k-1)}_t\left\{a^{(k)}_i(t)\right\}x^i}{\pi_{j\backslash k}(x)}. \quad (9)$$

Further, with

$$c_{j,k} = \lim_{x\to\infty} \frac{\pi_{j\backslash k}(x)}{x^{j-k}},$$

we have that

$$\begin{aligned} M^{(k-1)}_t\left\{a^{(k)}_{j-(k-1)}(t)\right\} &= \frac{c_{j,k-1}}{\overline{\alpha}_{k-1}} M^{(k-1)}_t\left\{\varphi^{(k-1)}_{j\backslash(k-1)}(t)\right\} \\ &= 0, \end{aligned}$$

so that

$$\psi_{j\backslash k}(x) = \frac{p^{(k)}_{j-k}(x)}{\pi_{j\backslash k}(x)} \in \mathcal{L}_{j\backslash k}.$$

For $j = k$ we find that

$$\kappa^{(k)}_0 = M^{(k-1)}_t\left\{\frac{\varphi^{(k-1)}_{k\backslash(k-1)}(t)q_2(x) - (1 - t/\overline{\alpha}_{k-1})p^{(k-1)}_1(x)}{-x(1-t/x)}\right\}.$$

Note that

$$\lim_{x\to\overline{\alpha}_{k-1}} -\frac{q_2(x)}{x} M^{(k-1)}_t\left\{\frac{\varphi^{(k-1)}_{k\backslash(k-1)}(t)}{1 - t/x}\right\} = 0,$$

so that

$$\begin{aligned}\kappa_0^{(k)} &= \lim_{x\to\overline{\alpha}_{k-1}} M_t^{(k-1)}\left\{\frac{1-t/\overline{\alpha}_{k-1}}{1-t/x}\right\}\frac{p_1^{(k-1)}(x)}{x}\\ &= \left|\kappa_0^{(k-1)}\right|^{-2}\lim_{x\to\overline{\alpha}_{k-1}}\frac{\varphi^{(k-1)}_{k\backslash(k-1)}(x)}{Z_k(x)}\\ &= \lim_{x\to\overline{\alpha}_{k-1}}\frac{1}{\overline{\kappa}_0^{(k-1)}}\left[E_k+\frac{F_k}{Z_{k-1}(x)}\right]\\ &= \left[E_k+F_k/Z_{k-1}(\overline{\alpha}_{k-1})\right]/\overline{\kappa}_0^{(k-1)}.\end{aligned}$$

Finally, in the special case in which $j=k-1$, we have that

$$M_t^{(k-1)}\left\{\frac{\Phi_{(k-1)\backslash(k-1)}(t,x)-\Phi_{(k-1)\backslash(k-1)}(x,t)}{t-x}\right\}=$$
$$\kappa_0^{(k-1)}M_t^{(k-1)}\left\{\frac{(1-x/\overline{\alpha}_{k-1})-(1-t/\overline{\alpha}_{k-1})}{t-x}\right\}=\overline{R}_{k-1},$$

where R_{k-1} is given by (8). □

The following theorem now shows that these $\psi_{j\backslash k}$ do satisfy the same three-term recurrence relation as the φ_j.

Theorem 3.2. *Let $\psi_{j\backslash k}$ be defined as before in (7). The rational functions $\psi_{j\backslash k}$, with $j=n-2,n-1,n$ and $n\geq k+1$, then satisfy the three-term recurrence relation given by*

$$\psi_{n\backslash k}(x)=Z_n(x)\left\{\left[E_n+\frac{F_n}{Z_{n-1}(x)}\right]\psi_{(n-1)\backslash k}(x)+\frac{C_n}{Z_{n-2*}(x)}\psi_{(n-2)\backslash k}(x)\right\}. \quad (10)$$

The initial conditions are $\psi_{(k-1)\backslash k}(x)\equiv 0$ and

$$\psi_{k\backslash k}(x)\equiv -\overline{E}_{k-1}C_k/\overline{\kappa}_0^{(k-1)}\neq 0.$$

Proof. First note that the ARFs $\varphi^{(k-1)}_{j\backslash(k-1)}$, with $j=n-2,n-1,n$, do satisfy the three-term recurrence relation given by (10), and hence, so do the $\Phi_{n\backslash(k-1)}$. We now have that

$$\begin{aligned}\frac{\psi_{n\backslash k}(x)}{1-x/\alpha_k} &= \\ &E_nM_t^{(k-1)}\left\{\frac{1}{t-x}\times\left[Z_n(t)\Phi_{(n-1)\backslash(k-1)}(t,x)-Z_n(x)\Phi_{(n-1)\backslash(k-1)}(x,t)\right]\right\}\\ &+F_nM_t^{(k-1)}\left\{\frac{1}{t-x}\times\left[\frac{Z_n(t)}{Z_{n-1}(t)}\Phi_{(n-1)\backslash(k-1)}(t,x)-\frac{Z_n(x)}{Z_{n-1}(x)}\Phi_{(n-1)\backslash(k-1)}(x,t)\right]\right\}\\ &+C_nM_t^{(k-1)}\left\{\frac{1}{t-x}\times\left[\frac{Z_n(t)}{Z_{n-2*}(t)}\Phi_{(n-2)\backslash(k-1)}(t,x)-\frac{Z_n(x)}{Z_{n-2*}(x)}\Phi_{(n-2)\backslash(k-1)}(x,t)\right]\right\},\end{aligned}$$

$$\begin{aligned}\psi_{n\backslash k}(x) &= Z_n(x)\left\{\left[E_n+\frac{F_n}{Z_{n-1}(x)}\right]\psi_{(n-1)\backslash k}(x)+\frac{C_n}{Z_{n-2*}(x)}\psi_{(n-2)\backslash k}(x)\right\}+M_t^{(k-1)}\left\{\frac{f_n(x,t)}{t-x}\right\}\\ &\quad+\delta_{n,k+1}\overline{R}_{k-1}C_{k+1}\left[(1-x/\alpha_k)\frac{Z_{k+1}(x)}{Z_{k-1*}(x)}\right],\end{aligned}$$

where $f_n(x,t)=(1-x/\alpha_k)g_n(x,t)$ and $g_n(x,t)$ is given by

$$\begin{aligned}g_n(x,t) &= E_n[Z_n(t)-Z_n(x)]\Phi_{(n-1)\backslash(k-1)}(t,x)\\ &+F_n\left[\frac{Z_n(t)}{Z_{n-1}(t)}-\frac{Z_n(x)}{Z_{n-1}(x)}\right]\Phi_{(n-1)\backslash(k-1)}(t,x)\\ &+C_n\left[\frac{Z_n(t)}{Z_{n-2*}(t)}-\frac{Z_n(x)}{Z_{n-2*}(x)}\right]\Phi_{(n-2)\backslash(k-1)}(t,x).\end{aligned}$$

Note that

$$\begin{aligned}Z_n(t)-Z_n(x) &= \frac{(t-x)}{(1-t/\alpha_n)(1-x/\alpha_n)}\\ \frac{Z_n(t)}{Z_{n-1}(t)}-\frac{Z_n(x)}{Z_{n-1}(x)} &= \frac{(t-x)/Z_{n-1}(\alpha_n)}{(1-t/\alpha_n)(1-x/\alpha_n)}\\ \frac{Z_n(t)}{Z_{n-2*}(t)}-\frac{Z_n(x)}{Z_{n-2*}(x)} &= \frac{(t-x)/Z_{n-2*}(\alpha_n)}{(1-t/\alpha_n)(1-x/\alpha_n)},\end{aligned}$$

so that

$$\begin{aligned}\frac{f_n(x,t)}{t-x} &= \left[(1-x/\alpha_k)\frac{Z_n(x)}{Z_{k-1*}(x)}\right](1-t/\alpha_n)^{-1}h_n(t)\\ &= \left[(1-x/\alpha_k)\frac{Z_n(x)}{Z_{k-1*}(x)}\right]\left(1+\frac{Z_n(t)}{\alpha_n}\right)h_n(t),\end{aligned}$$

where

$$h_n(t)=\left[E_n\varphi^{(k-1)}_{(n-1)\backslash(k-1)}(t)+\frac{F_n}{Z_{n-1}(\alpha_n)}\varphi^{(k-1)}_{(n-1)\backslash(k-1)}(t)+\frac{C_n}{Z_{n-2*}(\alpha_n)}\varphi^{(k-1)}_{(n-2)\backslash(k-1)}(t)\right].$$

It clearly holds that

$$M_t^{(k-1)}\{h_n(t)\}=\frac{\delta_{n,k+1}C_{k+1}}{\overline{\kappa}_0^{(k-1)}Z_{k-1*}(\alpha_{k+1})}.$$

Further, note that

$$\frac{Z_n(t)}{Z_{n-2*}(\alpha_n)}=\frac{Z_n(t)}{Z_{n-2*}(t)}-1$$

and

$$\frac{Z_n(t)}{Z_{n-1}(\alpha_n)}=\frac{Z_n(t)}{Z_{n-1}(t)}-1.$$

Hence,

$$Z_n(t)h_n(t)=\varphi^{(k-1)}_{n\backslash(k-1)}(t)-F_n\varphi^{(k-1)}_{(n-1)\backslash(k-1)}(t)-C_n\varphi^{(k-1)}_{(n-2)\backslash(k-1)}(t),$$

so that

$$\frac{M_t^{(k-1)}\{Z_n(t)h_n(t)\}}{\alpha_n} = -\frac{\delta_{n,k+1}C_{k+1}}{\overline{\kappa}_0^{(k-1)}\alpha_{k+1}}.$$

Consequently, we have that

$$M_t^{(k-1)}\left\{\frac{f_n(x,t)}{t-x}\right\} = \\ -\delta_{n,k+1}\overline{R}_{k-1}C_{k+1}\left[(1-x/\alpha_k)\frac{Z_{k+1}(x)}{Z_{k-1*}(x)}\right],$$

which ends the proof. □

The next theorem directly follows from Lemma 3.1 and Theorem 3.2.

Theorem 3.3. *Let $\psi_{j\backslash k}$ be defined as before in (7). These $\psi_{j\backslash k}$ are the ARFs $\varphi_{j\backslash k}^{(k)}$ of order k with initial conditions $\varphi_{(k-1)\backslash k}^{(k)}(x) \equiv 0$ and*

$$\varphi_{k\backslash k}^{(k)}(x) \equiv -\overline{E}_{k-1}C_k/\overline{\kappa}_0^{(k-1)} \neq 0.$$

In the above lemma and theorems we have assumed that the ARFs $\varphi_{j\backslash(k-1)}^{(k-1)}$ form an orthonormal system with respect to a HPDLF $M^{(k-1)}$. The assumption certainly holds for $k = 1$, and hence, the ARFs $\varphi_{j\backslash 1}^{(1)}$ are the rational functions of the second kind of the ORFs φ_j. The next question is then whether the ARFs $\varphi_{j\backslash 1}^{(1)}$ form an orthonormal system with respect to a HPDLF $M^{(1)}$. Therefore, we need the following lemma.

Lemma 3.4. *Let the ARFs $\varphi_{j\backslash k}^{(k)}$ of order k be defined by (7). Then the leading coefficient $K_{j-k}^{(k)}$, i.e. the coefficient of $b_{j\backslash k}$ in the expansion of $\varphi_{j\backslash k}^{(k)}$ with respect to the basis $\{b_{k\backslash k}, \ldots, b_{j\backslash k}\}$, is given by*

$$K_{j-k}^{(k)} = K_{j-(k-1)}^{(k-1)} M_t^{(k-1)}\left\{\frac{1-t/\overline{\alpha}_{k-1}}{1-t/\alpha_j}\right\}, \quad j \geq k.$$

Proof. Note that the leading $K_{j-k}^{(k)}$ is given by (see also [3, Thm. 3.1])

$$K_{j-k}^{(k)} = \lim_{x\to\alpha_j}\frac{\varphi_{j\backslash k}^{(k)}(x)}{b_{j\backslash k}(x)} = \lim_{x\to\alpha_j}\frac{p_{j-k}^{(k)}(x)}{x^{j-k}}.$$

Further, let $q_{j-(k-2)}$ be defined as before in Lemma 3.1. Clearly, for $j \geq k$ it then holds that

$$\lim_{x\to\alpha_j} -\frac{q_{j-(k-2)}(x)}{x^{j-(k-1)}} M_t^{(k-1)}\left\{\frac{\varphi_{k\backslash(k-1)}^{(k-1)}(t)}{1-t/x}\right\} = 0.$$

So, from (9) we deduce that

$$\begin{aligned} K_{j-k}^{(k)} &= \lim_{x\to\alpha_j}\frac{p_{j-(k-1)}^{(k-1)}(x)}{x^{j-(k-1)}} M_t^{(k-1)}\left\{\frac{1-t/\overline{\alpha}_{k-1}}{1-t/x}\right\} \\ &= K_{j-(k-1)}^{(k-1)} M_t^{(k-1)}\left\{\frac{1-t/\overline{\alpha}_{k-1}}{1-t/\alpha_j}\right\}. \end{aligned}$$

This proves the statement. □

As a consequence, we now have the following theorem.

Theorem 3.5. *Let the ARFs $\varphi_{j\backslash k}^{(k)}$ of order k be defined by (7) and assume that $\alpha_{k-1} \in \overline{\mathbb{R}}_0$. Further, suppose that*

$$M_t^{(k-1)}\left\{\frac{1-t/\alpha_{k-1}}{1-t/\alpha_j}\right\} \neq 0 \tag{11}$$

whenever $j > k$ and $\alpha_j \notin \{\alpha_{k-1}, \overline{\alpha}_k, \alpha_k\}$. Then it holds that the $\varphi_{j\backslash k}^{(k)}$ form an orthonormal system with respect to a HPDLF $M^{(k)}$.

Proof. As pointed out at the end of Section 2, it suffices to prove that the $\varphi_{j\backslash k}^{(k)} \in \mathcal{L}_{j\backslash k} \setminus \mathcal{L}_{(j-1)\backslash k}$ for $j > k$.

Note that $\varphi_{j\backslash k}^{(k)} \in \mathcal{L}_{j\backslash k} \setminus \mathcal{L}_{(j-1)\backslash k}$ iff $K_{j-k}^{(k)} \neq 0$. We now have that $K_{j-(k-1)}^{(k-1)} \neq 0$ for every $j > k$, due to the fact that the ARFs $\varphi_{j\backslash(k-1)}^{(k-1)}$ are regular. Moreover, as $M^{(k-1)}$ is a HPDLF and because $\varphi_{k\backslash(k-1)}^{(k-1)}$ is regular, we also have that

$$M_t^{(k-1)}\left\{\frac{1-t/\alpha_{k-1}}{1-t/\alpha_j}\right\} \neq 0$$

whenever $\alpha_j \in \{\alpha_{k-1}, \overline{\alpha}_k, \alpha_k\}$. Thus, together with the assumption given by (11), it follows from Lemma 3.4 that $\varphi_{j\backslash k}^{(k)} \in \mathcal{L}_{j\backslash k} \setminus \mathcal{L}_{(j-1)\backslash k}$ for every $j > k$. Consequently, the ARFs $\varphi_{j\backslash k}^{(k)}$ then satisfy the six conditions given in Theorem 2.2. □

Finally, in Theorem 3.7 we give a relation between ARFs of different order that holds independent of whether the ARFs involved form an orthonormal system with respect to a HPDLF. First we need the following lemma.

Lemma 3.6. *The ARFs $\varphi_{n\backslash s}^{(s)} = \kappa_0^{(s)}\hat{\varphi}_{n\backslash s}^{(s)}$, with $s = k, k+1, k+2$ and $n \geq k+1$, satisfy the relation given by*

$$\begin{aligned} \hat{\varphi}_{n\backslash k}^{(k)}(x) = Z_{k+1}(x)\left[E_{k+1} + \frac{F_{k+1}}{Z_k(x)}\right]\hat{\varphi}_{n\backslash(k+1)}^{(k+1)}(x) \\ + C_{k+2}\frac{Z_{k+2}(x)}{Z_{k*}(x)}\hat{\varphi}_{n\backslash(k+2)}^{(k+2)}(x). \end{aligned} \tag{12}$$

Proof. First, consider the case in which $n = k+1$. From Theorem 3.2 we deduce that

$$\hat{\varphi}_{(k+1)\backslash k}^{(k)}(x) = Z_{k+1}(x)\left[E_{k+1} + \frac{F_{k+1}}{Z_k(x)}\right]\hat{\varphi}_{k\backslash k}^{(k)}(x).$$

We also have that $\hat{\varphi}_{k\backslash k}^{(k)}(x) \equiv 1 \equiv \hat{\varphi}_{(k+1)\backslash(k+1)}^{(k+1)}(x)$, while $\hat{\varphi}_{(k-1)\backslash k}^{(k)}(x) \equiv 0 \equiv \hat{\varphi}_{(k+1)\backslash(k+2)}^{(k+2)}(x)$. Hence, the statement clearly holds for $n = k+1$.

Next, consider the case in which $n = k+2$. From Theorem 3.2 we now deduce that

$$\hat{\varphi}^{(k)}_{(k+2)\backslash k}(x) = Z_{k+2}(x)\left[E_{k+2} + \frac{F_{k+2}}{Z_{k+1}(x)}\right]\hat{\varphi}^{(k)}_{(k+1)\backslash k}(x) + C_{k+2}\frac{Z_{k+2}(x)}{Z_{k*}(x)}\hat{\varphi}^{(k)}_{k\backslash k}(x).$$

Furthermore, we have that $\hat{\varphi}^{(k)}_{k\backslash k}(x) \equiv 1 \equiv \hat{\varphi}^{(k+2)}_{(k+2)\backslash(k+2)}(x)$. While,

$$\begin{aligned} Z_{k+2}(x)&\left[E_{k+2} + \frac{F_{k+2}}{Z_{k+1}(x)}\right]\hat{\varphi}^{(k)}_{(k+1)\backslash k}(x) \\ &= Z_{k+1}(x)\left[E_{k+1} + \frac{F_{k+1}}{Z_k(x)}\right] \times \\ Z_{k+2}(x)&\left[E_{k+2} + \frac{F_{k+2}}{Z_{k+1}(x)}\right]\hat{\varphi}^{(k+1)}_{(k+1)\backslash(k+1)}(x) \\ &= Z_{k+1}(x)\left[E_{k+1} + \frac{F_{k+1}}{Z_k(x)}\right]\hat{\varphi}^{(k+1)}_{(k+2)\backslash(k+1)}(x). \end{aligned}$$

Consequently, the statement clearly holds for $n = k+2$ as well.

Finally, assume that the statement holds for $n-2$ and $n-1$. By induction, the statement is then easily verified for $n \geq k+3$ by applying the three-term recurrence relation, given by Theorem 3.2, to the left hand side of (12) for $\hat{\varphi}^{(k)}_{n\backslash k}$, as well as to the right hand side of (12) for $\hat{\varphi}^{(k+1)}_{n\backslash(k+1)}$ and $\hat{\varphi}^{(k+2)}_{n\backslash(k+2)}$. □

Theorem 3.7. *The ARFs $\varphi^{(s)}_{n\backslash s} = \kappa^{(s)}_0\hat{\varphi}^{(s)}_{n\backslash s}$, with $s = k, j+1, j+2$ and $k+1 \leq j \leq n-1$, are related by*

$$\hat{\varphi}^{(k)}_{n\backslash k}(x) = \hat{\varphi}^{(j)}_{n\backslash j}(x)\hat{\varphi}^{(k)}_{j\backslash k}(x) + C_{j+1}\frac{Z_{j+1}(x)}{Z_{j-1*}(x)}\hat{\varphi}^{(j+1)}_{n\backslash(j+1)}(x)\hat{\varphi}^{(k)}_{(j-1)\backslash k}(x). \quad (13)$$

Proof. Note that for every $l \geq 0$ it holds that

$$\hat{\varphi}^{(l)}_{l\backslash l}(x) \equiv 1 \quad \text{and} \quad \hat{\varphi}^{(l)}_{(l+1)\backslash l}(x) = Z_{l+1}(x)\left[E_{l+1} + \frac{F_{l+1}}{Z_l(x)}\right].$$

Thus, for $j = n-1$ or $j = k+1$, the relation given by (13) is nothing more than the three-term recurrence relation given by Theorem 3.2, respectively the relation given by (12).

So, suppose that the statement holds for j. By induction we then find for $j+1$ that

$$\hat{\varphi}^{(k)}_{(j+1)\backslash k}(x) = \hat{\varphi}^{(j)}_{(j+1)\backslash j}(x)\hat{\varphi}^{(k)}_{j\backslash k}(x) + C_{j+1}\frac{Z_{j+1}(x)}{Z_{j-1*}(x)}\hat{\varphi}^{(k)}_{(j-1)\backslash k}(x),$$

and

$$C_{j+2}\frac{Z_{j+2}(x)}{Z_{j*}(x)}\hat{\varphi}^{(j+2)}_{n\backslash(j+2)}(x) = \hat{\varphi}^{(j)}_{n\backslash j}(x) - \hat{\varphi}^{(j+1)}_{n\backslash(j+1)}(x)\hat{\varphi}^{(j)}_{(j+1)\backslash j}(x).$$

Consequently,

$$\begin{aligned} \hat{\varphi}^{(j+1)}_{n\backslash(j+1)}(x)\hat{\varphi}^{(k)}_{(j+1)\backslash k}(x) & \\ + C_{j+2}\frac{Z_{j+2}(x)}{Z_{j*}(x)}\hat{\varphi}^{(j+2)}_{n\backslash(j+2)}(x)\hat{\varphi}^{(k)}_{j\backslash k}(x) & \\ = C_{j+1}\frac{Z_{j+1}(x)}{Z_{j-1*}(x)}\hat{\varphi}^{(j+1)}_{n\backslash(j+1)}(x)\hat{\varphi}^{(k)}_{(j-1)\backslash k}(x) & \\ + \hat{\varphi}^{(j)}_{n\backslash j}(x)\hat{\varphi}^{(k)}_{j\backslash k}(x), & \end{aligned}$$

which ends the proof. □

4 Conclusion

In this paper, we have given a relation between associated rational functions (ARFs) of order $k-1$ and k in terms of rational functions of the second kind, assuming the ARFs of order $k-1$ form an orthonormal system with respect to a Hermitian positive-definite inner product. Further, we have given a relation between ARFs of different order that holds in general; i.e. the relation holds independently of whether the ARFs involved form an orthonormal system with respect to a Hermitian positive-definite inner product. If all the poles are at infinity, we again obtain the polynomial case.

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