A Fixed Point Theorem on Four Self Maps Under Weakly Compatible

V.Srinivas and R.Umamaheshwar Rao

Abstract- **This paper is to prove a common fixed point theorem for four self maps which generalizes the result of Brian Fisher [1] by a weaker conditions such as weakly compatible mappings and associated sequence instead of commuting mappings and completeness of a metric space.**

Index Terms: **Self maps, fixed point, associated sequence, weakly compatible mappings.**

I. INTRODUCTION

Two self maps S and T are said to be commutative if ST = TS. The concept of the commutativity has generalized in several ways. For this Sessa S [6] has introduced the concept of weakly commuting and Gerald Jungck [2] initiated the concept of compatibility.

A. Compatible Mappings

Two self maps S and T of a metric space (X,d) are said to be compatible mappings if $\lim\limits_{n\to\infty} d(STx_n,TSx_n)=0$, whenever $<x_n>$ is a sequence in X such that $\lim\limits_{n\to\infty} Sx_n = Tx_n = t$ for some $t\in X$.

One can be easily verified that when the two mappings are commuting then they are compatible but not conversely.

In 1998, Jungck and Rhoades [4] introduced the notion of weakly compatible and showed that compatible maps are weakly compatible but not conversely.

B. Weakly Compatible

A pair of maps A and S is called weakly compatible pair if they commute at coincidence points.

V.Srinivas and R.Umamaheshwar Rao are with Department of Science and Humanitites, Sreenidhi Institute of Science and Technology, Ghatkesar, Hyderabad, India-501 301.

C. Contractive Modulus

A function $\phi:[0,\infty)\to[0,\infty)$ is said to be a contractive modulus if $\phi(0)=0$ and $\phi(t) < t$ for $t>0$.

D. Upper Semi Continuous

A real valued function ϕ defined on $X\subseteq R$ is said to be upper semi continuous if $\lim\limits_{n\to\infty} \text{Sup}\ \phi(t_n)\leq \phi(t)$, for very sequence $<t_n>\in X$ with $t_n\to t$ as $n\to\infty$.

Obviously every continuous function is upper semi continuous but not conversely.

Brian Fisher [1] proved the following Common Fixed Point theorem.

II. Theorem: Suppose S, P, T and Q are four self maps of a metric space (X,d) satisfying the conditions.

$S(X)\subseteq Q(X)$ and $T(X)\subseteq P(X)$ (1) pairs (S,P) and (T,Q) are commuting (2)
one of S,P,T and Q is continuous, (3) and
$d(Sx,Ty)\leq c\,\lambda(x,y)$ (4)
where $\lambda(x,y)=\max\{d(Px,Qy),d(Px,Sx), d(Qy,Ty)\}$ for all $x,y\in X$ and $0\leq c<1$ (5)
Further if
X is complete, (6)
then S,P,T and Q have a unique common fixed point $z\in X$. Also z is the unique common fixed point of (S,P) and of (T,Q).

E. Associated Sequence:

Suppose S,P,Tand Q are four self maps of a metric space (X,d) satisfying $S(X)\subseteq Q(X)$ and $T(X)\subseteq P(X)$, then for any $x_0\in X$, we have $Sx_0\in S(X)$ and therefore $x_0\in Q(X)$ which gives a $x_1\in X$ such that $Sx_0=Qx_1$ again $Tx_1\in T(X)$ so that $Tx_1\in P(X)$ and hence there is a $x_2\in X$ such that $Tx_1 = Px_2$. Now $Sx_2\in S(X)$ gives an $x_3\in X$ such that $Sx_2 = Qx_3$. Again $Tx_3\in T(X)$ so that there is a $x_4\in X$ such that $Tx_3 = Px_4$. Repeating this process we get a sequence $<x_n>$ in X such that $Sx_{2n} = Qx_{2n+1}$ and

$Tx_{2n+1} = Px_{2n+2}$ for $n \geq 0$. This is called associated sequence of x_0 relative to the four self maps S,P,T and Q.

Now we prove a Lemma.

Lemma: Suppose S,P,T and Q are four self maps of a metric space (X,d) for which the conditions (1) and (4) hold. Further if (X,d) is a complete metric space then for any $x_0 \in X$ and for any of its associated sequence $<x_n>$ relative to the four self maps, the sequence

$Sx_0, Tx_1, Sx_2, Tx_3, \ldots, Sx_{2n}, Tx_{2n+1}, \ldots$ ...(7)

converges to some point $z \in X$.

Proof: Suppose S,P,T and Q are self maps of a metric space (X,d) for which the conditions (1) and (4) hold. Let $x_0 \in X$ and $<x_n>$ be an associated sequence of x_0 relative to the four Self-maps. Then, since $Sx_{2n} = Qx_{2n+1}$ and $Tx_{2n+1} = Px_{2n+2}$, for $n \geq 0$. From the condition (4), we have

$$d(Sx_{2n}, Tx_{2n+1}) \leq c\,\lambda(x_{2n}, x_{2n+1}) \qquad (8)$$

where

$$\begin{aligned}\lambda(x_{2n}, x_{2n+1}) &= \max\{d(Px_{2n}, Qx_{2n+1}), d(Px_{2n}, Sx_{2n}), d(Qx_{2n+1}, Tx_{2n+1})\}\\ &= \max\{d(Tx_{2n-1}, Sx_{2n}), d(Tx_{2n}, Sx_{2n}), d(Sx_{2n}, Tx_{2n+1})\}\\ &= \max\{d(Tx_{2n-1}, Sx_{2n}), d(Sx_{2n}, Tx_{2n+1})\}\end{aligned}$$

since $c<1$,

$$\max\{d(Tx_{2n-1}, Sx_{2n}), d(Sx_{2n}, Tx_{2n+1})\} = d(Tx_{2n-1}, Sx_{2n})$$

So that (8) gives,

$$d(Sx_{2n}, Tx_{2n+1}) \leq c\, d(Tx_{2n-1}, Sx_{2n}) \qquad (9)$$

Similarly we can prove that

$$d(Tx_{2n-1}, Sx_{2n}) \leq c\, d(Sx_{2n-2}, Tx_{2n-1}) \qquad (10)$$

Now, from (9) and (10) we get

$$d(Sx_{2n}, Tx_{2n+1}) \leq c^2\, d(Sx_{2n-2}, Tx_{2n-1})$$

which on repeated use gives

$$d(Sx_{2n}, Tx_{2n+1}) \leq c^{2n}\, d(Sx_0, Tx_1) \qquad (11)$$

Now, since $c<1$, $c^{2n} \to 0$, as $n \to \infty$. shows that the sequence given in (7) is a cauchy sequence in X and since (X,d) is a complete metric space, it converges to a point say $z \in X$.

The converse of the Lemma is not true, that is S,P,T and Q are self maps of a metric space (X,d) satisfying (1) and (4), even if for $x_0 \in X$ and for associated sequence $<x_n>$ of x_0, the sequence in (7) converges, the metric space(X,d) need not be complete.

III. Main Theorem:

Theorem: Let S,P,T and Q are four self maps of metric space (X,d) satisfying the conditions.

$S(X) \subseteq Q(X)$ and $T(X) \subseteq P(X)$ (12)

(S,P)and (T,Q) are weakly compatible (13)

$d(Sx, Ty) \leq \phi(\lambda(x,y))$ (14)

where ϕ is upper semi continuous, contractive modulus and $\lambda(x,y) = \max\{d(Px,Qy), d(Px,Sx), d(Qy,Ty)\}$ for all $x,y \in X$.

Further if

there is a point $x_0 \in X$ and an associated sequence $<x_n>$ of x_0 relative to the four self maps such that the sequence $Sx_0, Tx_1, Sx_2, Tx_3, \ldots Sx_{2n}, Tx_{2n+1}, \ldots$ converges to some point $z \in X$. (15)

then S,P,T and Q have a common fixed point $z \in X$. Further z is the unique common fixed point of (S,P) and (T,Q).

Proof: From (15), there is a associated sequence $<x_n>$ relative to x_o such that $Sx_{2n} = Qx_{2n+1}$ and $Tx_{2n+1} = Px_{2n+2}$ for n greater than or equal to zero.

The sequences $\{Sx_{2n}\}$, $\{Tx_{2n+1}\}$ converges to z as $n \to \infty$ (16)

Since $T(X) \subseteq P(X)$, there exists a point $u \in X$ such that $z = Pu$.

$$\begin{aligned}\text{Now } d(Su,z) &= \lim_{n\to\infty} d(Su, Tx_{2n+1}) \qquad (17)\\ &= \lim_{n\to\infty} \mathrm{Sup}(d(Su, Tx_{2n+1}))\\ &\leq \lim_{n\to\infty} \mathrm{Sup}\,(\phi(\lambda(u, x_{2n+1})))\end{aligned}$$

where $\lambda(u, x_{2n+1}) = \max\{d(Pu, Qx_{2n+1}), d(Pu,Su), d(Qx_{2n+1}, Tx_{2n+1})\}$

Letting $n \to \infty$,

$$\lim_{n\to\infty} \lambda(u, x_{2n+1}) = \max\{d(z,z), d(z,Su), d(z,z)\} = d(Su,z). \qquad (18)$$

In view of (18),(17) gives.

$$d(Su,z) \leq \phi(d(Su,z)) \qquad (19)$$

If $Su \neq z$ then, $d(Su,z)>0$, since ϕ is contractive modulus $\phi(d(Su,z))<d(Su,z)$, (19) gives $d(Su,z)<d(Su,z)$, which is a contradiction, thus $Su = z$.

Therefore, $Pu = Su = z$, showing that u is a coincidence point for S and P and z is a coincidence point. Since the pair of maps S and P are weakly compatible, $SPu = Su$, i.e., $Sz = Pz$. Since $S(X)\subseteq Q(X)$, there exists a point $v\in X$ such that $z = Qv$.

Now, $d(z,Tv)=d(Su,Tv) \leq \phi(\lambda(u,v))$ (20)

where $\lambda(u,v)=\max\{d(Pu,Qv),d(Pu,Su), d(Qv,Tv)\}$

$= \max \{d(z,z),d(z,z),d(z,Tv)\}$

$= d(z,Tv)$.

Therefore, $d(z,Tv)\leq \phi(d(z,Tv))$. If $z \neq Tv$ then, $d(z,Tv)>0$, since ϕ is contractive modulus $\phi(d(z,Tv))<d(z,Tv)$, gives $d(z,Tv) < d(z,Tv)$, which is a contradiction.

Thus $z = Tv$.

Hence, $z = Tv = Qv$, showing that v is a coincidence point for T and Q and z is a coincidence value, since the pair of maps T and Q are weakly compatible, $QTv = v$, i.e., $Tz = Qz$ .Now, we show that z is a fixed point of S.

$d(Sz,z) = d(Sz,Tv) \leq \phi(\lambda(z,v))$ (21)

where $\lambda(z,v)=\max\{d(Pz,Qv),d(Pz,Sz), d(Qv,Tv)\}$

$= \max \{d(Sz,z),d(Sz,Sz),d(z,z)\}$

$= d(Sz,z)$.

Therefore, $d(Sz,z)\leq\phi(d(Sz,z))$. If $z \neq Sz$ then, $d(Sz,z)>0$, since ϕ is contractive modulus

$\phi(d(Sz,z))<d(Sz,z)$, gives $d(Sz,z)< (Sz,z)$, which is a contradiction , thus $Sz = z$.

Therefore, $Sz =Pz = z$.

Now, we show that z is a fixed point of T.

$d(z,Tz) = d(Sz,Tz) \leq \phi(\lambda(z,z))$ (22)

where $\lambda(z,z) = \max \{d(Pz,Qz),d(Pz,Sz), d(Qz,Tz)\}$

$= \max \{d(z,Tz), d(z,z), d(Tz,Tz)\}$

$= d(z,Tz)$.

Therefore, $d(z,Tz)\leq \phi(d(z,Tz))$ .If $z \neq Tz$ then, $d(z,Tz)>0$, since ϕ is contractive modulus $\phi(d(z,Tz))<d(z,Tz)$, gives $d(z,Tz) < d(z,Tz)$, which is a contradiction, thus $Tz = z$. Therefore, $Tz = Qz = z$.

Hence, $Sz = Pz=Tz=Qz=z$. Showing that z is the common fixed point of S,P,T and Q. Now, we prove the uniqueness of this fixed point.

Suppose that z^1 is another common fixed point of S,P,T and Q. Then we have $Sz^1 = Pz^1 = Tz^1 = Qz^1 = z^1$.

Now,
$d(z,z^1) = d(Sz,Tz^1) \leq \phi(\lambda(z, z^1))$ (23)

Since $\lambda(z,z^1) = d(z,z^1)$, which gives

$d(z,z^1)\leq \phi(d(z,z^1))$ and If $z\neq z^1$, $d(z,z^1)>0$ so that $\phi(d(z,z^1))< d(z,z^1)$ and (23) gives $d(z,z^1)<d(z,z^1)$, is a contradiction, hence $z^1 = z$. Therefore z is the unique common fixed point of S,P,T and Q.

Now, we give the example to justify our result.

Example: Let $X = [0,1)$ with $d(x,y)= |x-y|$

$$Px = Qx = \begin{cases} \frac{1}{5} - x & if \quad 0 \leq x \leq 1/10 \\ \frac{1}{8} & if \quad 1/10 < x < 1 \end{cases}$$

$$Tx = Sx = \begin{cases} \frac{1}{10} & \text{if} \quad 0 \leq x \leq 1/10 \\ \frac{1}{8} & \text{if} \quad 1/10 < x < 1 \end{cases}$$

In fact SP(0) = $\frac{1}{8}$ ≠ PS(0) = $\frac{1}{10}$, so that SP ≠ PS on $\left[0, \frac{1}{10}\right)$, Similarly TQ ≠ QT on $\left[0, \frac{1}{10}\right)$ and also SPx = PSx and TQx = QTx for all $x \in \left[\frac{1}{10}, 1\right)$ which shows the pairs (S,P) and (T,Q) are weakly compatible.

Let $x_n = \left(\frac{1}{10} - \frac{1}{10^n}\right)$ be a sequence in X converges to $\frac{1}{10}$ as $n \to \infty$. Hence, for such $\langle x_n \rangle$ sequences Sx_n, Tx_n, Px_n, Qx_n converges to $\frac{1}{10}$ as $n \to \infty$. $PSx_n \to \frac{1}{10}$, $SPx_n \to \frac{1}{8}$ as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} d(SPx_n, PSx_n) = d\left(\frac{1}{8}, \frac{1}{10}\right) \neq 0$$

Showing that the pair (S,P) is not compatible. Similarly, the pair (T,Q) is not compatible.

Remark: In the above example, the mappings S,T,P and Q are not continuous the pairs (S,P) and (T,Q) are neither commuting nor compatible but they are weakly compatible.

It is easy to prove that the associated sequence relative to the above self maps such that the sequence (15) is converges to a point $\frac{1}{10}$ in X but the metric space is not complete. Moreover, $\frac{1}{10}$ is the point of discontinuity of four self maps. Hence, Theorem III is a generalization of Theorem II.

REFERENCES

[1] B.Fisher, "*Common Fixed Point of Four Mappings*",Bull. Inst. of .Math. Academia. Sinicia, vol.11, 1983, pp.103-113.

[2] G.Jungck, "*Compatible Mappings and Common Fixed Points*", Inst. J. Math. Math. Sci.,Vol.4, sept. 1986, pp.771-779.

[3] G.Jungck, "*Commuting Maps and Fixed Points*",Amer. Math. Monthly, Vol.83,1976, pp.261-263.

[4] G.Jungck, and B.E.Rhoades," *Fixed Point for set valued Functions Without Continuity*", Indian J. Pure Appl. Math, vol.29,no.3, 1998, pp.227-238.

[5] S.Leader, "*Fixed Points for general Contractions in Metric Spaces*", Math.Japonica,Vol.24,1979, pp.17-24.

[6] S.Sessa, "*On a weak Commutativity Condition of Mappings in a Fixed Point Considerations*", Publ. Inst Math. Debre., Vol.32,1982, pp.149–153