Vibration of the Euler-Bernoulli Beam with Allowance for Dampings

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Abstract—**The Euler-Bernoulli uniform elastically supported beam model with incorporated dissipation mechanisms is dealt with. Conditions are given to ensure oscillatory character of solutions.**

Keywords: Euler-Bernoulli beam equation, dissipation mechanisms, globally oscillatory solutions, uniform oscillatory time

1 Conservative systems

The classical linear theory of deformation yields the Euler-Bernoulli model for transverse vibrations of a beam. This typical linear elastic system is described by the partial differential equation for the function $(t,x) \mapsto u(t,x)$, $u\colon \mathbb{R}^+ \times (0,\,\ell) \to \mathbb{R}$:

$$\varrho A\,\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI\,\frac{\partial^2 u}{\partial x^2}\right) = 0. \tag{1}$$

It represents the (oldest yet commonly used) model for the motion of a straight elastic beam of length ℓ, cross-sectional area A, mass density ϱ, oriented so that in the rest configuration the x-axis lies along the neutral axis of the beam with the end points located at $x = 0$ and $x = \ell$. The one-dimensional model (1) where the displacement u depends only on one-dimensional spatial variable x (and time t) is obtained upon the use of Hooke's law and other simplifying assumptions: the thickness and width of the beam are small compared with the length, cross-sections of the beam remain plane during any deformation, each point of the axis performs only motion in a plane perpendicular to the axis and all points of the axis move in one common plane (plane of vibration). The quantity E is the modulus of elasticity of the beam material and I is the moment of inertia of the cross-section about an axis through the center of mass perpendicular to the plane of vibration (centroidal area moment of inertia). In the particular case of a uniform beam whose material and geometric properties are independent of x the quantities ϱ, A and the bending stiffness EI are constant and Eq. (1) assumes the (constant coefficients) form

$$\frac{\partial^2 u}{\partial t^2} + \frac{EI}{\varrho A}\,\frac{\partial^4 u}{\partial x^4} = 0. \tag{2}$$

The equation is complemented by boundary conditions. For the sake of definiteness and simplicity let us assume that the beam is simply supported (or hinged) at its ends. This means that the edges of the beam cannot translate in the transverse direction, but they are free to rotate about the axis perpendicular to the plain of vibration, i. e. the bending moment $M = -EI\partial^2 u/\partial x^2$ vanishes at the ends. Consequently, the boundary conditions are given by

$$u(t,0) = \frac{\partial^2 u(t,0)}{\partial x^2} = 0,\ u(t,\ell) = \frac{\partial^2 u(t,\ell)}{\partial x^2} = 0. \tag{3}$$

The Euler-Bernoulli beam model can be modified in various ways. For instance, if the beam rests on an elastic foundation (the modulus of which is γ) or the beam is subjected to an axial (tensile/compressive) force S we get

$$\varrho A\,\frac{\partial^2 u}{\partial t^2} + EI\,\frac{\partial^4 u}{\partial x^4} - S\,\frac{\partial^2 u}{\partial x^2} + \gamma\,u = 0. \tag{4}$$

In the Euler-Bernoulli model (and above mentioned modifications) only translation motion of cross-sections is taken into account and the effects of rotatory inertia are neglected. This is reasonable only for a slender beam the cross-sectional dimensions of which are small in comparison with its length. In contrast, these effects are involved in a more accurate model due to Rayleigh. A still more complete model (considered to be the most complete one-dimensional model) is due to Timoshenko where also shearing deformations are taken into account (for example rectangular beam elements are deformed into parallelograms or skew trapezoids). For more detail see [1], [2], [17], [21].

An important feature of the Euler-Bernoulli model is that the forces acting on the system can be derived from a potential energy and the total mechanical energy is conserved: the system is conservative.

2 Dissipation mechanisms

"Perhaps the most notable disadvantage associated with conservative systems is the fact that they do not occur in nature" ([5], p. 433).

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The presence of energy dissipation mechanisms is now generally accepted in all models used for simulation of mechanical vibrations in elastic systems. Two kinds of energy losses are distinguished: external (interaction with surrounding medium, interface with other physical systems) and internal (caused by processes within the system, e. g. increase of heat energy to the detriment of mechanical energy by means of internal friction, thermoelastic effects, etc.). Dissipation mechanisms are often suggested by experimental results. Dissipation mechanisms are called direct if they give rise to supplementary dissipation terms in the original conservative equations. On the other hand, indirect dissipation mechanisms “involve coupling the mechanical equations governing beam motion to related dissipation systems with additional dynamics, resulting in an overall system in which mechanical energy is dissipated” ([19], p. 379). Russell introduced two types of such coupled dissipative systems: the Euler-Bernoulli (and Timoshenko) beam with thermoelastic damping and with shear difussion damping (see [19]). Further source of damping is showed by the so-called hereditary materials, or material with memory (see e. g. [1]).

Here we shall follow up three types of direct dissipation mechanisms represented by direct insertion of the following terms

$$(a)\ 2\alpha_0\,\frac{\partial u}{\partial t},\quad (b)\ -2\alpha_1\,\frac{\partial^3 u}{\partial t\,\partial x^2},\quad (c)\ 2\alpha_2\,\frac{\partial^5 u}{\partial t\,\partial x^4} \tag{5}$$

into Eq. (2) (α_0, α_1 and α_2 are (small positive) constants, the factor 2 is written only for computational convenience).

The term (a) introduces the so-called external or *viscous* damping. The amplitudes of all normal modes of the vibration (the modal amplitudes) are attenuated at the same rate (contrary to experience). Normal modes are terms in the Fourier series expansion of solution u with respect to the orthogonal set $\{v_k\}_{k=1}^{\infty}$ of eigenfunctions of the the “elasticity” operator L,

$$Lv = \frac{d^4 v}{dx^4},\ v(0) = \frac{d^2 v(0)}{dx^2} = 0,\ v(\ell) = \frac{d^2 v(\ell)}{dx^2} = 0. \tag{6}$$

In [5] Chen and Russell proposed the so-called “square root” ($L^{\frac{1}{2}}$) model for which the so-called *structural* damping is achieved. Basic property (consistent with empirical studies) is that the amplitudes of the normal modes of vibration are attenuated at rates which are proportional to the oscillation frequencies (see also [6], [20]). It is shown in [18] that the positive square root $L^{\frac{1}{2}}$ of the operator (6) is a differential operator

$$L^{\frac{1}{2}} v = -\frac{d^2 v}{dx^2},\quad v(0) = v(\ell) = 0. \tag{7}$$

Hence, the term $2\alpha_1 L^{\frac{1}{2}}\frac{\partial u}{\partial t}$ is equal just to (b) in (5) (and, moreover, with very natural interpretation that the damping force is proportional to the bending rate). For other boundary conditions, in general, there is a difference between the square root $L^{\frac{1}{2}}$ of the fourth-order derivative operator and the negative second derivative, by [18]: $-\frac{d^2 v}{dx^2} = [I + P]\,L^{\frac{1}{2}} v$ where P is a bounded, but in general not compact, operator in $L_2(0, \ell)$.

The presence of an additional term (c) in (2) means that the damping rates of the normal modes of vibration depend quadratically on frequency at low frequencies (consistent with empirical studies for some materials), high frequency modes are overdamped (and difficult to observe at all). This type of damping is the so-called *internal* or *Kelvin-Voigt* damping. A general scheme of Kelvin-Voigt damping approach which can be applied to vibration problems of any linear elastic system: damping forces depend on velocity in the same way as restoring forces depend on displacement.

The Euler-Bernoulli beam model equation under presence of damping terms, axial force and (in general nonlinear) elastic foundation assumes the form

$$\begin{aligned}&\frac{\partial^2 u}{\partial t^2} + \frac{EI}{\varrho A}\,\frac{\partial^4 u}{\partial x^4} - \frac{S}{\varrho A}\,\frac{\partial^2 u}{\partial x^2} + \sigma(u) + \\ &+ 2\left(\alpha_0\,\frac{\partial u}{\partial t} - \alpha_1\,\frac{\partial^3 u}{\partial t\,\partial x^2} + \alpha_2\,\frac{\partial^5 u}{\partial t\,\partial x^4}\right) = 0.\end{aligned} \tag{8}$$

3 Setting of the problem

Let $I = (0, \ell)$, $H = L_2(I)$ and $V = W_2^2(I) \cap \overset{\circ}{W}{}_2^1(I)$. We identify H with its dual H' and H' with a dense subspace of the dual V' of V, thus $V \subset H \subset V'$, both embedding are continuous and dense and it is correct to denote the duality pairing on $V' \times V$ by the same symbol $\langle\cdot,\cdot\rangle$ as the scalar product in H.

The operator (6) may can be viewed either as an unbounded positive definite selfadjoint operator $L\colon D(L) \to H$ where $D(L) = \{v \in W_2^4(I) \mid v(0) = v(\ell) = 0,\ v''(0) = v''(\ell) = 0\}$, either as an isomorphism of V onto V'.

For the square root $L^{\frac{1}{2}}$ of L it holds $D(L^{\frac{1}{2}}) = V$ and $\langle Lw, v\rangle = \langle L^{\frac{1}{2}} w, L^{\frac{1}{2}} v\rangle$ for $w, v \in V$.

In terms of the operator L Eq. (8) can be subsumed in the following abstract differential equation

$$\begin{aligned}&\ddot{u} + \left(a_1\,L^{\frac{1}{2}} + a_2\,L\right) u + \sigma(u) + \\ &+ 2\left(\alpha_0 + \alpha_1\,L^{\frac{1}{2}} + \alpha_2\,L\right)\dot{u} = 0,\end{aligned} \tag{9}$$

where a_1, a_2, α_0, α_1, α_2 are constants. The function $u \mapsto \sigma(u)$ will be assumed (for definiteness and simplicity) in the form $\sigma(u) = \sigma_+\,u^+ - \sigma_-\,u^-$ where $u^{\pm} = \max\{\pm u, 0\}$ and σ_+ and σ_- are constants (in particular, if $\sigma_+ = \sigma_- = \sigma_0$ then $\sigma(u) = \sigma_0 u$).

By an energy solution of Eq. (9) we mean a function $u \in C(\mathbb{R}^+; V) \cap C^1(\mathbb{R}^+; H) \cap C^2(\mathbb{R}^+; V')$ such that for any compact interval $J \subset \mathbb{R}^+$ the equality

$$\int_J \Big[\langle -\dot{u} - 2\big(\alpha_0 + \alpha_1 L^{\frac{1}{2}} + \alpha_2 L\big)\, u,\, \dot{z}(t)\rangle + \\ + \langle \big(a_1 L^{\frac{1}{2}} + a_2 L\big)u + \sigma(u),\, z(t)\rangle\Big]\, dt = 0 \qquad (10)$$

holds for any $z \in \overset{\circ}{W}{}^1_2(J; V)$ (the test function z belongs to this space if and only if $z\colon J \to V$ is an absolutely continuous function, z vanishes at boundary points of J, the (strong) derivative $\dot{z}$ exists almost everywhere on J and $\dot{z} \in L_2(J; V)$). A solution u is (by embedding theorems) a continuous function on $\mathbb{R}^+ \times \bar{I}$. The uniqueness of a solution is achieved by prescribing values u and $\dot{u}$ (in V and H, respectively) for some $t_0 \in \mathbb{R}^+$.

4 Globally oscillatory solutions

For how long time period at most can a non-zero solution remain non-negative (non-positive) on the interval I? To answer this question we use the concepts of oscillatory time and globally oscillatory function (cf. [4]).

A continuous function $u\colon \mathbb{R}^+ \times I \to \mathbb{R}$ is called globally oscillatory (about zero at $+\infty$) if there exists the so-called oscillatory time $\Theta > 0$ such that for any interval $J \subset \mathbb{R}^+$ the length $|J|$ of which is greater than Θ the function u changes the sign on the set $J \times I$, i. e. there exist couples $(t_1, x_1), (t_2, x_2) \in J \times I$ such that $u(t_1, x_1) < 0 < u(t_2, x_2)$, and, consequently, $u(t_0, x_0) = 0$ for some $(t_0, x_0) \in J \times I$. (The concept of a globally oscillatory function is used even for functions which may fail to be continuous, cf. [4], [11].)

Eq. (9) is called uniformly globally oscillatory (about zero at $+\infty$) is there exists (the so-called uniform) oscillatory time Θ such that any non-zero energy solution is globally oscillatory with the same oscillatory time.

For examples of uniformly globally oscillatory equations of mathematical physics see [3], [4], [7]-[9], [11]-[15], [22].

We formulate conditions in order Eq. (9) be uniformly globally oscillatory. The evaluation of the uniform oscillatory time Θ can be done by means of the so-called summit function $(q, p) \mapsto \vartheta^q_p$; we refer to [10] where this function is introduced and studied.

The results are based on the special properties of the first frequency-mode pair, namely, on the existence of λ_1, the smallest eigenvalue of the eigenvalue problem for operator (6) that is *positive* and on the existence of the corresponding eigenfunction (modal function) v_1 that can be chosen *positive in* I. In our case $Lv_1 = \lambda_1 v_1$,

$$\lambda_1 = \left(\frac{\pi}{\ell}\right)^4 \quad \text{and} \quad v_1 = \sin\frac{\pi x}{\ell}. \qquad (11)$$

Theorem. *Let a_1, a_2, α_0, α_1, α_2, σ_+, $\sigma_- \in \mathbb{R}$, $a_2 > 0$. Denote*

$$\alpha = \alpha_0 + \alpha_1\sqrt{\lambda_1} + \alpha_2\lambda_1, \qquad (12)$$

$$E = \min\{\sigma_+,\, \sigma_-\} + a_1\sqrt{\lambda_1} + a_2\lambda_1 \qquad (13)$$

and assume

$$E > 0, \quad |\alpha| < \sqrt{E}. \qquad (14)$$

Then Eq. (9) *is uniformly globally oscillatory and the uniform oscillatory time is given by*

$$\Theta = \vartheta^E_{-|\alpha|} + \vartheta^E_0. \qquad (15)$$

Proof. We prove that any non-zero solution cannot remain of one sign in I for any time period $J \subset \mathbb{R}^+$ greater than Θ. By contradiction, let us assume that u is non-negative on $J \times I$ where $J \subset \mathbb{R}^+$ is any interval with the length $|J|$ greater than (15). We prove that $u \equiv 0$ in $\mathbb{R}^+ \times I$.

By (14) we can choose $\varepsilon > 0$ such that $|\alpha| < E - \varepsilon$ and

$$|J| \geq \vartheta^{E-\varepsilon}_{-|\alpha|} + \vartheta^{E-\varepsilon}_0 > \vartheta^E_{-|\alpha|} + \vartheta^E_0.$$

(the function ϑ^q_p is monotonically decreasing in the variable q if p is fixed, we refer again to [10]). Let us now take any subinterval (τ_1, τ_2) of the interval J of the length $\vartheta^{E-\varepsilon}_{-|\alpha|} + \vartheta^{E-\varepsilon}_0$. We choose a special test function z in (10) in the form $z = \gamma(t)\, v_1$ where v_1 is the eigenfunction of L from (11) and γ is the function possessing the following properties:

- $\gamma \in C^2([\tau_1,\, \tau_2])$,
- $\gamma > 0$ in (τ_1, τ_2), $\gamma(\tau_1) = \gamma(\tau_2) = 0$,
- $\dot{\gamma}(\tau_1) > 0$, $\dot{\gamma}(\tau_2) < 0$,
- $\ddot{\gamma} - 2\,(\alpha^+\dot{\gamma}^+ + \alpha^-\dot{\gamma}^-) + (E - \varepsilon)\,\gamma = 0$ in (τ_1, τ_2),

where $\alpha^\pm = \max\{\pm\alpha,\, 0\}$, $\dot{\gamma}^\pm(t) = \max\{\pm\dot{\gamma}(t),\, 0\}$. An explicit form of such a function can be found in [10]. (In terms introduced in [10] the function γ is the τ_1–shift $t \mapsto c(t - \tau_1)$ of the universal comparison function $c(t) = C(t, q, p, n)$ corresponding to $q = E - \varepsilon$, $p = -\alpha^+$ and $n = -\alpha^-$.)

Inserting $z(t) = \gamma(t)\, v_1$ (and zero outside of $(\tau_1,\, \tau_2)$) into (10) and using $L^{\frac{1}{2}} v_1 = \sqrt{\lambda_1}\, v_1$ we get

$$0 \geq \dot{\gamma}(\tau_1)\,\langle u(\tau_1, \cdot),\, v_1\rangle - \dot{\gamma}(\tau_2)\,\langle u(\tau_2, \cdot),\, v_1\rangle + \\ + \int_{\tau_1}^{\tau_2} \left[\ddot{\gamma} - 2(\alpha^+\dot{\gamma}^+ + \alpha^-\dot{\gamma}^-) + E\,\gamma\right]\langle u,\, v_1\rangle\, dt \geq \\ \geq \varepsilon \int_{\tau_1}^{\tau_2} \gamma\,\langle u,\, v_1\rangle\, dt.$$

Due to the fact $v_1 > 0$ in I and $\gamma > 0$ in $(\tau_1,\, \tau_2)$ we have $u \equiv 0$ in $(\tau_1,\, \tau_2) \times I$, hence $u \equiv 0$ in $J \times I$ because

(τ_1, τ_2) is arbitrary and consequently $u \equiv 0$ in $\mathbb{R}^+ \times I$ by virtue of the unique solvability of the initial boundary value problem for Eq. (9).

The same conclusion $u \equiv 0$ is obtained if we assume $u \leq 0$ on $J \times I$ where $J \subset \mathbb{R}^+$ is any interval the length $|J|$ of which is greater than (15). The proof is complete.

Remark. Many differential operators exhibit similar properties as the operator L, namely the positivity of the first eigenvalue and the positivity of the corresponding eigenfunction. Analogous suitable properties are encountered with a number of abstract operators in ordered Banach spaces (see e. g. [16]). Moreover, it is possible to define the notion of oscillation for abstract-valued functions with values in abstract ordered Banach spaces (nonnegativity of a function is replaced by its appearance in the cone defined by the ordering). This makes it possible to define and investigate oscillatory properties of Eq. (9) in a more general setting as an abstract evolution equation where L is a (positive definite, selfadjoint, with compact resolvent) operator L on an abstract Hilbert space H (see [13]) or even to cope with more general equations involving nonlinear operators (see [11]).

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