Wide Band Noise as a Distributed Delay of White Noise

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Abstract—**The paper presents a mathematical theory of handling and working with wide band noises. We demonstrate that a wide band noise can be represented as a distributed delay of a white noise. Based on this we deduce that the behavior of a wide band noise is same as the behavior of an infinite dimensional colored noise along the boundary line. All these are used to deduce a complete set of formulae for the Kalman type optimal filter for a wide band noise driven linear systems.**

Keywords: ***Wide band noise, white noise, linear stochastic system, Kalman filtering***

1 Introduction

Kalman filtering, originating from the famous works of Kalman and Bucy [1, 2], is one of the powerful methods of estimation and widely used in applications. The overwhelming majority of the results in Kalman filtering theory have been obtained for a pair of independent or correlated white noises corrupting the state and observation systems in finite [3] and infinite [4] dimensional spaces. However, real noises are only approximations to white noises. Fleming and Rishel [5] wrote that the real noises are wide band and white noises are an ideal case of wide band noises. Whenever the parameters of white and wide band noises are sufficiently close to each other, the wide band noise can be replaced by the white noise to make the respective mathematical model simpler. Therefore, in order to get more adequate version of the Kalman filter, a method of handling and working with wide band noises must be developed.

In this paper we are aimed to give a mathematical background for the wide band noises. We show that under general conditions a wide band noise can be modeled as a *distributed delay* of a white noise, demonstrating an important relationship between practical wide band noises and ideal white noises. We show that an equation, corrupted by a wide band noise, is indeed an infinite dimensional equation, corrupted by a white noise. Based on this, we modify the Kalman filter to linear systems corrupted by white and wide band noises.

2 White and wide band noises

A random process $\varphi(t)$, $t \geq 0$, is called a wide band noise if

$$\operatorname{cov}(\varphi(t+s), \varphi(t)) = \lambda(t,s) \neq 0$$

for $0 \leq s \leq \varepsilon$, and

$$\operatorname{cov}(\varphi(t+s), \varphi(t)) = 0$$

for $s \geq \varepsilon$, where ε is a small positive value. In case if φ has zero mean and λ is independent on t, i.e., $\lambda(t,s) = \lambda(s)$, φ is said to be stationary. The function λ is called the autocovariance function of φ. A white noise $w(t)$, $t \geq 0$, is a (generalized) random process with zero mean and

$$\operatorname{cov}(w(t+s), w(t)) = \delta(s),$$

where δ is Dirac delta function. Comparison of these two definitions shows that the white noise w is a limit case of the stationary wide band noise φ if the value of ε is infinitely small and, respectively, $\lambda(0)$ is infinitely large.

In engineering wide band noises are observed by their autocovariance functions. It can be easily calculated that if $\psi(r)$, $-\varepsilon \leq r \leq 0$, is a solution of the equation

$$\lambda(s) = \int_{-\varepsilon}^{-s} \psi(r)\psi(r+s)\,dr,\ 0 \leq s \leq \varepsilon, \tag{1}$$

then the random process φ defined by

$$\varphi(t) = \int_{\max(0,t-\varepsilon)}^{t} \psi(s-t)w(s)\,ds,\ t \geq 0, \tag{2}$$

where w is a white noise, is a wide band noise which is stationary starting the instant ε and its autocovariance function equals to λ. Based on this the existence of solution of the equation (1) becomes very important. In [6] it is proved that under positive definiteness of the function λ and other general conditions the equation (1) has a solution though the number of solutions is infinite. Here, the positive definiteness is a natural condition satisfied by autocovariance functions, and the existence of infinite number of solutions is a consequence of the fact

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that the covariance function provide only a partial information about the wide band noise. The solution ψ of the equation (1) is called a relaxing function of the wide band noise φ, represented in the form of (2).

The relaxing function is more significant parameter than the autocovariance function associated with wide band noises because it allows express wide band noises in terms of white noises. Below we follow to the representation (2) and construct Kalman type optimal filter for linear partially observable systems corrupted by white and wide band noises. The main idea of such construction is that a wide band noise in the form of (2) behaves as a boundary value of an infinite dimensional colored noise.

3 Setting the filtering problem

Consider the linear partially observable system

$$\begin{cases} x'(t) = Ax(t) + \varphi(t) + \Phi w(t),\ t \ge 0,\ x(0) = x_0, \\ z'(t) = Cx(t) + w(t),\ t \ge 0,\ z(0) = 0, \end{cases} \tag{3}$$

where x and z are the vector-valued state and observation processes, respectively, x_0 is a Gaussian random vector with zero mean and $\mathrm{cov} x_0 = P_0$, w is a Gaussian vector-valued white noise with zero mean and $\mathrm{cov}(w(t+s), w(t)) = I\delta(s)$, I is the identity matrix, A, Φ and C are matrices, φ is a vector-valued wide band noise, defined by

$$\varphi(t) = \int_{\max(0,t-\varepsilon)}^{t} \Psi(s-t)w(s)\,ds,\ t \ge 0, \tag{4}$$

with Ψ being continuously differentiable matrix-valued function on $[-\varepsilon, 0]$ satisfying $\Psi(-\varepsilon) = 0$. We assume that the dimensions of vectors and matrices mentioned above are consistent so that the system (3) is well-defined. The aim is to find the Kalman type optimal linear filter for the best estimation $\hat{x}(t)$ of $x(t)$ on the base of observations $z(s)$, $0 \le s \le t$.

It is seen that the system (3) differs from the Kalman's model by the appearance of the wide band noise φ in the signal system.

4 Reduction

Introduce

$$\tilde{\varphi}(t,\theta) = \int_{\max(0,t-\varepsilon-\theta)}^{t} \Psi'(s-t+\theta)w(s)\,ds, \tag{5}$$

where $-\varepsilon \le \theta \le 0$ and $t \ge 0$. If Γ is an integral operator, defined by

$$\Gamma h = \int_{-\varepsilon}^{0} h(\theta)\,d\theta$$

on the space L_2 of all square integrable vector-valued functions on $[-\varepsilon, 0]$, then one can easily evaluate that

$$\Gamma\tilde{\varphi}(t,\cdot) = \int_{-\varepsilon}^{0} \tilde{\varphi}(t,\theta)\,d\theta = \varphi(t).$$

On the other hand,

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \theta}\right)\tilde{\varphi}(t,\theta) = \Psi'(\theta)w(t). \tag{6}$$

This equality can be verified straightforwardly by calculation of the partial derivatives of $\tilde{\varphi}$, but its strong justification needs deeper mathematical considerations because the integral in (5) is a stochastic integral defined as a limit in the mean square sense (not in the pointwise sense). We refer to [7, 8] for mathematics of this issue. The equation (6) demonstrates that $\tilde{\varphi}$ is a colored noise with distributed (infinite dimensional) values.

Thus if we introduce a new infinite dimensional state process by

$$\tilde{x}(t) = \begin{bmatrix} x(t) \\ \tilde{\varphi}(t,\cdot) \end{bmatrix},\ \tilde{x}(0) = \begin{bmatrix} x_0 \\ 0 \end{bmatrix},$$

along with the linear transformations

$$\tilde{A} = \begin{bmatrix} A & \Gamma \\ 0 & -d/d\theta \end{bmatrix},\ \tilde{C} = \begin{bmatrix} C & 0 \end{bmatrix},\ \tilde{\Phi} = \begin{bmatrix} \Phi \\ \Psi'(\cdot) \end{bmatrix},$$

then

$$\begin{cases} \tilde{x}'(t) = \tilde{A}\tilde{x}(t) + \tilde{\Phi}w(t),\ t \ge 0,\ \tilde{x}(0) = \tilde{x}_0, \\ z'(t) = \tilde{C}\tilde{x}(t) + w(t),\ t \ge 0,\ z(0) = 0, \end{cases} \tag{7}$$

where the first component of $\tilde{x}$ is exactly the state process of the system (3). Thus the system (3) is reduced to the Kalman's model (7) with infinite dimensional state process.

5 Kalman type optimal filter

Let

$$\hat{\tilde{x}}(t) = \begin{bmatrix} \hat{x}(t) \\ \phi(t,\cdot) \end{bmatrix},$$

be the best estimate of $\tilde{x}(t)$ on the base of observations $z(s)$, $0 \le s \le t$, for the system (7). Then its first component $\hat{x}(t)$ is the best estimate of $x(t)$ on the base of observations $z(s)$, $0 \le s \le t$, for the system (3). Therefore, writing the Kalman filter for the system (7) in terms of parameters of the system (3), we obtain the following system of equations for $\hat{x}(t)$:

$$\begin{cases} \hat{x}'(t) = A\hat{x}(t) + \phi(t,0) + (P(t)C^* + \Phi)(z'(t) - C\hat{x}(t)), \\ (\frac{\partial}{\partial t} + \frac{\partial}{\partial \theta})\phi(t,\theta) = (Q^*(t,\theta)C^* + \Phi(\theta))(z'(t) - C\hat{x}(t)), \\ \hat{x}(0) = 0,\ \phi(0,\theta) = \phi(t,-\varepsilon) = 0,\ -\varepsilon \le \theta \le 0,\ t > 0, \end{cases}$$

where C^* is the transpose of C. Additionally, the mean square error of estimation equals to

$$e(t) = \mathbf{E}\|x(t) - \hat{x}(t)\|^2 = \mathrm{tr}P(t),$$

where $\mathbf{E}$ is a symbol for expectation, $\|\cdot\|$ is the Euclidean norm and $\mathrm{tr}P$ is the trace of the matrix P. It would be interesting to mention here that ϕ is a function on $G = [0,\infty) \times [-\varepsilon, 0]$ so that it has zero values on the

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boundary lines $t = 0$ and $\theta = -\varepsilon$ of G, at interior points of G it satisfies a partial differential equation that form its values on the boundary line $\theta = 0$ which in turn affect to the differential equation for the best estimate.

Moreover, the functions P and Q are solutions of the differential equations:

$$\begin{cases} P'(t) = P(t)A^* + AP(t) + Q(t,0) + Q^*(t,0) + \Phi\Phi^* \\ \qquad -(P(t)C^* + \Phi)(CP(t) + \Phi^*), \\ P(0) = P_0,\ t > 0, \end{cases}$$

and

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \theta}\right)Q(t,\theta) = AQ(t,\theta) + R(t.0,\theta) + \Phi\Psi^*(\theta) \\ \qquad -(P(t)C^* + \Phi)(CQ(t,\theta) + \Psi^*(\theta)), \\ Q(0,\theta) = Q(t,-\varepsilon) = 0,\ -\varepsilon \le \theta \le 0,\ t > 0, \end{cases}$$

where R is a solution of

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \tau}\right)R(t,\theta,\tau) = \Psi(\theta)\Psi^*(\tau) \\ \qquad -(Q(t,\theta)C^* + \Psi(\theta))(CQ(t,\tau) + \Psi^*(\tau)), \\ R(0,\theta,\tau) = R(t,-\varepsilon,\tau) = R(t,\theta,-\varepsilon) = 0, \\ \qquad -\varepsilon \le \theta \le 0,\ -\varepsilon \le \tau \le 0,\ t > 0. \end{cases}$$

Indeed if $\tilde{Q}$ is an integral operator from L_2 to the state space with the kernel function Q and $\tilde{R}$ is an integral operator from L_2 to L_2 with the kernel function R, then P, $\tilde{Q}$ and $\tilde{R}$ are the components of the solution $\tilde{P}$ of the operator Riccati equation related to the system (7) in the form

$$\tilde{P}(t) = \begin{bmatrix} P(t) & \tilde{Q}(t) \\ \tilde{Q}^*(t) & \tilde{R}(t) \end{bmatrix},$$

where Q^* stands for the adjoint of Q.

6 Concluding remarks

Kalman filters are used in Global Positioning Systems (GPS), systems of satellites positioning the objects on the Earth. Presently, there are three such satellite systems: the United States already completed and fully operating system NAVSTAR, the former Soviet Union system GLONASS, functioning partially, and the European Union highly ambitious system GALILEO in the stage of project and aiming precise positioning. Kalman type filters for linear systems with delayed white noises can significantly contribute to reach preciseness of positioning since the radio signals spend a visible time to run from ground radars to satellites and vice versa. Therefore the noises, corrupting radio signals in open space, affect to signal and observation systems with some delay.

A delay may be distributed (leading to the concept of wide band noise) or pointwise, time dependent or independent, random or not. A delayed noise can affect to signal as well as to observation system. Among different filtering problems with delay only one problem has been discussed in this paper. For the other problems we refer to [8, 9, 10].

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