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Points of Intersection of Symmetrical Solutions of Reaction-Diffusion Processes

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***Abstract*—In the symmetric case of the Gelfand problem involves the study of the points of intersection of the sequence of classical stationary solutions for an existing special value of the parameter λ. The bell-shaped solutions are being used in order to establish a first relation between the points of inflection of the difference of such two solutions and their points of intersection. The final result relates not only to the existence but also to the uniqueness of the point of intersection of any two different classical stationary solutions.**

***Index Terms*—Reaction-diffusion, symmetric problem, stationary solution, bell-shaped solution.**

I. Introduction

We consider the symmetric case of the steady-state of the Gelfand problem [7] for reaction-diffusion processes:

$$\begin{aligned} &\ddot{A}_r w + \lambda_\infty f(w) = 0, \quad r \in \Omega \\ &B_w = 0, \quad r \in \partial\Omega \end{aligned} \tag{1}$$

where Ω is a ball of R^N where $3 \le N \le 9$, with a smooth boundary $\partial\Omega$ and B is the operator which expresses the boundary conditions of the problem.

Let f satisfy the following properties:

$$\begin{aligned} &f(u) > 0, f(u) > 0, f''(u) > 0, \quad \text{for every } u \\ &\int_b^\infty ds/f(s) < \infty \end{aligned} \tag{2}$$

Then according to [1]–[4], there exists a value λ_∞ of the positive parameter λ for which the problem (1) has an infinite number of classical solutions $\{w_k(x)\}$.

It is already known from [5] that for the symmetric case of the Gelfand problem there exists a value $\ddot{e}^* = 2(N-2)$ of the positive parameter λ for which there exist infinite classical solutions with the last one being concave in the interval $[0,1]$ with $u''(1) = 0$ while the rest being bell-shaped.

We shall examine the points of intersection between the bell-shaped solutions of this problem.

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II. Existence of points of intersection

Let us consider two bell-shaped solutions $\overline{w}, \underline{w}$ of the sequence $\{w_k(x)\}$, such that the first has a greater supremum than the second. We shall prove the following proposition.

Proposition 1: The functions $\overline{w}, \underline{w}$ have at least one point of intersection.

Proof: We define the function $u = \overline{w} - \underline{w}$. Then, it follows that:

$$\Delta_r u = \Delta_r \overline{w} - \Delta_r \underline{w} = \lambda_\infty \left(e^{\overline{w}} - e^{\underline{w}}\right), \text{ or}$$
$$\ddot{A}_r u + \lambda_\infty e^{\underline{w}} \left(e^u - 1\right) = 0$$

This means that there exist some $s \in \langle \underline{w}, \overline{w} \rangle$, such that

$\Delta_r u + \lambda_\infty e^s u = 0$.

If there is no point of intersection of $\overline{w}, \underline{w}$ then $\overline{w} < \underline{w}$ and: $\Delta_r u + \lambda_\infty e^{\overline{w}} u > 0$ and $\Delta_r u + \lambda_\infty e^{\underline{w}} u < 0$ since $\underline{w} < s < \overline{w}$. Then, we conclude that $\Delta_r u + \lambda_\infty e^{\overline{w}} u = ku$ and $\Delta_r u + \lambda_\infty e^{\underline{w}} u = mu$, with $k > 0, m < 0$. Hence, we have a contradiction, because of the following Lemma [6], which states that for the linearized problem:

$$\begin{aligned} &\Delta\phi + \lambda f'(w)\phi = \mu\phi, \quad D \\ &B\phi = 0, \quad \partial D \end{aligned}$$

with w the stationary solution, we have that

$$\begin{aligned} &\mu_1 < 0, \quad \text{if } \lambda \in [0, \lambda^*) \text{ and } w = w_{\min} \\ &\mu_1 = 0, \quad \text{if } \lambda = \lambda^* \\ &\mu_1 > 0, \quad \text{if} \lambda \in (0, \lambda^*) \text{ and } w \neq w_{\min} \end{aligned}$$

The following Proposition clarifies the relation of the intersection points of the functions $\overline{w}, \underline{w}$ with the points of inflection of u.

Proposition 2: If the functions $\overline{w}, \underline{w}$ have at least two points of intersection, then their difference u has at least two points of inflection.

Proof: Since $u(r_1) = u(r_2) = 0$, using Rolle's proposition, we obtain the existence of z_1 such that $u'(z_1) = 0$, $r_1 < z_1 < r_2$. Using again Rolle's proposition for the derivative of u, since $u'(0) = \overline{w}' - \underline{w}' = 0$ and $u'(z_1) = 0$, we obtain that there exists y_1 such that $u''(y_1) = 0$, $0 < y_1 < z_1$.

Similarly, since $u(r_2) = u(1) = 0$, there exists z_2 such that $u'(z_2) = 0$, $r_2 < z_2 < 1$ and a y_2 such that $u''(y_2) = 0$, $z_1 < y_2 < z_2$.

In the next proposition we prove the existence of points of inflection of the function u.

Proposition 3: The difference u of the functions $\overline{w}, \underline{w}$ has at least one point of inflection.

Proof: According to Proposition 1 we know that there is at least one point of intersection r_0 between the functions $\overline{w}, \underline{w}$. It then follows that $u(r_0) = u(1) = 0$, therefore there exists a z such that $u'(z) = 0$, $r_0 < z < 1$.

Then $u'(0) = u'(z) = 0$, thus there exists a y such that $u''(y) = 0$, $y \in (0, z)$.

III. Uniqueness of points of intersection

The uniqueness of the point of intersection of two arbitrary classical bell-shaped solutions is closely dependent on the number of the points of inflection of their difference.

Lemma: The difference u of the functions $\overline{w}, \underline{w}$ satisfies the following inequalities :

a) $u''(0) = \overline{w}''(0) - \underline{w}''(0) > 0$

b) $u''(1) = \overline{w}''(1) - \underline{w}''(1) < 0$

Proof:

a) Using L' Hospital's rule we obtain that

$$\lim_{r \to 0} u'(r)/r = u''(0).$$

Hence, we have:

$$u''(0) + (N-1)u''(0) + \lambda_\infty e^s u(0) = 0$$

which implies that

$$u''(0) = -\ddot{e}_\infty e^s u(0)/m < 0.$$

b) Again, we get from the equation of the problem that

$$u''(1) + (N-1)u'(1) + \lambda_\infty e^w u(1) = 0$$

which implies that

$$u''(1) = -(N-1)u'(1) < 0$$

as required.

Proposition 4: The functions $\overline{w}, \underline{w}$ have exactly one point of intersection.

Proof: We already know that the points of intersection of the functions $\overline{w}, \underline{w}$ lie on the interval of their points of inflection. From the previous Lemma it follows that u can only have zero or two points of inflection. From Proposition 3 we know that u has at least one.

Therefore u has exactly two points of inflection, i.e., the functions $\overline{w}, \underline{w}$ have exactly one point of intersection.

IV. Discussion

In this paper the entire behavior of the infinite bell-shaped classical solutions of the symmetric case of the Gelfand problem is studied. In particular, it is proved that any two solutions have a unique point of intersection. It still remains an open question whether all these points of intersection are different or if some of them coincide. We expect however that the k-th term of the infinite sequence of solutions has exactly k different points of intersection with the previous terms.

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