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Lie Symmetry Methods in Finance - An Example of the Bond Pricing Equation

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Abstract—**Using Lie's classical method of group invariants, new solutions are found for the valuation of zero-coupon bonds. In finding the solutions, the model for the underlying short-term interest rate assumes a realistic time-dependent, mean-reverting drift form and a power-law volatility.**

Keywords: Lie symmetries, bond pricing

1 Introduction

Lie's classical method of group invariants, initiated by Sophus Lie [1] over 100 years ago, can provide new and simple solutions to differential equations arising from any application. In this paper we demonstrate the method on the one-factor interest-rate bond-pricing problem consisting of the partial differential equation and the final condition. We assume that the short-term interest rate follows a random walk which allows the volatility of interest rate changes to be highly sensitive to the level of riskless rate, and the market price of risk is arbitrary. The Lie symmetry classification method systematically uncovers those one-factor models that lead to variable reductions and exact solutions of the bond-pricing equation. In the simple solution presented, the nonlinear drift of the risk-neutral interest rate contains an arbitrary function of time, which may be freely chosen and the volatility has the desirable form $cr^{\frac{3}{2}}$. Using Generalised Method of Moments (GMM) the model was shown to outperform many of the current popular interest rate models.

2 Lie's Classical Method

A symmetry of a differential equation is a transformation mapping an arbitrary solution to another solution of the differential equation. The classical Lie groups of point invariance transformations depend on continuous parameters and act on the system's graph space that is co-ordinatised by the independent and dependent variables. As these symmetries can be determined by an explicit computational algorithm (known as Lie's algorithm or Lie's classical method), many automated computer algebra packages (see e.g Sherring [2], Kersten [3], Schwartz [4]) have been developed to find them. Thus they are the most extensively used of all symmetries. Recent accounts of Lie's algorithm, and its extensions, may be found in many excellent texts such as those of Ibragimov [5] and Bluman and Kumei [6].

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If a partial differential equation (PDE) is invariant under a point symmetry, one can often find **similarity solutions** or **invariant solutions** which are invariant under some subgroup of the full group admitted by the PDE. These solutions result from solving a reduced equation in fewer variables.

In essence, the **classical method** for finding symmetry reductions of a second order PDE in one dependent variable V and 2 independent variables (r, t),

$$\Delta(r, t, V_r, V_t, V_{rr}, V_{rt}, V_{tt}) = 0, \tag{1}$$

is to find a one-parameter Lie group of transformations, in infinitesimal form

$$\begin{aligned} r^* &= r + \epsilon\rho(r,t,V) + O(\epsilon^2) \\ t^* &= t + \epsilon\tau(r,t,V) + O(\epsilon^2) \\ V^* &= V + \epsilon\nu(r,t,V) + O(\epsilon^2), \end{aligned} \tag{2}$$

which leaves (1) invariant. The coefficients ρ, τ and ν of the infinitesimal symmetry are often referred to as the 'infinitesimals'. This invariance requirement is determined by

$$\mathbf{\Gamma}^{(2)}\Delta|_{\Delta=0} = 0, \tag{3}$$

where

$$\mathbf{\Gamma} = \rho(r,t,V)\frac{\partial}{\partial r} + \tau(r,t,V)\frac{\partial}{\partial t} + \nu(r,t,V)\frac{\partial}{\partial V} \tag{4}$$

are vector fields which span the associated Lie algebra, and are called the **infinitesimal generators** of the transformation (2), and $\mathbf{\Gamma}^{(2)}$ is the second extension (or second prolongation) of $\mathbf{\Gamma}$, extended to the second jet space, co-ordinatised by $(r, t, V, V_r, V_t, V_{rr}, V_{rt}, V_{tt})$.

Equation (3) is a polynomial equation in a set of independent functions of the derivatives of V. As the equation must be true for arbitrary values of these independent functions, their coefficients must vanish, leading to an overdetermined linear system of equations, known as the determining equations for the coefficients $\rho(r,t,V), \tau(r,t,V)$ and $\nu(r,t,V)$. Then for known functions ρ, τ, ν, invariant solutions V corresponding to (2) satisfy the invariant surface condition (ISC)

$$\Omega = \rho(r,t,V)\frac{\partial V}{\partial r} + \tau(r,t,V)\frac{\partial V}{\partial t} - \nu(r,t,V) = 0, \quad (5)$$

which when solved as a first order PDE, by the method of characteristics, yields the functional form of the similarity solution in terms of an arbitrary function, i.e.

$$V = f\left(r, t, \phi(z)\right), \quad z = z(r,t) \quad (6)$$

where ϕ is an arbitrary function of an invariant z for the symmetry. Substituting this functional form into (1) produces a quotient ordinary differential equation, which one solves for the function $\phi(z)$.

3 Bond Pricing Problem

The value of interest rate derivatives, such as bonds and swaps, naturally depends on the interest rates. It can be shown (see e.g. Wilmott [7]) that when the short-term interest rate (or *spot rate* as it is often called), follows a stochastic differential equation of the form

$$dr = u(r,t)dt + w(r,t)dX, \quad (7)$$

where dX is an increment in a Wiener process, the price of a zero-coupon bond $V(r,t;T)$, with expiry at $t = T$, will satisfy the partial differential equation (PDE)

$$\frac{\partial V}{\partial t} + \frac{w^2}{2}\frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r} - rV = 0 \quad (8)$$

subject to $V(r,T) = 1$. In equation (8), $\lambda(r,t)$ is defined as the market price of risk.

Obviously the choice of coefficients in the interest-rate model (7) is important for subsequent modelling of bond prices. Over the past 30 years, many different models describing the dynamics of the short rate have been proposed. Many of these can be embedded in the form

$$dr = (\alpha + \beta r)dt + \sigma r^\gamma dX, \quad (9)$$

where α, β, γ, and σ are constants. Such models include those of Vasicek [8] ($\gamma = 0$), Cox, Ingersoll and Ross [9] ($\gamma = \frac{1}{2}$), Brennan and Schwartz [10] ($\gamma = 1$) and Dothan [11] ($\alpha = \beta = 0, \gamma = 1$). Chan *et al* [12] performed a comprehensive empirical analysis on such one factor interest rate models using the Generalised Method of Moments (GMM). They found that the most successful models in capturing the dynamics of the short rate were those that allowed the volatility of interest rate changes to be highly sensitive to the level of the interest rate, in particular, those with $\gamma \geq 1$. Their unconstrained estimate of γ was 1.5. Campbell, Lo and MacKinlay [13] agree with this result and show that the heteroscedasticity of the short rate is markedly reduced as γ increases from 1 to 1.5. Chan *et al* [12] also found only weak evidence of a long-run level of mean reversion, suggesting that the short-rate may revert to a short-rate mean which may be time-dependent.

However the affine form for the drift in (9) has proven controversial. In [14], Aït-Sahalia tests parametric models by comparing their implied parametric density to the same density estimated nonparametrically. His studies show that the linearity of the drift as in (9) appears to be the main source of misspecification. More recently, Ahn and Gao [15] presented a one-factor model which includes nonlinearity in the drift term, as well as a realistic $r^{\frac{3}{2}}$ dependence in the volatility.

Many of the current popular one-factor models, including those of Vasicek [8] and Cox, Ingersoll and Ross [9] are appealing because of their tractability i.e. they lead to analytic solutions of the bond pricing equation (8). These solutions have proved to be useful in providing models for the term structure of interest rates which can be developed by means of the relation

$$y = \frac{-\log\ V}{T - t}; \quad (10)$$

for r fixed. However, as Chan *et al* [12] found, many of these well-known interest rate models perform poorly in their ability to capture the actual behaviour of the short rate because of their implicit restrictions on term structure volatility.

Interest rate models such as those of Ho and Lee [16], and Hull and White [17], incorporate time-dependent parameters. This has the added advantage that it allows a yield curve to be fitted. Thus a ‘no-arbitrage’ yield curve model can be developed so that it is perfectly consistent with the current market data. However, often in the cases for which the model has a time-dependent parameter, the corresponding solutions of (8) usually still involve numerical evaluation of solutions of coupled ordinary differential equations or of an integral.

4 Classical Lie Symmetry Classification of Equation (8)

In this paper, we assume that the short-term risk-neutral interest rate r follows the stochastic process

$$\begin{aligned} dr &= [u(r,t) - \lambda(r,t)\ w(r,t)]\ dt + w(r,t)\ dX \\ &= b(r,t)\ w(r,t)^2\ dt + w(r,t)\ dX \quad (11) \end{aligned}$$

with

$$w = cr^n, \quad b(r,t) = a(t)r^p - qr^m \quad (12)$$

and where c, q, n, m and p are constants with n, m and c non-zero, and $\lambda(r,t)$ is the market price of risk. Hence the dynamics of the short-term interest rate with real drift

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follows the process

$$dr = [b(r,t)\ w^2 + \lambda(r,t)\ w]dt + w\ dX. \qquad (13)$$

Thus we may write PDE (8) as

$$\frac{\partial^2 V}{\partial r^2} + \frac{2}{w^2}\frac{\partial V}{\partial t} + 2b(r,t)\frac{\partial V}{\partial r} - \frac{2r}{w^2}V = 0. \qquad (14)$$

With the aid of the automated computer-algebra symmetry package DIMSYM by Sherring [2], we find that for arbitrary $a(t), n, p, c, q$ and m, PDE (14) has a one-dimensional symmetry generated by the vector-field

$$V\frac{\partial}{\partial V} \qquad (15)$$

and the infinite-dimensional symmetry generated by

$$g(r,t)\frac{\partial}{\partial V}, \qquad (16)$$

where g is any solution to (14). Both of these superposition and scaling symmetries are typical of linear PDEs.

However, as indicated by DIMSYM, for certain special choices of $a(t), n, p$ and/or m, PDE (14) has, in addition 23 extra symmetries (see Goard [18] for the full list).

5 Bond Price Solutions

In this section we use one of the symmetries found to the bond pricing equation to construct a corresponding invariant solution. The search for this solution reduces PDE (14) to an ordinary differential equation (ODE). In particular as our solution needs to necessarily satisfy the final condition for a bond price i.e. $V(r,T) = 1$, we use a special linear combination of generators such that

$$\begin{aligned} \mathbf{\Gamma}(V-1) &= 0|_{V=1,t=T} \\ \text{and} \quad \mathbf{\Gamma}(t-T) &= 0|_{t=T}. \end{aligned} \qquad (17)$$

This condition (17) is in fact over-restrictive, and weaker conditions for the symmetry generator have been found by Goard [19, 20]. In the example below the drift of the interest rate process includes a free function of t, while the chosen $r^{\frac{3}{2}}$ - dependence of volatility conforms to actual data. As well, the drift term is consistent with the empirical findings of Aït-Sahalia [14].

Example - We consider the process followed by the short-rate r is

$$dr = [c^2 r\ (a(t) - qr) + \lambda(r,t) c r^{\frac{3}{2}}]dt + c r^{\frac{3}{2}} dX \qquad (18)$$

$$(\text{so} \quad w = c r^{\frac{3}{2}}, \quad b = \frac{a(t)}{r^2} - \frac{q}{r}) \qquad (19)$$

and thus the risk-neutral process is

$$dr = [c^2 r\ (a(t) - qr)]dt + c r^{\frac{3}{2}} dX. \qquad (20)$$

PDE (14) with w and b as in (19) has the added symmetry with generator

$$\mathbf{\Gamma} = (g(t) + \frac{\tau a'}{r} + \frac{\tau''}{rc^2} + \frac{\tau' a}{r})V\frac{\partial}{\partial V} + \tau\frac{\partial}{\partial t} - \tau' r\frac{\partial}{\partial r} \qquad (21)$$

where

$$g(t) = D_1 \int^t R\ dt + D_2, \qquad (22)$$

$$\tau(t) = \frac{-D_1}{2(1+q)}\frac{(\int^t Rdt)^2}{R} + \frac{(K - D_2)}{1+q}\frac{\int^t Rdt}{R} + \frac{\gamma}{R} \qquad (23)$$

and where

$$R = e^{c^2 \int^t a(t)dt}, \qquad (24)$$

and D_1, D_2, K and γ are arbitrary constants. After consideration of (17), we consider the case where

$$\begin{aligned} g(t) &= \alpha, \quad \text{constant} \\ \tau(t) &= \frac{\gamma}{R} + \beta\frac{\int^t Rdt}{R}, \\ \text{and where} \\ \beta &= \frac{-\gamma}{\int^T Rdt}. \end{aligned} \qquad (25)$$

Hence the generator we use is

$$\mathbf{\Gamma} = \alpha V\frac{\partial}{\partial V} + \tau(t)\frac{\partial}{\partial t} - r\tau'\frac{\partial}{\partial r} \qquad (26)$$

with τ given in $(25)_2$. The corresponding invariant solution is

$$V = \phi(z)e^{\alpha \int^t \frac{1}{\tau}dt}, \qquad z = r\tau, \qquad (27)$$

where ϕ needs to satisfy

$$c^2 z^3 \phi'' + 2\ (\beta z - c^2 q z^2)\phi' + 2\ (\alpha - z)\phi = 0. \qquad (28)$$

With $\alpha = 0$ (otherwise $V = 0$ at $t = T$), the solution to (28) is

$$\begin{aligned} \phi(z) &= z^{-k} e^{\frac{2\beta}{c^2 z}} \left[C_1 M\left(k + 2q + 2, 2k + 2q + 2, \frac{-2\beta}{c^2 z}\right) \right. \\ &\quad \left. + C_2 U\left(k + 2q + 2, 2k + 2q + 2, \frac{-2\beta}{c^2 z}\right)\right], \end{aligned} \qquad (29)$$

where k satisfies

$$k^2 + (1+2q)k - \frac{2}{c^2} = 0, \qquad (30)$$

C_1 and C_2 are arbitrary constants and where $M(a,b,x)$ and $U(a,b,x)$ are the Kummer-M and Kummer-U functions respectively. Hence a solution to (14) is

$$\begin{aligned} V(r,t) &= \frac{e^{\frac{2\beta}{c^2 r\tau}}}{(r\tau)^k}\left(C_1 M\left(k + 2q + 2, 2k + 2q + 2, \frac{-2\beta}{c^2 r\tau}\right)\right. \\ &\quad \left. + C_2 U\left(k + 2q + 2, 2k + 2q + 2, \frac{-2\beta}{c^2 r\tau}\right)\right), \end{aligned} \qquad (31)$$

where k satisfies equation (30) and τ is given in (25). Using the result (see e.g. [21]) that as $|x| \to \infty$,

$$M(a,b,x) = \frac{\Gamma(b)}{\Gamma(a)} e^x x^{a-b}[1 + O(|x|^{-1})],$$

we set in (31)

$$C_1 = \left(\frac{-2\beta}{c^2}\right)^k \frac{\Gamma(k+2q+2)}{\Gamma(2k+2q+2)} \quad \text{and} \quad C_2 = 0, \qquad (32)$$

so that the final condition $V(r,T) = 1$ is satisfied.

6 Performance of Solution

Both Chan *et al* [12] and Ahn and Gao [15] used the Generalised Method of Moments technique (GMM) of Hansen [22] to test models nested within a more general model. In this paper some results of a GMM investigation are included to show the comparison of the performance of the CKLS (Chan *et al*), AG (Ahn and Gao) and LS (Lie symmetry) models, which are nested in the unrestricted model:

$$dr = \{\alpha_1 + \beta(t)r + \alpha_2 r^2\}dt + \alpha_3 r^{3/2} dX$$

where

$$\begin{aligned} \beta(t) &= \beta_1 + \beta_2 \sin(h\pi t) + \beta_3 \cos(h\pi t) \\ &\quad + \beta_4 \sin(2h\pi t) + \beta_5 \cos(2h\pi t). \end{aligned}$$

The three nested models are constructed by placing certain restrictions on the parameters. For CKLS $\alpha_2 = 0, \beta_2 \ldots \beta_5 = 0$; for AG $\alpha_1 = 0, \beta_2 \ldots \beta_5 = 0$; for LS $\alpha_1 = 0$. For each nested model a hypothesis test to test if the nested model was not imposing overidentifying restrictions was conducted using an appropriate test statistic developed by Newey and West [23]. This statistic is asymptotically distributed χ^2 with degrees of freedom equal to the number of restrictions imposed on the unrestricted model to obtain the nested model. A number of different data sets were used mostly with similar results. As an example, in Table 1 below is the result for US 1-month treasury bill yields (see Figure 1) for 12/1946 to 02/1991.

The χ^2 values for the AG and CKLS models imply that they are both rejected at the 5% level of significance so that the AG and CKLS models are misspecified at the 5% level of significance in terms of their overidentifying restrictions. On the other hand, the LS model is not even rejected at the 20% level of significance, with χ^2 values of 0.3744. A point worth noting is that few of the trigonometric coefficients of $\beta(t)$ are individually, statistically significantly different from zero, however jointly, they are statistically significantly different from zero. This fact provides evidence that some variation in the short term riskless rate must be explained by an explicit function of time. Further evidence can be seen in the simulations of the US interest rate using CKLS, AG and LS models in Figure 2 below. Comparisons with Figure 1 show how the LS model most closely resembles the US series.

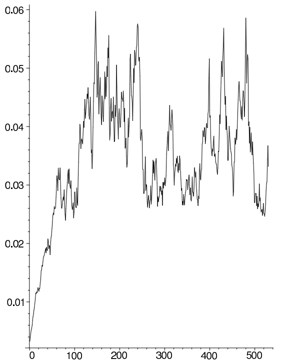
Table 1: Parameter estimates for the US 1 month T-Bill yields 12/46 - 02/91 for h=1/20

	Unrest.	LS	AG	CKLS
α_1	-0.0042	0	0	0.0084
	(0.541)			(0.069)
β_1	0.9121	0.6490	0.3271	-0.1606
	(0.049)	(0.000)	(0.012)	(0.983)
β_2	-0.3474	-0.3355	0	0
	(0.008)	(0.010)		
β_3	0.1786	0.1485	0	0
	(0.058)	(0.065)		
β_4	-0.1830	-0.1684	0	0
	(0.105)	(0.128)		
β_5	-0.0370	-0.0277	0	0
	(0.736)	(0.800)		
α_2	-13.5216	-10.7779	-4.9845	0
	(0.014)	(0.001)	(0.018)	
α_3	1.3051	1.2787	1.2196	1.2164
	(0.000)	(0.000)	(0.000)	(0.122)
$\chi^2(j)$		0.3744	13.9925	13.1000
		(0.541)	(0.016)	(0.022)
		DF=1	DF=5	DF=5

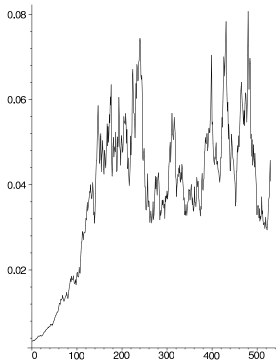
Figure 1: US 1-month T-Bill yields 12/1946-02/1991

7 Conclusion

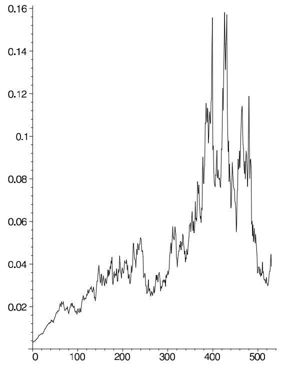
A Lie symmetry classification can significantly expand the class of analytically solvable models in mathematical finance. The additional solvable models, as shown here,



CKLS Model



AG model



LS model

Figure 2: Simulations of the US interest rate using the CKLS, AG and LS models.

may allow a better match to the underlying source of variation.

Although Lie symmetry reductions can be perceived to fail to capture the boundary conditions of real-world problems due to condition (17), it has been shown, [19], that (17) is in fact over-restrictive and the final condition need not be left invariant by the symmetry generator. Further, as shown in [20], the final condition can be used as a side condition to determine an even wider set of symmetries for the problem, thus possibly leading to more solutions.

References

[1] Lie, S., "Uber die Integration durch bestimmte Integrals von einer Klasse Linearer partieller Differentialgleichungen", *Archiv der Mathematik*, V6, pp. 328-368, /81

[2] Sherring, J., *DIMSYM Users Manual*, (Melbourne: La Trobe University), 1993.

[3] Kersten, P., Infinitesimal Symmetries: A Computational Approach, *Centrum voor Wiskunde en Informatica*, Amsterdam, 1987.

[4] Schwartz, F., "Symmetries of Differential Equations: from Sophus Lie to Computer Algebra", *Siam Review*, V30, pp. 450-481, /88

[5] Ibragimov, N.N., *CRC Handbook of Lie Group Analysis of Differential Equations*, CRC Press, 1994.

[6] Bluman, G.W. and Kumei, S., *Symmetries and Differential Equations*, Springer-Verlag, New York, 1989.

[7] Wilmott, P., *Derivatives : The Theory and Practice of Financial Engineering*, John Wiley and Sons Ltd, West Sussex, England, 1998.

[8] Vasicek, O., "An Equilibrium Characterization of the Term Structure", *Journal of Financial Economics*, V5, pp. 177-188, /77

[9] Cox, J.C., Ingersoll, J.E. and Ross, S.A., "The Theory of the Term Structure of Interest Rates", *Econometrica*, V53, pp. 385-407, /85

[10] Brennan, M.J. and Schwartz, E.S., "Analyzing Convertible Bonds", *Journal of Financial and Quantitative Analysis*, V15, pp. 907-929, /80

[11] Dothan, U.L., "On the Term Structure of Interest Rates", *Journal of Financial Economics*, V6, pp. 59-69, /78

[12] Chan, K.C., Karolyi, G.A., Longstaff, F. and Sanders, A.B., "An Empirical Comparison of Alternative Models of the Short-Term Interest Rate", *Journal of Finance*, V47, N3, pp. 1209-1227, /92

[13] Campbell, J.Y., Lo, A.W. and MacKinlay, A.C., *The Econometrics of Financial Markets*, Princeton University Press, Princeton, N.J., 1996.

[14] Aït-Sahalia, Y., "Testing Continuous-Time Models of the Spot Interest Rate", *Review of Financial Studies*, V9, pp. 385-426, /96

[15] Ahn, D. and Gao, B., "A Parametric Nonlinear Model of Term Structure Dynamics", *Review of Financial Studies*, V12, N4, pp. 721-762, /99

[16] Ho, T.S.Y. and Lee, S., "Term Structure Movements and Pricing Interest Rate Contingent Claims", *Journal of Finance*, V42, pp. 1129-1142, /86

[17] Hull, J.C. and White, A., "Pricing Interest Rate Derivative Securities", *Review of Financial Studies* V3, pp. 73-592, /90

[18] Goard, J., "New Solutions of the Bond-Pricing Equation via Lie's Classical Method", *Mathematical and Computer Modelling*, V32, pp. 299-313, /00

[19] Goard, J., "Non-Invariant Boundary Conditions and Lie's Classical Method", *Applicable Analysis*, V83, N5, pp. 473-481, /03

[20] Goard, J. "Finding Symmetries by Incorporating Initial Conditions as Side Conditions", preprint, University of Wollongong, School of Mathematics and Applied Statistics, 2008.

[21] Abramowitz, M. and Stegun, I.A. *Handbook of Mathematical Functions*, Dover Publications Inc., New York, 1965.

[22] Hansen, L.P. "Large Sample Properties of Generalized method of Moments estimators", *Econometrica*, V50, pp. 1029-1054, /82

[23] Newey, W. and West, K. "A Simple Positive Semi-definite, Heteroskedasticity and Autocorrelation Consistent Covariance Matrix", *Econometrica*, V55, pp. 703-708, /87