On Numerical Accuracy of Gauss-Chebyshev Integration Rules Using the Stochastic Arithmetic

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Abstract—**In this paper, the evaluation of $I = \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}dx$ is proposed by using the opened and closed Gauss - Chebyshev integration rules in the stochastic arithmetic. For this purpose, a theorem is proved to show the accuracy of the Gauss-Chebyshev rules. Then, the CESTAC [1] method and the stochastic arithmetic are used to validate the results and implement the numerical example.**

Keywords: Stochastic Arithmetic, CESTAC method, Gauss-Chebyshev integration rules.

1 Introduction

The basic idea of the CESTAC method [5,11] is to replace the usual floating-point arithmetic with a random arithmetic. Consequently, each result appears as a random variable. It has been explained in [1,2,4,7,10], one can use the CESTAC method which is a method based on the stochastic arithmetic, in order to evaluate a definite or an improper integral by using Newton-Cotes integration methods. In this paper, we are going to evaluate $I = \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}dx$ numerically using Gauss-Chebyshev integration rules [9] in the stochastic arithmetic.

In section 2, the idea of the stochastic arithmetic and the CESTAC method are introduced. In section 3, the numerical accuracy of the Gauss-Chebyshev integration rules is given. In section 4, a numerical example is given which is computed by using the stochastic arithmetic and the CESTAC method.

We show that it is possible during the run of the code of the Gauss-Chebyshev integration rules, to determine the optimal number of the points, to correctly stop the process, and to estimate the accuracy of the computed result.

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[1]Control et Estimation STochastique des Arrondis de Calculs

2 CESTAC Method-Stochastic Arithmetic

Let F be the set of all the values representable in the computer. Thus, any value $r \in \mathbf{R}$ is represented in the form of $R \in F$ in the computer. It has been mentioned in [11] that in a binary floating-point arithmetic with p mantissa bits, the rounding error stems from assignment operator is

$$R = r - \epsilon 2^{E-p}\alpha. \tag{1}$$

In this relation ϵ is the sign of r and $2^{-p}\alpha$ is the lost part of the mantissa due to round-off error and E is the binary exponent of the result. In single precision case, $p = 24$ and in double precision case, $p = 53$. Also if the floating-point arithmetic is as rounding to $+\infty$ or $-\infty$ then $-1 \leq \alpha \leq 1$.

According to (1) , if we want to perturb the last mantissa bit of the value r, it is sufficient that we change α in the interval $[-1, 1]$. In the CESTAC method if the arithmetic is considered as rounding to $+\infty$ or $-\infty$, α can be considered as a random variable uniformly distributed on $[-1, 1]$. Thus R, the calculated result, is a random variable and its precision depend s on its mean (μ) and its standard deviation (σ).

The idea of the CESTAC method is to consider that every result $R \in F$ of a floating-point operation corresponds to two informatical results, one rounded off from below (R^-) , the second rounded off from above (R^+), each of them representing the exact arithmetical result $r \in \mathbf{R}$, with equal validity. If a computer program is performed N times, the distribution of the results $R_i, i = 1, ..., N$ is quasi-Gaussian which their mean is equal to the exact value r, that is $E(R) = r$ [8,11]. These N samples are used for estimating the values μ and σ.

In practice, the samples R_i are obtained by perturbation of the last mantissa bit (or previous bits if necessary) of every result R, then the mean of random samples R_i, that is $\overline{R} = \frac{\sum_{i=1}^{N} R_i}{N}$, is considered as the result of an arithmetical operation. If $N = 3$, it has been proved in [11] that the number of exact significant digits common to $\overline{R}$ and to the exact value r can be estimated by,

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$$C_{\overline{R},r} = \log_{10} \frac{|\,\overline{R}\,|}{\sigma} - 0.39. \tag{2}$$

In relation (2), σ is the standard deviation of the samples R_i which is given by,

$$\sigma = \sqrt{\frac{\sum_{i=1}^{N}(R_i-\overline{R})^2}{N-1}}.$$

In the CESTAC method if $C_{\overline{R},r} \leq 0$ or $\overline{R} = 0$ then R is called an informatical or stochastic zero. In this case, we write $R = @0$ and it means the informatical result R is insignificant.

In order to simultaneous implementation of the CESTAC method we should substitute a stochastic arithmetic in place of the floating-point arithmetic. In this way every arithmetical operation is performed N times synchronously before running the next operation. If $N = 3$, the relation (2) can be used to estimate the number of exact significant digits of any result of any arithmetical operation. By using of the stochastic arithmetic, sudden losses of accuracy, numerical instabilities, and the appearance of an insignificant result (stochastic zero) are detected [3,6,8,11].

3 Numerical accuracy of the Gauss-Chebyshev rules

As mentioned in [7], to correctly quantify the accuracy of a computed result, one must estimate the number of its exact significant digits, i. e., the number of significant digits that are common to the computed result and the exact result. Therefore, we need the following definition.

Definition 1 Let a and b be two real numbers, the number of significant digits that are common to a and b, denoted $C_{a,b}$ can be defined by

$$C_{a,b} = \log_{10}\left|\frac{a+b}{2(a-b)}\right| = \log_{10}\left|\frac{a}{a-b} - \frac{1}{2}\right|, \qquad a \neq b \tag{3}$$

and for all real numbers a, $C_{a,a} = +\infty$.

One can use the relation (3) in order to find the accuracy of the Gauss-Chebyshev integration rules. Then, one can use the CESTAC method to find the optimal number of points using these rules.

Let f be a function which its Chebyshev expansion is rapidly convergent. Let $I = \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}dx$ and I_N be an approximation of I using closed or opened Gauss-Chebyshev integration rules. The aim is to find N_{opt} such that, $I_N - I_{2N} = @0$.

It has been proved in [9], the errors of I_N and I_{2N} are:

$$E_N f = I_N - I = \pi a_{2N} - \pi a_{4N} + \pi a_{6N} - \ldots,$$

$$E_{2N} f = I_{2N} - I = \pi a_{4N} - \pi a_{8N} + \pi a_{12N} - \ldots,$$

where, $a_i, i = kN, k = 2, 4, 6, \ldots,$ are the coefficients of Chebyshev expansion of f. Since, f has a rapidly convergent Chebyshev expansion, the error can be found by

$$E_N f \approx \pi a_{2N} \quad , \quad E_{2N} f \approx \pi a_{4N}. \tag{4}$$

If $a_i, i = 0, 1, \ldots$ are the coefficients of Chebyshev expansion of f, the real numbers $r > 1$ and C_f exist such that [9],

$$|a_i| \leq C_f \hat{i}^{-r}, \tag{5}$$

where, $\hat{i} = \max\{i, 1\}, i \geq 0$. Therefore from (5), $a_i = O(i^{-r})$. Hence from (4),

$$E_N f = O((2N)^{-r}), E_{2N} f = O((4N)^{-r}). \tag{6}$$

The following theorem shows the numerical accuracy of the Gauss-Chebyshev integration rules.

Theorem 1 let f be a function which its Chebyshev expansion be rapidly convergent. Let $I = \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}dx$ and I_N be an approximation of I by using closed or opened Gauss - Chebyshev integration method then,

$$C_{I_N,I_{2N}} = C_{I_N,I} - \log_{10}\left|1 - \frac{a_{4N}}{a_{2N}}\right| + O(\frac{1}{(2N)^r}), \tag{7}$$

where, a_{2N}, a_{4N}, are the first coefficients of the Chebyshev expansion of f and $r > 1$ is a real value.

Proof. According to definition 1,

$$C_{I_N,I_{2N}} = \log_{10}\left|\frac{I_N}{I_N - I_{2N}} - \frac{1}{2}\right| =$$

$$\log_{10}\left|\frac{I_N}{I_N - I_{2N}}\right| + \log_{10}\left|1 - \frac{1}{2I_N}(I_N - I_{2N})\right| =$$

$$\log_{10}\left|\frac{I_N}{I_N - I_{2N}}\right| + O(I_N - I_{2N}).$$

Since, $I_N - I_{2N} = I_N - I - (I_{2N} - I) = E_N f - E_{2N} f$, thus from (6), $I_N - I_{2N} = O(\frac{1}{(2N)^r})$.

Therefore,

$$C_{I_N,I_{2N}} = \log_{10}\left|\frac{I_N}{I_N - I_{2N}}\right| + O(\frac{1}{(2N)^r}). \tag{8}$$

Furthermore,

$$C_{I_N,I} = \log_{10}\left|\frac{I_N}{I_N - I} - \frac{1}{2}\right| =$$

$$\log_{10}\left|\frac{I_N}{I_N - I}\right| + O(I_N - I) = \log_{10}\left|\frac{I_N}{I_N - I}\right| + O(\frac{1}{(2N)^r}).$$

According to (8),

$$C_{I_N,I_{2N}} = \log_{10}\left|\frac{I_N}{I_N - I - (I_{2N} - I)}\right| + O(\frac{1}{(2N)^r}) =$$

$$\log_{10}\left|\frac{I_N}{(I_N - I)(1 - \frac{I_{2N}-I}{I_N - I})}\right|$$

$$+O(\frac{1}{(2N)^r}) =$$

$$\log_{10}\left|\frac{I_N}{I_N - I}\right| - \log_{10}\left|1 - \frac{I_{2N} - I}{I_N - I}\right| + O(\frac{1}{(2N)^r}).$$

Consequently,

$$C_{I_N,I_{2N}} = C_{I_N,I} - \log_{10}\left|1 - \frac{I_{2N} - I}{I_N - I}\right| + O(\frac{1}{(2N)^r})$$

Since, $\frac{I_{2N}-I}{I_N-I} = \frac{a_{4N}}{a_{2N}} + O(\frac{1}{(4N)^r})$. Therefore,

$$C_{I_N,I_{2N}} = C_{I_N,I} - \log_{10}\left|1 - \frac{a_{4N}}{a_{2N}}\right| + O(\frac{1}{(2N)^r}).$$

The relation (7) shows that, if the Gauss-Chebyshev integration rules are used in order to estimate I then, for N large enough, the number of common significant digits between I_N and I_{2N} are almost equal to the number of common significant digits between I and I_N. The term $\log_{10}\left|1 - \frac{a_{4N}}{a_{2N}}\right|$ is near zero when N increases, because $0 < \frac{a_{4N}}{a_{2N}} << 1$. Also, $O(\frac{1}{(2N)^r})$ is small and negligible.

Therefore, if the CESTAC method is used then, the computations of the sequence I_N's are stopped when for an index like $N_{opt}, I_{N_{opt}} - I_{2N_{opt}} = @0$. In this case, $I_{N_{opt}}$ is an approximation of I.

4 Numerical Example

In this section, we evaluate a numerical example which has been provided by Visual Fortran in double precision in the Stochastic arithmetic. The computed values have obtained by using the opened and closed Gauss-Chebyshev rules with step size $h = \frac{b-a}{N}, N = 2^n, n \geq 1$. The successive values I_N and I_{2N} are computed and at each iteration, the number of significant digits of $|I_N - I_{2N}|$ and $|I_N - I|$ can be estimated. When $|I_N - I_{2N}| = @0$, I_N and I_{2N} are equal stochastically. The computations of the sequence I_N's are stopped when for an index like N_{opt} the number of common significant digits in the difference between $I_{N_{opt}}$ and $I_{2N_{opt}}$ become zero. In this case, one can say, before N_{opt}th iteration, $|I_N - I_{2N}|$ and $|I_N - I|$ has exact significant digits. But, the computation after N_{opt}th iteration are useless. In other words, the number of iteration in N_{opt} has been optimized. Also, according to theorem 1, the significant digits of the last approximation $I_{N_{opt}}$ are in common with the mathematical value of the integral I. Therefore, $I_{N_{opt}}$ is an approximation of I with optimal step size $h_{opt} = \frac{b-a}{N}$.

Example 1 In this example, the numerical solution of the integral $I = \int_{-1}^{1} \frac{\sqrt{(1+x)^3}}{\sqrt{1-x^2}} dx = \frac{8\sqrt{2}}{3} \approx$ 3.77123616632825 is considered [9]. The results are shown in tables 1 and 2 by using the opened and closed Gauss-Chebyshev rules in the stochastic arithmetic.

The last values of n in the tables are the optimal numbers of the points. The values C_{E_N} and C_{errN} are the number of the significant digits of $E_N = |I_N - I_{2N}|$ and the absolute error of I_N respectively. As we observe the optimal value of I using the closed Gauss-Chebyshev rule is $I_N = 3.77123616632851$ and using the opened Gauss-Chebyshev rule is $I_N = 3.77123616632799$ with $N_{opt} = 2^{10} = 1024$. In both cases, $C_{I_N,I} \simeq 12$.

5 Conclusion

In this paper, we have explained that by using the CESTAC method based on stochastic arithmetic one can use the Gauss-Chebyshev integration rules to approximate I and validate the result step by step. According to theorem 1, one can find an optimal value N_{opt} so that $I_{N_{opt}}$ is the best approximation for I from the computer point of view.

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n	I_N	$E_N = \|I_N - I_{2N}\|$	C_{E_N}	$errN = \|I_N - I\|$	C_{errN}
1	3.79223779587407	1.983789877463875E-002	12.59	2.100162954582041E-002	12.59
2	3.77239989709944	1.093014948547892E-003	11.00	1.163730771181666E-003	11.24
3	3.77130688215089	6.632656578453577E-005	9.72	7.071582263377489E-005	10.21
4	3.77124055558510	4.115400662474400E-006	8.46	4.389256849239113E-006	8.72
5	3.77123644018444	2.567475423376209E-007	7.71	2.738561867647131E-007	7.50
6	3.77123618343690	1.603947404523372E-008	5.91	1.710864442709218E-008	6.10
7	3.77123616739742	1.002347094214429E-009	4.71	1.069170381858460E-009	5.25
8	3.77123616639508	6.264988527959758E-011	3.98	6.682328764403185E-011	3.57
9	3.77123616633243	3.920567574292970E-012	2.81	4.173402364434272E-012	2.42
10	3.77123616632851	2.341830433275997E-013	1.13	2.528347901413023E-013	1.28
11	3.77123616632827	@0	-3.94	1.865174681370263E-014	0.33

Table 1 (Closed Gauss-Chebyshev rule)

n	I_N	$E_N = \|I_N - I_{2N}\|$	C_{E_N}	$errN = \|I_N - I\|$	C_{errN}
1	3.75256199832475	1.765186887755347E-002	11.80	1.867416800350104E-002	11.69
2	3.77021386720231	9.603618169763036E-004	10.02	1.022299125947572E-003	10.32
3	3.77117422901928	5.809576446728452E-005	8.73	6.193730897126788E-005	9.14
4	3.77123232478375	3.601905590085626E-006	7.48	3.841544503983367E-006	7.78
5	3.77123592668934	2.246686025368187E-007	6.86	2.396389138977402E-007	6.44
6	3.77123615135794	1.403475128706570E-008	4.96	1.497031136092157E-008	5.14
7	3.77123616539269	8.770747091565075E-010	3.94	9.355600738558678E-010	4.31
8	3.77123616626977	5.480875013101164E-011	2.98	5.848536469936032E-011	2.78
9	3.77123616632458	3.411641339804798E-012	1.87	3.676614568348668E-012	1.60
10	3.77123616632799	2.147911478308136E-013	0.21	2.649732285438707E-013	0.33
11	3.77123616632822	@0	-0.26	@0	-0.15

Table 2 (Opened Gauss-Chebyshev rule)