Proceedings of the World Congress on Engineering 2008 Vol II
WCE 2008, July 2 - 4, 2008, London, U.K.

The Inverse Eigenvalue Problem for Some Especial Kinds of Matrices

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***Abstract*—In recent paper [1](Jann Peny, Xi-YonHu. Lie Zhany) two inverse eigenvalue problems are solved and in the order article [2](Hubert paickmann. Juan Egana, Ricordo. L. sofo), a correction, for one of the problems stated in the first article, has been presented as well. In this article, according to the article [2], a solution which is different from the one in the article [1] has been presented for one of the problems which are in article [1]. The solution in the article [1] and the one which is presented by us, in the main diagonal, are similar, but instead of first column and row, we valued second column and row, furthermore other element of the matrix are considered null.**

Index Terms—Symmetric bordered diagonal matrices; Matrix inverse, eigenvalue problem (AMS classification: 65F15; 65F18; 15A18)

I. Introduction

In recent paper [1], an inverse eigenvalue problem is solved, a part of which, considering

$$\lambda_1^{(n)} < \lambda_1^{(n-1)} < \ldots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \ldots < \lambda_n^{(n)},$$

finds an $n \times n$ matrix B_n, such that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are the minimal and maximal eigenvalues of B_j respectively for all $j = 1, 2, 3, \cdots, n$,

in which B_n is as below:

$$B_n = \begin{pmatrix} a_1 & b_1 & b_2 & b_3 & \cdots & b_{n-1} \\ b_1 & a_2 & 0 & 0 & \cdots & 0 \\ b_2 & 0 & a_3 & 0 & \cdots & 0 \\ b_3 & 0 & 0 & a_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ b_{n-1} & 0 & 0 & 0 & \cdots & a_n \end{pmatrix}.$$

where a_i are distinct for all $i = 1, 2, \cdots, n$ and all b_i are positive.

Then consider the following matrix:

$$A_n = \begin{pmatrix} a_1 & b_1 & 0 & 0 & \cdots & 0 \\ b_1 & a_2 & b_2 & b_3 & \cdots & b_{n-1} \\ 0 & b_2 & a_3 & 0 & \cdots & 0 \\ 0 & b_3 & 0 & a_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & b_{n-1} & 0 & 0 & \cdots & a_n \end{pmatrix}. \qquad (1)$$

where a_i are distinct for all $i = 1, 2, \cdots, n$ and all b_i are positive. Throughout this paper, we use A_n to denote a special kind of matrices defined as in (1) and A_j to denote the $j \times j$ leading principal submatrix of A_n.

In this paper we, like paper [1], construct a matrix A_n as the following condition:

For $2n-1$ given real numbers $\lambda_1^{(n)} < \lambda_1^{(n-1)} < \ldots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \ldots < \lambda_n^{(n)}$, we find an $n \times n$ matrix A_n, such that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are the minimal and maximal eigenvalues of A_j respectively for all $j = 1, 2, 3, \cdots, n$.

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II. Properties of the matrix A_n

Similar paper [1] we assume later on, $b_0 = 1$ and let $\varphi_j(\lambda) = det(\lambda I_j - A_j)$ and $\varphi_0(\lambda) = 1$.

Lemma 1 . *For a given matrix A_n, the sequence $\{\varphi_j(\lambda)\}$ satisfies the following recurrence relation*

$$\varphi_j(\lambda) = (\lambda - a_j)\varphi_{j-1}(\lambda) - b_{j-1}^2 \prod_{i=1, i\neq 2}^{j-1} (\lambda - a_j) \qquad j = 3, 4, \cdots, n \quad (2)$$

Lemma 2 . *The characteristic polynomial sequence $\{\varphi_j(x)\}$ have some properties of a Sturm sequence, satisfying the following properties:*

1) All roots of $\varphi_n(x)$ are real and simple.

2) roots of $\varphi_{j-1}(x)$ and $\varphi_{j+1}(x)$ are distinct and if ξ is a root of $\varphi_j(x)$, then$\varphi_{j+1}(\xi)$ $\varphi_{j-1}(\xi)$ <0.

3) $\varphi_0(x)$ has no real root.

According to what we mentioned above all $\varphi_i(x)$ have simple roots and since in i intervals the sign of each φ_i changes and also φ_i have i roots, then all roots are real. Since $\varphi_0 = 1$, considering what we said above $\{\varphi_i\}$ have some properties of a Sturm sequence.

Lemma 3 . *Assume*

$$\lambda_1^{(n)} < \ldots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \ldots < \lambda_n^{(n)}$$

are the eigenvalues of $\varphi_i(x)$ for i=1,2,. . .,n, then we have

$$\lambda_1^{(j)} < a_i < \lambda_j^{(j)} \qquad \text{for} j = 2, \ldots, n, \qquad i = 1, 2, \ldots, j$$

corollary 1 . *If $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are the minimal and maximal zeros of $\varphi_j(\lambda)$ respectively, then*

1) for $\mu < \lambda_1^{(j)}$ we have $(-1)^j\varphi_j(\mu) > 0$,

2) for $\mu > \lambda_j^{(j)}$ we have $\varphi_j(\mu) > 0$ j=1,2,. . .,n.

III. Existence and Uniqueness

Theorem 1 . *(Existence and uniqueness of matrix A)*

Let $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ for j=1,2,. . .,n are real and satisfy in the following relation:

$$\lambda_1^{(n)} < \ldots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \ldots < \lambda_n^{(n)}$$

then there exist the unique matrix A in form (1) with $a_i \neq a_j$ $(i, j = 1, 2, \ldots, n)$ and $b_i > 0$, where $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are minimal and maximal eigenvalues of A_j respectively.

If

$$\lambda_1^{(2)} + \lambda_2^{(2)} \neq 2\lambda_1^{(1)} \qquad (3)$$

and

$$\frac{\lambda_{j-1}^{(j-1)}-\lambda_j^{(j)}}{\lambda_{j-1}^{(j-1)}-\lambda_1^{(j)}}<\frac{\varphi_{j-1}(\lambda_1^{(j)})\prod_{i=1,i\neq 2}^{j-1}(\lambda_j^{(j)}-a_i)}{\varphi_{j-1}(\lambda_j^{(j)})\prod_{i=1,i\neq 2}^{j-1}(\lambda_1^{(j)}-a_i)} \tag{4}$$

for $j=3,4,\ldots,n$, *or*

$$\frac{\lambda_1^{(j-1)}-\lambda_j^{(j)}}{\lambda_1^{(j-1)}-\lambda_1^{(j)}}>\frac{\varphi_{(j-1)}(\lambda_1^{(j)})\prod_{i=1,i\neq 2}^{j-1}(\lambda_j^{(j)}-a_i)}{\varphi_{j-1}(\lambda_j^{(j)})\prod_{i=1,i\neq 2}^{j-1}(\lambda_1^{(j)}-a_i)} \tag{5}$$

for $j=3,4,\ldots,n$, *then we can find* a_i, b_j *by the following relations :*

$$a_1=\lambda_1^{(1)},\qquad a_2=\lambda_2^{(2)}+\lambda_1^{(2)}-\lambda_1^{(1)},\qquad b_1^2=(\lambda_1^{(2)}-\lambda_1^{(1)})(\lambda_1^{(1)}-\lambda_2^{(2)}),$$

$$a_j=\frac{\lambda_1^{(j)}\varphi_{i-1}(\lambda_1^{(j)})\prod_{i=1,i\neq 2}^{j-1}(\lambda_j^{(j)}-a_i)-\lambda_j^{(j)}\varphi_{i-1}(\lambda_j^{(j)})\prod_{i=1,i\neq 2}^{j-1}(\lambda_1^{(j)}-a_i)}{\varphi_{i-1}(\lambda_1^{(j)})\prod_{i=1,i\neq 2}^{j-1}(\lambda_j^{(j)}-a_i)-\varphi_{i-1}(\lambda_j^{(j)})\prod_{i=1,i\neq 2}^{j-1}(\lambda_1^{(j)}-a_i)} \tag{6}$$

$$b_{j-1}^2=\frac{(\lambda_j^{(j)}-\lambda_1^{(j)})\varphi_{i-1}(\lambda_1^{(j)})\varphi_{i-1}(\lambda_j^{(j)})}{\varphi_{i-1}(\lambda_1^{(j)})\prod_{i=1,i\neq 2}^{j-1}(\lambda_j^{(j)}-a_i)-\varphi_{i-1}(\lambda_j^{(j)})\prod_{i=1,i\neq 2}^{j-1}(\lambda_1^{(j)}-a_i)} \tag{7}$$

$j=3,4,\ldots,n.$

At first we prove that a_j and b_{j-1} exist for all j, if we denote:

$$h_j=\varphi_{j-1}(\lambda_1^{(j)})\prod_{i=1,i\neq 2}^{j-1}(\lambda_j^{(j)}-a_i)-\varphi_{j-1}(\lambda_j^{(j)})\prod_{i=1,i\neq 2}^{j-1}(\lambda_1^{(j)}-a_i)$$

h_j is the denominator of a_j and b_{j-1}^2, and we prove that it is always nonzero. The sign of $\varphi_{j-1}(\lambda_1^{(j)})\prod_{i=1,i\neq 2}^{j-1}(\lambda_j^{(j)}-a_i)$ is $(-1)^{j-1}$ and since $a_i<\lambda_j^{(j)}$ for $i=1,2,\ldots,j$, then $\prod_{i=1,i\neq 2}^{j-1}(\lambda_j^{(j)}-a_i)>0$ and the sign of $\varphi_{j-1}(\lambda_1^{(j)})$ according to which we proved is $(-1)^{j-1}$.

Furthermore the sign of $-\varphi_{j-1}(\lambda_j^{(j)})\prod_{i=1,i\neq 2}^{j-1}(\lambda_1^{(j)}-a_i)$ is $(-1)^{j-1}$ and it is nonzero, then denominator of both terms with same sign and nonzero, is nonzero.

Then a_j , b_{j-1}^2 exist. Furthermore

$$b_{j-1}^2=\frac{(\lambda_j^{(j)}-\lambda_1^{(j)})\varphi_{i-1}(\lambda_1^{(j)})\varphi_{i-1}(\lambda_j^{(j)})}{h_j} \tag{8}$$

in numerator of (8) sign of $(\lambda_j^{(j)}-\lambda_1^{(j)})$ and $\varphi_{j-1}(\lambda_1^{(j)})$ is positive and $\varphi_{j-1}(\lambda_j^{(j)})$ has sign $(-1)^{j-1}$, then the sign of numerator is $(-1)^{j-1}$ and the denominator of this rational expression has sign $(-1)^{j-1}$, therefore b_{j-1}^2 is positive.

Now we prove that a_i which attained are distinct. From $\lambda_1^{(2)}+\lambda_2^{(2)}\neq 2\lambda_1^{(1)}$, we have $\lambda_1^{(2)}+\lambda_2^{(2)}-\lambda_1^{(1)}\neq\lambda_1^{(1)}$ consequently $a_2\neq a_1$. Let

$$u_j=\varphi_{j-1}(\lambda_1^{(j)})\prod_{i=1,i\neq 2}^{j-1}(\lambda_j^{(j)}-a_i),\qquad v_j=\varphi_{j-1}(\lambda_j^{(j)})\prod_{i=1,i\neq 2}^{j-1}(\lambda_1^{(j)}-a_i)$$

for $j=3,4,\ldots,n$.

Now we explain $j=3$:

The relation (4) includes

$$\frac{\lambda_2^{(2)}-\lambda_3^{(3)}}{\lambda_2^{(2)}-\lambda_1^{(3)}}<\frac{(-1)^2u_3}{(-1)^2v_3},$$

since v_3 is negative, then $(-1)^2(\lambda_1^3u_3-\lambda_3^3v_3)>\lambda_2^{(2)}(-1)^2(u_3-v_3)$, finally

$$a_3=\frac{(-1)^2(\lambda_1^3u_3-\lambda_3^3v_3)}{(-1)^2(u_3-v_3)}>\lambda_2^{(2)}$$

whereas $\lambda_1^{(2)}<a_1,a_2<\lambda_2^{(2)}$, then $a_3\neq a_1\neq a_2$.

Now we assume a_i , for $i=1,2,\ldots,j-1$ are distinct, by relation (4) we have:

$$\frac{\lambda_{j-1}^{(j-1)}-\lambda_j^{(j)}}{\lambda_{j-1}^{(j-1)}-\lambda_1^{(j)}}<\frac{((-1)^{j-1})u_j}{((-1)^{j-1})v_j},$$

note that $(-1)^{j-1}v_j$ is negative, then

$$\frac{(-1)^{j-1}[\lambda_1^{(j)}u_j-\lambda_j^{(j)}v_j]}{(-1)^{j-1}[u_j-v_j]}>\lambda_{j-1}^{(j-1)}$$

This means $a_j>\lambda_{j-1}^{(j-1)}$ and since $\lambda_1^{(j-1)}<a_i<\lambda_{j-1}^{(j-1)}$ for $i=1,\ldots,j-1$, then we have:

$$a_j\neq a_{j-1}\neq\ldots\neq a_1.$$

If we use relation (5) we conclude that $a_j<\lambda_1^{(j-1)}$, in which we take distinct a_i for $i=1,2,\ldots,j$, then the problem has solution and equivalently the following equations:

$$\varphi_j(\lambda_1^{(j)})=0,\qquad \varphi_j(\lambda_j^{(j)})=0$$

which have solutions distinct a_j for all $j=1,2,\ldots,n$ and b_{j-1} satisfying $b_{j-1}>0$ for all $j=2,3,\ldots,n$.

If problem has solution, then

$$\varphi_1(\lambda_1^{(1)})=(\lambda_1^{(1)}-a_1)=0\Rightarrow a_1=\lambda_1^{(1)},$$

$$\begin{aligned}\varphi_2(\lambda_1^{(2)})&=(\lambda_1^{(2)}-a_1)(\lambda_1^{(2)}-a_2)-b_1^2=0,\\ \varphi_2(\lambda_2^{(2)})&=(\lambda_2^{(2)}-a_1)(\lambda_2^{(2)}-a_2)-b_1^2=0,\end{aligned} \tag{9}$$

then by simplifying we get:

$$a_2=\frac{(\lambda_2^{(2)})^2-(\lambda_1^{(2)})^2+\lambda_1^{(1)}(\lambda_1^{(2)}-\lambda_2^{(2)})}{\lambda_2^{(2)}-\lambda_1^{(2)}}=\lambda_2^{(2)}+\lambda_1^{(2)}-\lambda_1^{(1)}$$

with substituting a_2 in (9) we have: $b_1^2=(\lambda_1^{(2)}-\lambda_1^{(1)})(\lambda_1^{(1)}-\lambda_2^{(2)})$, and since

$$\varphi_j(\lambda_1^{(j)})=(\lambda_1^{(j)}-a_j)\varphi_{j-1}(\lambda_1^{(j)})-b_{j-1}^2\prod_{i=1,i\neq 2}^{j-1}(\lambda_1^{(j)}-a_i)=0, \tag{10}$$

and

$$\varphi_j(\lambda_j^{(j)})=(\lambda_j^{(j)}-a_j)\varphi_{j-1}(\lambda_1^{(j)})-b_{j-1}^2\prod_{i=1,i\neq 2}^{j-1}(\lambda_j^{(j)}-a_i)=0, \tag{11}$$

For $3<j\leq n$ note that with

$$(10)\times\prod_{i=1,i\neq 2}^{j-1}(\lambda_j^{(j)}-a_i)\text{ - }(11)\times\prod_{i=1,i\neq 2}^{j-1}(\lambda_1^{(j)}-a_i)$$

we can find (6) and with

$$(10)\times\varphi_{j-1}(\lambda_j^{(j)})-(11)\times(\lambda_{j-1}^{(j)})$$

we find (7). Finally uniqueness matrix A_n by (6) and (7) is trivial.

References

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