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Entrapped Gas Action for One-Dimensional Models of Capillary Dynamics

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Abstract—**This paper is concerned with a mathematical and numerical study of the effect of gas entrapment on liquid dynamics in a closed-end horizontal capillary. This problem is important in order to understand how the presence of a gas inside the capillary can influence the dynamics of capillary flows and the non destructive test procedures carried out through liquid penetrant testing. In this context, by considering the most relevant approaches in modeling the gas entrapment inside a cylindrical capillary, some numerical simulations are carried out in order to deduce peculiar features arising in such a problem.**

Keywords: capillary dynamics, entrapped gas action, mathematical modeling.

1 Introduction

Capillarity is a well known physical phenomenon directly related to the free energy present at the interface liquid-air. Whenever the liquid gets in touch with a solid capillary, the motion of the air-liquid interface meniscus takes place, according to the wettability of the liquid. The force responsible for such motion is just the so-called surface tension.

The first ones to study the surface tension and capillarity, at the beginning of the nineteenth century, were Laplace and Young. In particular, their work started from the awareness that the static pressure on the liquid side of a liquid-air interface is reduced by the effect of the surface tension. Some time later, Hagen and Poiseuille, studying the flow of viscous liquids in circular pipes (and capillary tubes in particular), derived the well-known Poiseuille flow profile for a, fully developed, Newtonian fluid. Then Reynolds tested experimentally the stability of the Poiseuille profile, finding that it is valid in the case of laminar flow.

The first ones to set up a model for the dynamics of liquid flow into a capillary, dating from the early twentieth century, were Bell and Cameron [2], Lucas [20], Washburn [33] and Rideal [25]. From these works it was derived the well-known Washburn solution. The Washburn solution was derived by considering that the motion of a liquid penetrating a capillary is determined by a balance among capillarity, gravitational, and viscous forces under the assumption of a Poiseuille profile as the velocity profile.

For sufficiently long time, the Washburn solution describes excellently the dynamics of capillary flow and this is also proved by experimental results. Moreover, a further validation of the Washburn solution has been given by, both simulations of molecular dynamics, see for instance Martic et al. [23, 22, 21], and the lattice-Boltzman statistical-mechanical description, mainly used by physicists, see Chibbaro [6]. Unfortunately, the Washburn solution is defective in describing the initial transient, because of the fact that the model neglects the inertial effects. On the contrary, those inertial effects were considered in a model proposed by Bosanquet [3]. The SNC model, introduced by Szekely et al. [30], takes into account the outside flow effects including an apparent mass parameter within the inertial terms.

Recently, many experimental and theoretical studies on liquids flowing under capillary action have pointed out the limitations of the Washburn solution and its validity only as an asymptotic approximation. For example, liquids flowing in thin tubes were considered by Fisher and Lark [14]. Nonuniform cross-sectional capillaries have been studied, for instance by Erickson et al. [12] and by Young [32]. As far as surface grooves are concerned, see, for instance, Mann et al. in [16], Romero and Yost in [26], Rye et al. [27], and Yost et al. in [35], Microstrips were investigated by Rye et al. [28]. On the other hand, several studies have been devoted to capillary rise, dynamics of menisci, wetting and spreading, see, for instance, the papers by Clanet and Quéré [7], Zhmud et al. [36], Xiao et al. [34], Chebbi [5], Fries and Dreyer [15] and the recent books by de Gennet et al. [9] or by Karniadakis et al. [17] and the plenty of references quoted therein. Moreover, useful reviews appeared within the

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specialized literature, made by: Dussan [11], de Gennes [8], and Leger and Joanny [19].

Our main concern is devoted to report and test the mathematical modeling of entrapped gas action in a horizontal closed-end capillary. In this context the effect of the entrapped gas on the liquid dynamics was first studied by Deutsch [10] from a theoretical viewpoint and more recently by Pesse et al. [24] from an experimental one.

This study is of interest for the non-destructive technique named "liquid penetrant testing" used in the production of airplane parts, for instance, as well as in many industrial applications where the detection of open defects is of crucial interest.

2 Mathematical modeling

Firstly, let us consider a horizontal cylindrical capillary put in touch with a reservoir, as it is depicted in figure 1. We assume that the only physical entities active on bulk of liquid are: the surface tension, the viscosity, the pressure of the trapped gas and the atmospheric pressure. For the validity of our one-dimensional analysis we must assume that the fluid has a quasi-steady Poiseuille velocity profile. This consists in assuming that the fluid motion is a laminar flow, i.e. the liquid is considered to be moving in circular concentric circles (for a cylindrical geometry) with a parabolic velocity profile null at the wall (no-slip boundary condition). Moreover, it is assumed that $Re = 2RU/\mu \ll 1$, $Bo = \rho g R^2/\gamma \ll 1$, $Ca = \mu U/\gamma \ll 1$, and $We = 2\rho R U^2/\gamma \ll 1$, where U, R, ρ, γ, μ and g are the average velocity, the capillary radius, the liquid density, the surface tension, the viscosity and the acceleration due to gravity, respectively. Before going on, it is worth explaining the meaning of these assumptions:

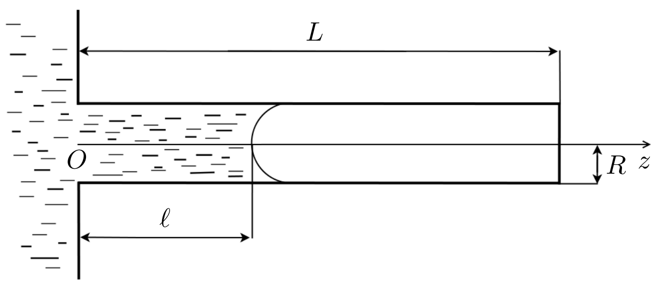

Figure 1: Physical setup and notation.

- A very low Reynolds number Re expresses the physical condition for laminar flow.
- The lower is the Bond number Bo the less is the deformation induced by the gravitational acceleration on the spherical shape of the meniscus interface due to the surface tension, in a horizontal cylinder.
- A low capillary number Ca expresses that the surface tension is predominant over the viscous effect for slow fluid motion.
- A low Weber number We means that the inertia of fluid is negligible with respect to its surface tension.

Finally, we assume also as a further simplification that contact angle is constant. Such a simplification fits well the assumption of low capillary numbers and/or $R/L \ll 1$, where L is the length of the capillary. A detailed discussion of the dynamic contact angle simplification is provided elsewhere, see for instance Tokaty [31] or Adamson [1]. A more complex model, involving two different liquids can be found by the interested reader in [4].

Assuming that the above assumptions are fulfilled, our mathematical model is given by the Newtonian equation of motion plus natural initial conditions:

$$\begin{cases} \rho(\ell + cR)\dfrac{d^2\ell}{dt^2} + \rho\left(\dfrac{d\ell}{dt}\right)^2 = 2\dfrac{\gamma\cos\vartheta}{R} - 8\dfrac{\mu\ell}{R^2}\dfrac{d\ell}{dt} \\ \ell(0) = \dfrac{d\ell}{dt}(0) = 0 \ . \end{cases} \qquad (1)$$

Here we have taken into account the coefficient of apparent mass $c = O(1)$, introduced by Szekely et al. [30] in order to get a well-posed problem, see [13]. The model (1) accounts for the displacement of a liquid due to the surface tension action inside a closed end horizontal capillary. The prescribed initial conditions are discussed at length by Kornev and Neimark [18]. As far as the authors knowledge is concerned, no analytical solutions are available for the model (1).

Moreover, in the case of a vertical capillary, the gravity action should be taken into account, by adding to the left hand side of equation (1) the term

$$\pm(\rho\ell)g \ , \qquad (2)$$

where the plus or minus sign have to be used when the liquid reservoir is over or below the cavity, respectively. Furthermore, for a closed-end capillary the entrapped gas action should be taken into account, by adding to the right hand side of equation (1) a term

$$\Omega(\ell, L) \ ,$$

depending on the lengths involved.

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2.1 Entrapped gas modeling

Two different ways to model the entrapped gas action are available in literature: the first, given by the following formula

$$\Omega(\ell, L) = p_a - p_a \frac{L}{L-\ell} \,, \tag{3}$$

is due to Deutsch [10], here p_a is the atmospheric pressure; the second, according to Zhmud et al. [36] and Chibbaro [6], takes into account only the viscous drag produced by the entrapped gas as follows

$$\Omega(\ell, L) = \frac{8\mu_e(L-\ell)}{R^2}\frac{d\ell}{dt} \,, \tag{4}$$

where μ_e is the viscosity of the entrapped gas.

By getting closer to the physics of the phenomenon, the gas action occurring inside the capillary can be seen as an adiabatic compression. This implies the assumption of no heat exchange during the liquid penetration. Under this assumption, we can use the adiabatic equation of gas expressed by

$$p\, V^\lambda = constant \,,$$

where p is the pressure, V is the gas volume, and $\lambda > 0$ is the well known gas constant ($\lambda = 1.4$ for bi-atomic gases). In order to deduce the value for the constant, we can assume that p_a is the pressure at time $t = 0$ when the volume occupied by the gas is equal to $A \cdot L$, where $A = \pi R^2$ denotes the cross sectional area of the capillary. As a result, we have that the constant is equal to $p_a A^\lambda L^\lambda$ and the adiabatic equation becomes

$$p(t)(L-\ell)^\lambda A^\lambda = p_a L^\lambda A^\lambda \,.$$

The entrapped gas action can be taken into account by adding to the right-hand side of equation (1) the pressure that this gas applies to the penetrant liquid meniscus, that is

$$\Omega(\ell, L) = p_a - p_a \left(\frac{L}{L-\ell}\right)^\lambda \,. \tag{5}$$

It is evident that the model by Deutsch is a specific case of the adiabatic model obtained by fixing $\lambda = 1$, that is about the value for dry air.

Let us consider here equation (5), and remark that the entrapped gas pressure

$$p_e = p_a \left(\frac{L}{L-\ell}\right)^\lambda$$

verifies the initial condition $p_e(\ell = 0) = p_a$, the asymptotic condition

$$\lim_{L\to\infty} p_e = p_a \,,$$

and the limit condition

$$\lim_{\ell\to L} p_e = +\infty$$

corresponding to the common intuition that no action is expected if the capillary is not closed and that the internal pressure will increase if we let ℓ increase for a closed capillary.

If we consider the case $\lambda = 1$, then the velocity can be found to be

$$u(r,z) = f(r)\frac{2\gamma\cos\vartheta(L-z) - p_a z R}{LR(L-z)} \tag{6}$$

with

$$f(r) = \frac{R^2}{4\mu}(1 - r^2/R^2) \tag{7}$$

Equations (6) and (7) show that the flow develops slowly enough to retain the parabolic profile of Poiseuille flow, while continuously adjusting its magnitude (and flow rate) in proportion to a constantly diminishing pressure gradient. From equation (6) we note that the flow will cease when

$$\frac{\ell_{\max}}{L} = \frac{2\gamma\cos\vartheta}{R\, p_a + 2\gamma\cos\vartheta} \,. \tag{8}$$

According to Deutsch [10], for p_a equal to 1 atmosphere ($\approx 1\times 10^6$ dyncs/cm^2), $R = 10^2$ μm and an air-water interface with $\gamma = 71.8$ mN/m and $\vartheta = 0°$, equation (8) shows that the flow will cease at

$$\frac{\ell_{\max}}{L} \approx 1.5\% \,, \tag{9}$$

and there will be about 98.5% of the capillary depth left to be filled.

3 Numerical results

As a simple test case, figure 2 shows the numerical solution corresponding to a closed-end capillary and water, obtained for the same values used by Deutsch [10] and reported in the end of the previous section.

Figure 3 shows a sample behavior for the inclusion velocity (first derivative of the front length).

The case $\lambda = 1$ was treated already in [4]. We remark that the obtained result $\ell_{\max}/L \approx 1.4\%$ is thinly different from Deutsch one, because we have taken into account also the effect due to the apparent mass, that is $c \neq 0$, which is related to the outside flow dynamics. Depending on the values used for R and L, we have also observed the occurrence of oscillatory damped solutions. The possibility to obtain this kind of solution was pointed out

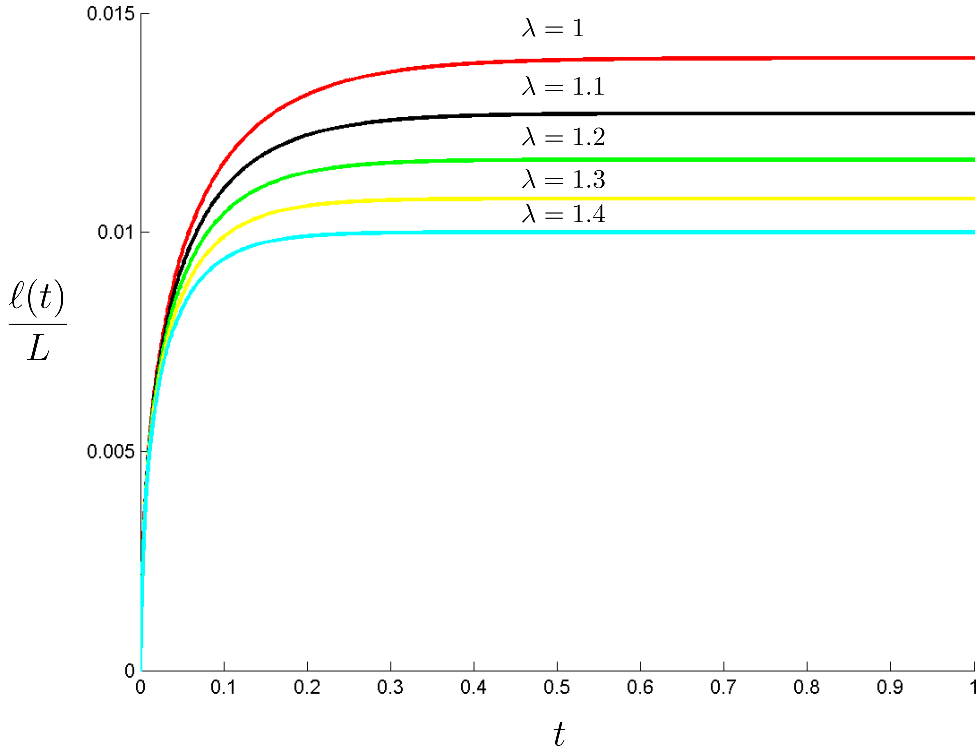

Figure 2: Water simulation in a close-end capillary. We used the equation (5) with different values of λ.

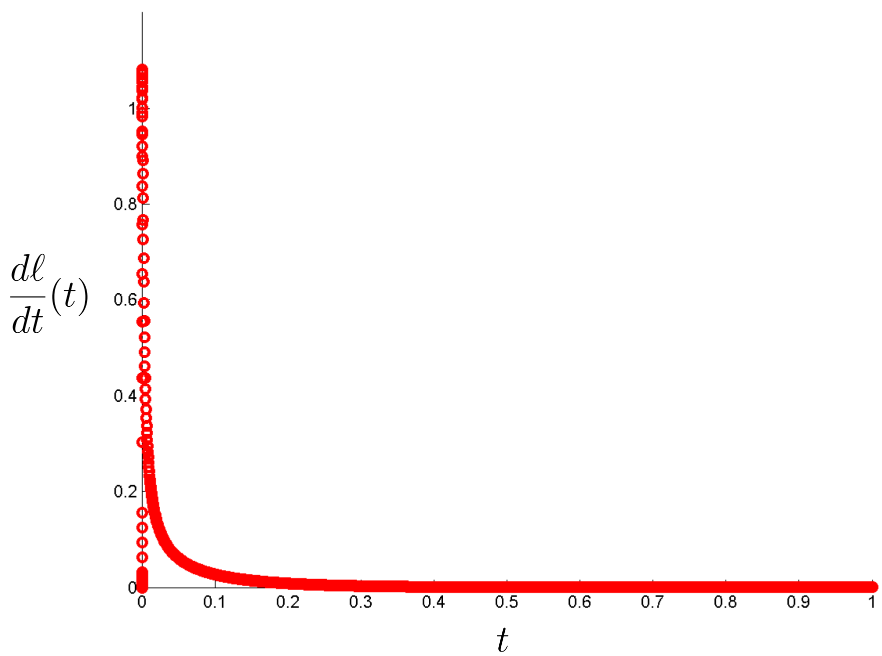

Figure 3: Transitory of the first derivative of $\ell(t)$ for $\lambda = 1$.

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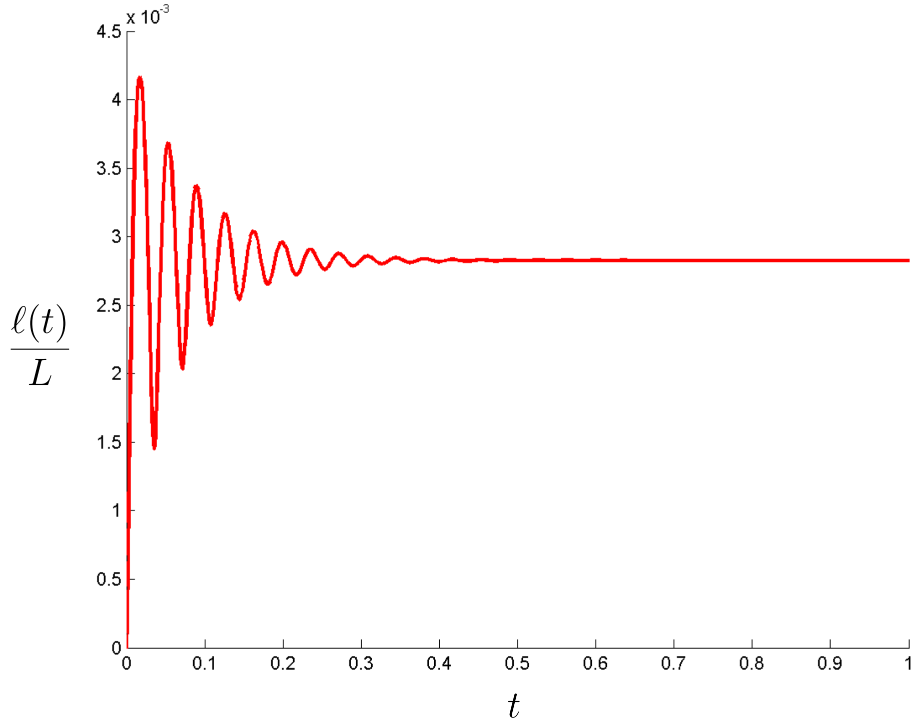

Figure 4: Water meniscus oscillations inside a close-end capillary for $\lambda = 1$, and $R = 0.005$ m.

also by Zhmud et al. [36]. In particular, such solutions oscillate just around the stationary level reached by the liquid. A sample of this behavior is depicted in figure 4. From a qualitative viewpoint, the term responsible for the presence, of the damped oscillations observed in figure 4 rather than the non oscillatory behavior in figure 2, is the viscous term, i.e. the one in (1) containing the constant μ. Indeed, having fixed a finite capillary length L, from numerical experiments it can be deduced the existence of a threshold on the capillary radius, so that, for higher values there is a non oscillatory behavior instead of the oscillatory one obtained for lower values. As a specific case, for a capillary long $L = 0.1$ m it is observed a threshold radius $R = 7.2\ 10^{-5}$ m below which we have non oscillatory solutions whereas above it the solutions start to oscillate.

The numerical results reported in this section were obtained by the `ODE45` solver of the MATLAB ODE suite developed by Shampine and Reichelt [29].

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