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Approximation Results For q-Parametric BBH Operators

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Abstract—**The paper introduces q-parametric Bleimann, Butzer and Hahn (q-BBH) operators as a rational transformation of q-Bernstein-Lupaş operators. On their basis, a set of new results on q-BBH operators can be obtained easily from the corresponding properties of q-Bernstein-Lupaş operators. Furthermore, convergence properties of q-BBH operators are studied.**

Keywords: Bernstein polynomials, BBH operators, Lupaş operators.

1 Introduction

The linear operator U_n defined by

$$U_n\left(f,x\right):=\frac{1}{\left(1+x\right)^n}\sum_{k=0}^{n}f\left(\frac{k}{n-k+1}\right)\left(\begin{array}{c}n\\k\end{array}\right)x^k,$$

where $f\in\mathbb{R}^{[0,\infty)}$ was introduced by Bleimann, Butzer, and Hahn [5] to approximate continuous functions. In [5], [1] the authors pointed out some formal similarities and differences between U_n and the classical Bernstein operator B_n. Connection suggested in [1] can be formulated by means of the following identity

$$U_n=\Phi^{-1}\circ B_{n+1}\circ\Phi,$$

Φ^{-1} and Φ are suitable positive linear operators which will be defined below. This idea was used in [8] to define new q-analogue of the Bleimann, Butzer, and Hahn operators as follows:

$$U_{n,q}\left(f;x\right):=\left(\Phi^{-1}\circ B_{n+1,q}\circ\Phi\right)\left(f;x\right),$$

where $B_{n+1,q}$ is a Philips q-analogue of the Bernstein operators.

Using the classical connection between Bernstein and BBH operators we propose the following q-analogue of the Bleimann, Butzer and Hahn operators in $C^0_{1+x}\left[0,\infty\right)$:

$$U_{n,q}\left(f;x\right):=\left(\Phi^{-1}\circ R_{n+1,q}\circ\Phi\right)\left(f;x\right), \tag{1}$$

where $R_{n,q}$ is the Lupaş q-Bernstein operator on $C\left[0,1\right]$ defined by

$$R_{n,q}(f,x)=\sum_{k=0}^{n}f\left(\frac{[k]}{[n]}\right)\left[\begin{array}{c}n\\k\end{array}\right]\times\frac{q^{\frac{k(k-1)}{2}}x^k(1-x)^{n-k}}{(1-x+qx)...(1-x+q^{n-1}x)}.$$

Thanks to (1), different properties of $R_{n+1,q}$ can be transferred to $U_{n,q}$ with some extra effort. Thus, the limiting behaviour of $U_{n,q}$ can be immediately derived from (1) and the well known properties of $R_{n+1,q}$.

2 Construction and some properties of the operators

Let $q>0$. For any $n\in N\cup\{0\}$, the q-integer $[n]=[n]_q$ is defined by

$$[n]:=1+q+...+q^{n-1},$$
$$[0]:=0;$$

and the q-factorial $[n]!=[n]_q!$ by

$$[n]!:=[1]\,[2]\,...\,[n]\,,$$
$$[0]!:=1.$$

For integers $0\leq k\leq n$, the q-binomial is defined by

$$\left[\begin{array}{c}n\\k\end{array}\right]:=\frac{[n]!}{[k]!\,[n-k]!}.$$

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Also, we use the following notations:

$$(1-x)_q^n := \prod_{j=0}^{n-1}\left(1-q^j x\right),$$

$$(1-x)_q^\infty := \prod_{j=0}^{\infty}\left(1-q^j x\right),$$

$$b_{n,k}(q;x) := \left[\begin{array}{c} n \\ k \end{array}\right] \times \frac{q^{\frac{k(k-1)}{2}} x^k (1-x)^{n-k}}{(1-x+qx)...(1-x+q^{n-1}x)},$$

$$u_{n,k}(q;x) := \left[\begin{array}{c} n \\ k \end{array}\right] \frac{q^{k(k+1)/2} x^k}{(1+qx)_q^n},$$

$$b_{\infty,k}(q;x) := \frac{q^{\frac{k(k-1)}{2}}(x/1-x)^k}{(1-q)^k [k]! \prod_{j=0}^{\infty}(1+q^j(x/1-x))},$$

$$u_{\infty,k}(q;x) := \frac{q^{k(k+1)/2} x^k}{(1+qx)_q^\infty (1-q)^k [k]!},$$

$$C[0,\infty] = \{f \in C[0,\infty) \mid f(x) \text{ has a finite limit at } \infty\},$$

$$C_{1+x}^0[0,\infty) = \{f \in C[0,\infty) \mid f(x) = o(1+x), x \to \infty\}.$$

It is assumed that $C_{1+x}^0[0,\infty)$ is endowed with the norm

$$\|f\|_{1+x} = \sup_{x\geq 0}\frac{|f(x)|}{1+x}.$$

We consider the operators $\Phi : \mathbb{R}^{[0,\infty)} \to \mathbb{R}^{[0,1]}$,

$$\Phi(f,t) := \begin{cases} (1-t) f\left(\frac{t}{1-t}\right), & t \in [0,1), \\ 0, & t = 1, \end{cases}$$

and $\Phi^{-1} : \mathbb{R}^{[0,1)} \to \mathbb{R}^{[0,\infty)}$,

$$\Phi^{-1}(g,x) := (1+x)\, g\left(\frac{x}{1+x}\right), \quad x \in [0,\infty).$$

Theorem 1 *[10] We have the following relations:*

1. *$\Phi^{-1} \circ \Phi$ is the identity operator on $\mathbb{R}^{[0,\infty)}$.*
2. *$f \in C_{1+x}^0[0,\infty)$ if and only if $\Phi f \in C[0,1]$.*
3. *If $f \in C_{1+x}^0[0,\infty)$, then f is convex if and only if f is convex and nonincreasing.*

We introduce Bleimann, Butzer and Hahn type operators based on q-integers as follows.

Definition 2 *For $f \in \mathbb{R}^{[0,\infty)}$ the q-Bleimann, Butzer and Hahn operators are given by*

$$\begin{aligned} U_{n,q}(f;x) &:= \left(\Phi^{-1} \circ R_{n+1,q} \circ \Phi\right)(f;x) \\ &= \sum_{k=0}^{n} f\left(\frac{[k]}{q^k[n-k+1]}\right) u_{n,k}(q;x) \\ &= \frac{1}{(1+qx)_q^n}\sum_{k=0}^{n} f\left(\frac{[k]}{q^k[n-k+1]}\right) \\ &\times \left[\begin{array}{c} n \\ k \end{array}\right] q^{k(k+1)/2} x^k, \quad n \in N. \end{aligned}$$

Definition 3 *Let $0 < q < 1$. The linear operator defined on $\mathbb{R}^{[0,\infty)}$ given by*

$$\begin{aligned} &U_{\infty,q}(f;x) \\ &:= \sum_{k=0}^{\infty} f\left(\frac{1-q^k}{q^k}\right) u_{\infty,k}(q;x) \\ &= \frac{1}{(1+qx)_q^\infty}\sum_{k=0}^{\infty} f\left(\frac{1-q^k}{q^k}\right)\frac{q^{k(k+1)/2}}{(1-q)^k[k]!}x^k \end{aligned}$$

is called the limit q-BBH operator.

Lemma 4 *$U_{n,q}, U_{\infty,q} : C_{1+x}^0[0,\infty) \to C_{1+x}^0[0,\infty)$ are linear positive operators and*

$$\|U_{n,q}(f)\|_{1+x} \leq \|f\|_{1+x},$$
$$\|U_{\infty,q}(f)\|_{1+x} \leq \|f\|_{1+x}.$$

Lemma 5 *We have*

$$U_{n,q}(1;x) = 1,$$
$$U_{n,q}(t;x) = x - \frac{q^{n(n+1)/2} x^{n+1}}{(1+qx)_q^n}.$$

Theorem 6 *If $f \in C_{1+x}^0[0,\infty)$ is a convex function, then the sequence $\{U_{n,q}(f;x)\}$ is nonincreasing in n for each $q \in (0,1]$ and $x \in [0,\infty)$.*

Proof. We start by writing

$$\begin{aligned}
&U_{n,q}(f;x) - U_{n+1,q}(f;x) \\
&= \frac{1}{(1+qx)_q^{n+1}} \sum_{k=0}^{n} f\left(\frac{[k]}{q^k[n-k+1]}\right) \\
&\times \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k+1)/2} x^k \left(1+q^{n+1}x\right) \\
&- \frac{1}{(1+qx)_q^{n+1}} \sum_{k=0}^{n+1} f\left(\frac{[k]}{q^k[n-k+2]}\right) \\
&\times \begin{bmatrix} n+1 \\ k \end{bmatrix} q^{k(k+1)/2} x^k \\
&= -\frac{1}{(1+qx)_q^{n+1}} q^{(n+1)(n+2)/2} x^{n+1} \\
&\times \left(f\left(\frac{[n+1]}{q^{n+1}}\right) - f\left(\frac{[n]}{q^n}\right)\right) \\
&+ \frac{1}{(1+qx)_q^{n+1}} \sum_{k=0}^{n-1} f\left(\frac{[k]}{q^k[n-k+1]}\right) \\
&\times \begin{bmatrix} n \\ k \end{bmatrix} q^{(k+1)(k+2)/2} q^{n-k} x^{k+1} \\
&+ \frac{1}{(1+qx)_q^{n+1}} \sum_{k=0}^{n-1} f\left(\frac{[k+1]}{q^{k+1}[n-k]}\right) \\
&\times \begin{bmatrix} n \\ k+1 \end{bmatrix} q^{(k+1)(k+2)/2} x^{k+1} \\
&- \frac{1}{(1+qx)_q^{n+1}} \sum_{k=0}^{n-1} f\left(\frac{[k+1]}{q^{k+1}[n-k+1]}\right) \\
&\times \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} q^{(k+1)(k+2)/2} x^{k+1}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&U_{n,q}(f;x) - U_{n+1,q}(f;x) \\
&= -\frac{1}{(1+qx)_q^{n+1}} \\
&\times q^{(n+1)(n+2)/2} x^{n+1} \left(f\left(\frac{[n+1]}{q^{n+1}}\right) - f\left(\frac{[n]}{q^n}\right)\right) \\
&+ \frac{1}{(1+qx)_q^{n+1}} \sum_{k=0}^{n-1} a_k \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} q^{(k+1)(k+2)/2} x^{k+1},
\end{aligned} \tag{2}$$

where

$$\begin{aligned}
a_k = {} & \frac{q^{n-k}[k+1]}{[n+1]} f\left(\frac{[k]}{q^k[n-k+1]}\right) \\
&+ \frac{[n-k]}{[n+1]} f\left(\frac{[k+1]}{q^{k+1}[n-k]}\right) \\
&- f\left(\frac{[k+1]}{q^{k+1}[n-k+1]}\right).
\end{aligned} \tag{3}$$

Now from Theorem 1 since f is nonincreasing the first term is nonnegative. Thus to show monotonicity of $U_{n,q}$ it suffices to show nonnegativity of a_k, $0 \le k \le n$. Let us write

$$\begin{aligned}
\alpha &= \frac{q^{n-k}[k+1]}{[n+1]}, 1-\alpha = \frac{[n-k]}{[n+1]}, \\
x_1 &= \frac{[k]}{q^k[n-k+1]}, x_2 = \frac{[k+1]}{q^{k+1}[n-k]}.
\end{aligned}$$

Then it follows that

$$\begin{aligned}
&\alpha x_1 + (1-\alpha) x_2 \\
&= \frac{q^{n-k}[k+1]}{[n+1]} \frac{[k]}{q^k[n-k+1]} \\
&+ \frac{[n-k]}{[n+1]} \frac{[k+1]}{q^{k+1}[n-k]} \\
&= \frac{[k+1]}{q^{k+1}[n+1]} \frac{q^{n-k+1}[k] + [n-k+1]}{[n-k+1]} \\
&= \frac{[k+1]}{q^{k+1}[n+1]} \frac{[n+1]}{[n-k+1]} = \frac{[k+1]}{q^{k+1}[n-k+1]}.
\end{aligned}$$

We see immediately that

$$\begin{aligned}
a_k &= \alpha f(x_1) + (1-\alpha) f(x_2) \\
&- f(\alpha x_1 + (1-\alpha) x_2) \ge 0
\end{aligned}$$

which proves the theorem. ■

3 Convergence properties of $U_{n,q}$

For $f \in C[0,1]$, $t > 0$, the modulus of continuity is defined by

$$\omega(f,t) = \sup_{|x-y| \le t} |f(x) - f(y)|.$$

Theorem 7 *Let $q = q_n$ satisfies $0 < q_n < 1$ and let $q_n \to 1$ as $n \to \infty$. For any $x \in [0,\infty)$ and for any $f \in C_{1+x}^0[0,\infty)$ the following inequality holds*

$$\begin{aligned}
&\frac{1}{1+x} |U_{n,q}(f;x) - f(x)| \\
&\le 2\omega\left(\Phi f, \sqrt{\lambda_n(x)}\right), \\
&\textit{where } \lambda_n(x) = \frac{x}{(1+x)^2} \frac{1}{[n+1]_{q_n}}.
\end{aligned}$$

Proof. Positivity of R_{n+1,q_n} implies that for any $g \in C[0,1]$,

$$\begin{aligned}
&|R_{n+1,q_n}(g;x) - g(x)| \\
&\le R_{n+1,q_n}(|g(t) - g(x)|; x).
\end{aligned} \tag{4}$$

On the other hand

$$\begin{aligned}
&|(\Phi f)(t) - (\Phi f)(x)| \\
&\le \omega(\Phi f, |t-x|) \\
&\le \omega(\Phi f, \delta)\left(1 + \frac{1}{\delta}|t-x|\right), \ \delta > 0.
\end{aligned}$$

This inequality and (4) imply that

$$\begin{aligned}&\left|R_{n+1,q_n}\left(\Phi f;x\right)-\left(\Phi f\right)(x)\right|\\&\leq\omega\left(\Phi f,\delta\right)\left(1+\frac{1}{\delta}R_{n+1,q_n}\left(\left|t-x\right|;x\right)\right)\end{aligned}$$

and

$$\begin{aligned}&\left|U_{n,q}\left(f;x\right)-f\left(x\right)\right|\\
&=(1+x)\left|R_{n+1,q_n}\left(\Phi f;\frac{x}{1+x}\right)-\left(\Phi f\right)\left(\frac{x}{1+x}\right)\right|\\
&\leq(1+x)\,\omega\left(\Phi f,\delta\right)\\
&\times\left(1+\frac{1}{\delta}R_{n+1,q_n}\left(\left|t-\frac{x}{1+x}\right|;\frac{x}{1+x}\right)\right)\\
&\leq(1+x)\,\omega\left(\Phi f,\delta\right)\left(1+\frac{1}{\delta}\right.\\
&\times\left.\left(R_{n+1,q_n}\left(\left|t-\frac{x}{1+x}\right|^2;\frac{x}{1+x}\right)\right)^{1/2}\right)\\
&=(1+x)\,\omega\left(\Phi f,\delta\right)\left(1+\frac{1}{\delta}\left(\frac{x}{1+x}\frac{1}{[n+1]_{q_n}}\right.\right.\\
&\left.\left.+\frac{x}{1+x}\frac{q_nx}{1+q_nx}\left(1-\frac{1}{[n+1]_{q_n}}\right)-\left(\frac{x}{1+x}\right)^2\right)^{1/2}\right)\\
&\leq(1+x)\,\omega\left(\Phi f,\delta\right)\left(1+\frac{1}{\delta}\left(\frac{x}{1+x}\frac{1}{[n+1]_{q_n}}\right.\right.\\
&\left.\left.-\left(\frac{x}{1+x}\right)^2\frac{1}{[n+1]_{q_n}}\right)^{1/2}\right)\\
&=(1+x)\,\omega\left(\Phi f,\delta\right)\left(1+\frac{1}{\delta}\left(\frac{x}{(1+x)^2}\frac{1}{[n+1]_{q_n}}\right)^{1/2}\right),\end{aligned}$$

where we have used the explicit formula for $R_{n+1,q_n}\left(\left|t-\frac{x}{1+x}\right|^2;\frac{x}{1+x}\right)$, which can be found in [9]. Now by choosing $\delta=\sqrt{\lambda_n(x)}$, we obtain desired result. ■

Corollary 8 *Let $q=q_n$ satisfies $0<q_n<1$ and let $q_n\to1$ as $n\to\infty$. For any $f\in C^0_{1+x}[0,\infty)$ it holds that*

$$\lim_{n\to\infty}\left\|U_{n,q}\left(f;x\right)-f\left(x\right)\right\|_{1+x}=0.$$

It is proved in [9] that, $b_{n,k}(q;x)\to b_{\infty,k}(q;x)$ uniformly in $x\in[0,1)$ as $n\to\infty$. In the next lemma we give an estimate for $\left|b_{n,k}(q;\frac{x}{1+x})-b_{\infty,k}(q;\frac{x}{1+x})\right|$ for $x\in[0,\infty)$.

Lemma 9 *Let $0<q<1$, $k\geq0$, $n\geq1$.For any $x\in[0,\infty)$ we have*

$$\begin{aligned}&\left|b_{n,k}(q;\frac{x}{1+x})-b_{\infty,k}(q;\frac{x}{1+x})\right|\\&\leq b_{n,k}(q;\frac{x}{1+x})\frac{xq^n}{1-q}+b_{\infty,k}(q;\frac{x}{1+x})\frac{q^{n-k+1}}{1-q}.\end{aligned}$$

Using Lemma 9 we prove the following quantitative result for the rate of local convergence of $U_{n,q}\left(f;x\right)$ in terms of the first modulus of continuity.

Theorem 10 *Let $0<q<1$ and $f\in C^0_{1+x}[0,\infty)$. Then for all $0\leq x<\infty$ we have*

$$\begin{aligned}&\left|U_{n,q}\left(f;x\right)-U_{\infty,q}\left(f;x\right)\right|\\&\leq2\left(1+x\right)\left(\frac{1+x}{1-q}+1\right)\omega\left(\Phi f,q^{n+1}\right).\end{aligned}$$

Proof. Consider

$$\begin{aligned}\Delta\left(x\right)&:=U_{n,q}(f;x)-U_{\infty,q}(f;x)\\&=\left(\Phi^{-1}\circ R_{n+1,q}\circ\Phi\right)(f;x)\\&\quad-\left(\Phi^{-1}\circ R_{\infty,q}\circ\Phi\right)(f;x)\\&=\left(\Phi^{-1}\circ\left(R_{n+1,q}-R_{\infty,q}\right)\circ\Phi\right)(f;x)\\&=\left(\Phi^{-1}\circ\left(R_{n+1,q}-R_{\infty,q}\right)\right)(\Phi f;x).\end{aligned}$$

Since $U_{n,q}(f;x)$ and $U_{\infty,q}(f;x)$ possess the end point interpolation property, $\Delta(0)=0$. For all $x\in(0,\infty)$ we rewrite Δ in the following form

$$\begin{aligned}&\Delta\left(x\right)\\
&=\Phi^{-1}\circ\sum_{k=0}^{n+1}\left[\left(\Phi f\right)\left(\frac{[k]}{[n+1]}\right)-\left(\Phi f\right)\left(1-q^k\right)\right]\\
&\times b_{n+1,k}(q;x)\\
&+\Phi^{-1}\circ\sum_{k=0}^{n+1}\left[\left(\Phi f\right)\left(1-q^k\right)-\left(\Phi f\right)(1)\right]\\
&\times\left(b_{n+1,k}(q;x)-b_{\infty,k}(q;x)\right)\\
&-\Phi^{-1}\circ\sum_{k=n+2}^{\infty}\left[\left(\Phi f\right)\left(1-q^k\right)-\left(\Phi f\right)(1)\right]\\
&\times b_{\infty,k}(q;x)=:I_1+I_2+I_3.\end{aligned}$$

We start with estimation of I_1 and I_3. Since

$$\begin{aligned}\frac{[k]}{[n+1]}-\left(1-q^k\right)&=\frac{1-q^k}{1-q^{n+1}}-\left(1-q^k\right)\\&=q^{n+1}\frac{1-q^k}{1-q^{n+1}}\leq q^{n+1},\end{aligned}$$

$$0\leq1-\left(1-q^k\right)=q^k\leq q^{n+1},\qquad k>n+1,$$

we get

$$\begin{aligned}&|I_1| &(5)\\ &\leq (1+x)\,\omega\left(\Phi f, q^{n+1}\right)\sum_{k=0}^{n+1} b_{n+1,k}(q;\frac{x}{1+x})\\ &= (1+x)\,\omega\left(\Phi f, q^{n+1}\right),\end{aligned}$$

$$\begin{aligned}&|I_3| &(6)\\ &\leq (1+x)\,\omega\left(\Phi f, q^{n+1}\right)\sum_{k=n+2}^{\infty} b_{\infty,k}(q;\frac{x}{1+x})\\ &\leq (1+x)\,\omega\left(\Phi f, q^{n+1}\right).\end{aligned}$$

Finally we estimate I_2. Using the property,

$$\omega\left(f,\lambda t\right)\leq(1+\lambda)\,\omega\left(f,t\right),\ \lambda>0,$$

and Lemma 9 we get

$$\begin{aligned}|I_2| &\leq (1+x)\sum_{k=0}^{n+1}\omega\left(\Phi f, q^k\right)\\ &\times\left|b_{n+1,k}(q;\frac{x}{1+x})-b_{\infty,k}(q;\frac{x}{1+x})\right|\\ &\leq (1+x)\,\omega\left(\Phi f, q^{n+1}\right)\sum_{k=0}^{n+1}\left(1+q^{k-n-1}\right)\\ &\times\left|b_{n+1,k}(q;\frac{x}{1+x})-b_{\infty,k}(q;\frac{x}{1+x})\right|\\ &\leq 2\,(1+x)\,\omega\left(\Phi f, q^{n+1}\right)\frac{1}{q^{n+1}}\\ &\times\sum_{k=0}^{n+1}q^k\left|b_{n+1,k}(q;\frac{x}{1+x})-b_{\infty,k}(q;\frac{x}{1+x})\right|\\ &\leq 2\,(1+x)^2\,\frac{1}{1-q}\omega\left(\Phi f, q^{n+1}\right). \qquad (7)\end{aligned}$$

From (5), (6), and (7), we conclude the desired estimation. ■

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