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Solving Second Order Linear Differential Equations via Algebraic Invariant Curves

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Abstract—**The idea to find first integrals for polynomial vector fields via algebraic invariant curves can be traced back to Darboux in the 19th century. In 1983, this approach was further developed by Prelle and Singer to become a semi-decision algorithm for finding elementary first integrals. In this paper, we describe how to extend the Prelle-Singer method to deal with second order linear differential equations via Kovacic's results on algebraic solutions of Riccati equations. Some illustrative examples on using this approach are provided.**

Index Terms—**Algebraic invariant curves, the Prelle-Singer method, Kovacic's theorem, Riccati equations, second order linear differential equations.**

I. Introduction

The method of integrating planar polynomial vector fields by means of algebraic invariant curves was introduced by Darboux in 1878. Then, Prelle and Singer extended his work in [1] and proposed a procedure for computing elementary first integrals of first order differential equations. While Darboux described his results with a strong geometric flavour for polynomial differential equations on the projective plane, the results of Prelle and Singer were written in the language of differential algebra. Since then, it aroused some researchers' interest to use the ideas of Darboux or Prelle-Singer to study the properties of algebraic invariant curves, first integrals and integrability of planar polynomial differential systems; or to compute first integrals, quasi-rational first integrals or closed form solutions of first order ordinary differential equations, with the aid of computer algebra systems. For instance, the works described in [3-10] are along such directions. On the other hand, finding closed form solutions of linear ordinary differential equations is based on differential Galois theory, which can be traced back to the 19th century. In [2], Kovacic gave a useful algorithm based on this theory to compute closed form solutions (known as Liouvillian solutions) of differential equations of the form $y''+a(x)y'+b(x)y=0$, where $a(x)$, $b(x)$ are rational functions of x. A good survey of Kovacic's work or its extensions to higher order linear differential equations can be found in [6], [11].

In this paper, we will give a brief review of the Prelle-Singer method for finding first integrals, and hence the closed form solutions of first order differential equations, as well as Kovacic's results on second order linear ordinary differential equations. Then, we describe how to integrate Kovacic's results with the Prelle-Singer method to compute Liouvillian solution of second order linear ordinary differential equations. Examples on using this approach will be included. We hope our discussions would be found useful to researchers working on similar research areas or lecturers who are involved in teaching differential equations, advanced engineering or applied mathematics courses.

II. The Prelle-Singer Method

.Given an autonomous planar system of first order differential equations $\dot{x}_i=\omega_i(x)$, $1\le i\le 2$, where t is an independent variable, x_i are dependent variables and ω_i are polynomials in x_i, we are interested to know whether it has first integrals or not. In 1983, Prelle and Singer introduced a procedure for finding elementary first integrals of planar autonomous systems of differential equations. Despite the fact that it is a semi-decision algorithm, as described in [4-7], it has been proved experimentally to be a useful and practical procedure for solving first order ordinary differential equations. This method works as follows:

Step 1: Set $N=1$.

Step 2: Find irreducible algebraic invariant curves (also known as eigenpolynomials or Darboux polynomials) f_i with degree ≤ N, such that $f_i \mid Df_i$, where D denotes the differential operator $\omega_1\partial/\partial x_1+\omega_2\partial/\partial x_2$.

Step 3: Let $Df_i=f_i g_i$, where g_i are called the cofactors of the corresponding f_i. Decide if there are constants n_i, not all zeros, such that $\sum_{i=1}^{m} n_i g_i=0$. If such n_i exist, then $\prod_{i=1}^{m} f_i^{n_i}$ is a first integral. If such n_i do not exist, then proceed to the next step.

Step 4: Decide if there are constants n_i, such that

$$\sum_{i=1}^{m} n_i g_i=-\left(\frac{\partial\omega_1}{\partial x_1}+\frac{\partial\omega_2}{\partial x_2}\right).$$

If such n_i exist, then $R=\prod_{i=1}^{m} f_i^{n_i}$ is an integrating factor and an elementary first integral I can be obtained by integrating the following pair of equations:

$$\frac{\partial I}{\partial x_1}=R\omega_2;\ \frac{\partial I}{\partial x_2}=-R\omega_1.$$

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If such such n_i do not exist, then proceed to the next step.

Step 5: Increase the value of N by 1. If N is greater than a preset bound, then return failure, otherwise repeat the whole procedure.

Some remarks related to this method are summarized as follows:

(a) The determination of the algebraic invariant curves f_i can be accomplished by the method of undetermined coefficients. In most cases, it is the most involved step in this method because one needs to solve a system of polynomial equations, which are often non-linear. For more details on computation of f_i, please refer to [4] or [7].

(b) The first integrals obtained in Step 3 are usually called rational first integrals, so as to distinguish them from those obtained in Step 4 via integration.

(c) This method is still a semi-decision algorithm for finding elementary first integrals because it cannot tell us how to determine the degree bound N for algebraic invariant curves effectively, but rather requires us to preset a degree bound for N at the beginning. Despite such drawback, it is quite a useful approach to solve first order ordinary differential equations, as reported in [4].

In the next section, we will give an overview of Kovacic's work on second order linear differential equations and describe how to integrate it with the Prelle-Singer method to determine the closed form solutions of second order linear differential equations.

III. KOVACCI'S WORK ON SECOND ORDER ODES

In [2], Kovacic proposed an algorithm to determine the Liouvillian solutions of second order linear differential equations of the form $y''+a(x)y'+b(x)y=0$, where $a(x)$, $b(x)$ $\in C(x)$, the set of complex rational functions of x, or to state that no Liouvillian solution exists. By using the transformation $z = ye^{\int a/2\,dx}$, one can reduce this equation into the form $z''=rz$, where $r = a'/2 + a^2/4 - b$. Then, Kovacic proved that if this reduced equation has Liouvillian solutions, at least one of them is of the following form:

(a) $e^{\int \omega\,dx}$, where $\omega \in C(x)$; or.

(b) $e^{\int \omega\,dx}$, where ω is algebraic of degree 2 over $C(x)$; or

(c) algebraic over $C(x)$.

Let $\omega = (\log z)'$. Then, $z''=rz$ can be converted into a Riccati equation $\omega'+\omega^2 = r$ and Kovacic deduced the following important result:

Theorem 2.1: The second order differential equation $z''=rz$ has a Liouville solution if and only if the associated Riccati equation $\omega'+\omega^2 = r$ has an algebraic solution in $C(x)$, with degree $n \in \{1,2,4,6,12\}$.

This result suggests that we can use the Prelle-Singer method to look for an algebraic solution for $\omega'+\omega^2 = r$ if it exists. The defining minimal polynomial for this algebraic solution will appear as an irreducible algebraic invariant curves described in Step 2 of the Prelle-Singer method. In fact, this approach can be used to find the general solution of $\omega'+\omega^2 = r$ since we can use the transformation $\omega = z + 1/v$, where v is a particular solution of the Riccati equation, to convert it into a linear first order differential equation, whose integrating factor is well-known.

IV. SOME EXAMPLES

We now illustrate how to use the method proposed in the last section to solve second order linear differential equations in the examples below.

Example 4.1. Find a Liouvillian solution of $y''-2xy/(x^3+1)^2 = 0$.

Solution. The associated Riccati equation of the given second order differential equation is $\omega'+\omega^2 = r$, where $r = 2x/(x^3+1)^2$. Using the Prelle-Singer method with $N = 4$, we obtain an algebraic invariant curve $(x^3+1)\omega - x^2$, and the associated cofactor is $-(x^3+1)(x^3\omega+\omega-2x^2)$. The algebraic invariant curve corresponds to the defining minimal polynomial of an algebraic solution of $\omega'+\omega^2 = r$. So, we can solve ω in terms of x to obtain:

$$\omega = \frac{x^2}{x^3+1}.$$

Hence, $y = e^{\int \omega\,dx} = (x^3+1)^{1/3}$ is a Liouvillian solution of $y''-2xy/(x^3+1)^2 = 0$.

Example 4.2. Find a Liouvillian solution of $y''+(3-x)y'/x - 5y/x = 0$.

Solution. The reduced second order differential equation is $z''=rz$ and the associated Riccati equation is $\omega'+\omega^2 = r$, where $r = (x^2+14x+3)/4x^2$. Using the Prelle-Singer method with $N = 4$, we obtain an algebraic invariant curve $(2x^3\omega - x^3 + 16x^2\omega - 15x^2 + 24x\omega - 52x - 36)/2$, and the associated cofactor is $2x(2x\omega+x+1)$. The algebraic invariant curve corresponds to the defining minimal polynomial of an algebraic solution of $\omega'+\omega^2 = r$. So, we can solve ω in terms of x to obtain:

$$\omega = \frac{x^3+15x^2+52x+36}{2x^3+16x^2+24x}.$$

Hence, $y = e^{\int(\omega - a/2)dx} = (x^2+8x+12)e^x$, where $a = (3-x)/x$, is a Liouvillian solution of $y''+(3-x)y'/x - 5y/x = 0$.

V. CONCLUDING REMARKS

A method to compute second order linear differential equations via algebraic invariant curves has been introduced in this paper. The main idea is to use Kovacic's results on second order differential equations and the associated first order Riccati equation, and then apply the Prelle-Singer method to compute the algebraic invariant curves, which will provide an algebraic solution of the Riccati equation and hence a Liouvillian solution of the given second order linear differential equation. With the aid of computer algebra systems to help compute the unknown coefficients of the algebraic invariant curves or even by hand calculations sometimes, this method can be a useful alternative to the classical techniques used to solve second order linear differential equations. We hope our discussions would be found useful to researchers working on similar research areas or lecturers who are involved in teaching differential equations, advanced engineering or applied mathematics courses.

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