Explicit Finite Difference Method For Convection-Diffusion Equations

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Abstract—**In this paper, we present an integral form of convection-diffusion equation. Then a class of alternating group explicit finite difference method (AGE) is constructed based on several asymmetric schemes. The AGE method is unconditionally stable and has the property of parallelism. Results of numerical examples show the AGE method is of high accuracy.**

Keywords: ***convection-diffusion equations, parallel computing, alternating group, finite difference, numerical method***

1 Introduction

Research on parallel finite difference methods for convection -diffusion equations is an interesting topic. We notice that a so-called AGE method is widely cared because of its intrinsic parallelism and absolute stability. The AGE method was originally presented for solving diffusion equations by Evans [1], and was applied to convection-diffusion equations in [2]. Based on the AGE method, many alternating group methods and domain-split methods were developed such as in [3-6]. In [7] the AGE method for parabolic equations with periodic boundary conditions was Applied, but the method will lead to numerical vibration in the case of convection dominant equations. Based on Samarskii's scheme [8], Lu [9] presented a class of alternating block explicit-implicit method. Tian [10] presented a new group explicit method using the theory of exponential type transformation. Both of the two methods are effective to solve convection dominant problems, but we notice that the two methods have only accurate of order two in spatial step size. On the other hand, researches on periodic boundary problems have been scarcely presented.

In this paper we will consider the following convection-diffusion equation:

$$\frac{\partial u}{\partial t}+k\frac{\partial u}{\partial x}=\varepsilon\frac{\partial^2 u}{\partial x^2}, \qquad 0\le t\le T,\ k>0,\varepsilon>0 \quad (1.1)$$

with initial and boundary conditions:

$$\begin{cases} u(x,0)=f(x), \\ u(x,t)=u(x+1,t). \end{cases} \quad (1.2)$$

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Results about existence and uniqueness of theoretic solution for convection-diffusion equations can be found in [11-12].

We will organize this paper as follows: In section 2, a kind of exponential-type transformation [10] will be used to get the integral conservative form of convection-diffusion equations. Then we present a group of asymmetric schemes. Based on the schemes, a class of unconditionally stable alternating group explicit finite difference method will be derived. Stability analysis for the alternating group method is given in section 3. In section 4, Results of numerical experiments on stability and accuracy are presented. Some conclusions are given at the end of the paper.

2 The Alternating Group Method

The domain $\Omega:(0,1)\times(0,T)$ will be divided into $(m\times N)$ meshes with spatial step size $h=\frac{1}{m}$ in x direction and the time step size $\tau=\frac{T}{N}$. Grid points are denoted by (x_i,t_n) or $(i,\ n)$, $x_i=ih(i=0,\ 1,\cdots,m)$, $t_n=n\tau(n=0,\ 1,\cdots,\frac{T}{\tau})$. The numerical solution of (2.1) is denoted by u_i^n, while the exact solution $u(x_i,\ t_n)$.

We notice (1.1) is equivalent to $e^{-\frac{kx}{\varepsilon}}\frac{\partial u}{\partial t}=\varepsilon\frac{\partial}{\partial x}(e^{-\frac{kx}{\varepsilon}}\frac{\partial u}{\partial x})$. Integral from $x_{i-\frac{1}{2}}$ to $x_{i+\frac{1}{2}}$ at $t=(n+\frac{1}{2})\tau$, then we give the expression of the integral in equation form $\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}e^{-\frac{kx}{\varepsilon}}\frac{\partial u}{\partial t}dx=\varepsilon\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}e^{-\frac{kx}{\varepsilon}}\frac{\partial u}{\partial x}dx$.

The proposed alternating group method is used in computing by applying the special combination of several saul'yev asymmetry schemes to a group of grid points. Then the numerical solutions at each group of points can be obtained independently, and the computation in the whole domain can be divided into many sub-domains.

Let $p=e^{-\frac{kh}{2\varepsilon}}, q=e^{\frac{kh}{2\varepsilon}}, r=\frac{k\tau}{48h(q-p)}$. In order to construct the alternating group method, first we can establish the following saul'yev asymmetry finite difference schemes to approach (1.1) at $(i,\ n+\frac{1}{2})$:

$$[1+(26p+q)r]u_i^{n+1}-(27p+q)ru_{i+1}^{n+1}+pru_{i+2}^{n+1}=-2qru_{i-2}^n+(2p+54q)ru_{i-1}^n+[1-(28p+53q)r]u_i^n+(q+27p)ru_{i+1}^n-pru_{i+2}^n \quad (2.1)$$

ISBN:978-988-18210-1-0

$$-(p+25q)ru_{i-1}^{n+1}+[1+(27p+26q)r]u_i^{n+1}-(q+27p)ru_{i+1}^{n+1}+pru_{i+2}^{n+1}=-2qru_{i-2}^{n}+(p+29q)ru_{i-1}^{n}+[1-(27p+28q)r]u_i^{n}+(q+27p)ru_{i+1}^{n}-pru_{i+2}^{n} \quad (2.2)$$

$$qru_{i-2}^{n+1}-(p+27q)ru_{i-1}^{n+1}+[1+(28p+27q)r]u_i^{n+1}-(q+29p)ru_{i+1}^{n+1}+2pru_{i+2}^{n+1}=-qru_{i-2}^{n}+(p+27q)ru_{i-1}^{n}+[1-(26p+27q)r]u_i^{n}+(25p+q)ru_{i+1}^{n} \quad (2.3)$$

$$qru_{i-2}^{n+1}-(p+27q)ru_{i-1}^{n+1}+[1+(53p+28q)r]u_i^{n+1}-(2q+54p)ru_{i+1}^{n+1}+2pru_{i+2}^{n+1}=-qru_{i-2}^{n}+(p+27q)u_{i-1}^{n}+[1-(p+26q)r]u_i^{n} \quad (2.4)$$

$$2qru_{i-2}^{n+1}-(2p+54q)ru_{i-1}^{n+1}+[1+(28p+53q)r]u_i^{n+1}-(q+27p)ru_{i+1}^{n+1}+pru_{i+2}^{n+1}=[1-(26p+q)r]u_i^{n}+(q+27p)ru_{i+1}^{n}-pru_{i+2}^{n} \quad (2.5)$$

$$2qru_{i-2}^{n+1}-(p+29q)ru_{i-1}^{n+1}+[1+(27p+28q)r]u_i^{n+1}-(q+27p)ru_{i+1}^{n+1}+pru_{i+2}^{n+1}=(p+25q)ru_{i-1}^{n}+[1-(27p+26q)r]u_i^{n}+(27p+q)ru_{i+1}^{n}-pru_{i+2}^{n} \quad (2.6)$$

$$qru_{i-2}^{n+1}-(p+27q)ru_{i-1}^{n+1}+[1+(26p+27q)r]u_i^{n+1}-(q+25p)ru_{i+1}^{n+1}=-qru_{i-2}^{n}+(p+27q)ru_{i-1}^{n}+[1-(28p+27q)r]u_i^{n}+(q+29p)ru_{i+1}^{n}-2pru_{i+2}^{n} \quad (2.7)$$

$$qru_{i-2}^{n+1}-(p+27q)ru_{i-1}^{n+1}+[1+(p+26q)r]u_i^{n+1}=-qru_{i-2}^{n}+(p+27q)ru_{i-1}^{n}+[1-(53p+28q)r]u_i^{n}+(2q+54p)ru_{i+1}^{n}-2pru_{i+2}^{n} \quad (2.8)$$

Using the schemes mentioned above, we will have three basic point groups:
(1)"G1" group: eight inner points are involved, and (2.1) – (2.8) are used at each grid point respectively.
(2)"G2" group: four inner points are involved, and (2.1) – (2.4) are used respectively.
(3)"G3" group: four inner points are involved, and (2.5) – (2.8) are used respectively.

Let $m = 8s$, here s is an positive integer. Based on the basic point groups above, the alternating group method will be presented as following:

First at the $(n+1)$-th time level, we will have s point groups. "G1" is used in each group. Second at the $(n+2)$-th time level, we will have $(s+1)$ point groups. "G3" is used in the left four grid points. "G1" is used in the following $s-1$ point groups, while "G2" is used in the right four grid points.

From the alternating use of (2.1)-(2.8), grouping explicit computation can be obviously obtained. Thus computing in the whole domain can be divided into many subdomains, and can be worked out with several parallel computers. So the method can shorten the computing time compared to implicit methods, and has the property of parallelism.

Let $U^n = (u_1^n,\ u_2^n,\ \cdots,\ u_m^n)^T$, then we can denote the alternating group method as follows:

$$\begin{cases} (I+rA)U^{n+1}=(I-rB)U^{n} \\ (I+rB)U^{n+2}=(I-rA)U^{n+1} \end{cases} \quad (2.9)$$

$$A=\begin{pmatrix} A_1 & & & & \\ & A_1 & & & \\ & & \cdots & & \\ & & & A_1 & \\ & & & & A_1 \end{pmatrix}_{m\times m}$$

$$B=\begin{pmatrix} A_3 & & & & E \\ & A_1 & & & \\ & & \cdots & & \\ & & & A_1 & \\ D & & & & A_2 \end{pmatrix}_{m\times m},$$

$$A_1=\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$$A_{11}=\begin{pmatrix} 26p+q & -(27p+q) & p & 0 \\ -(p+25q) & 27p+26q & -(q+27p) & p \\ q & -(p+27q) & 28p+27q & -(q+29p) \\ 0 & q & -(p+27q) & 53p+28q \end{pmatrix}$$

$$A_{12}=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2p & 0 & 0 & 0 \\ -(2q+54p) & 2p & 0 & 0 \end{pmatrix}$$

$$A_{21}=\begin{pmatrix} 0 & 0 & 2q & -(2p+54q) \\ 0 & 0 & 0 & 2q \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A_{22}=\begin{pmatrix} 28p+53q & -(q+27p) & p & 0 \\ -(p+29q) & 27p+28q & -(q+27p) & p \\ q & -(p+27q) & 26p+27q & -(q+25p) \\ 0 & q & -(p+27q) & p+26q \end{pmatrix}$$

$$A_{2}=\begin{pmatrix} 26p+q & -(27p+q) & p & \\ -(p+25q) & 27p+26q & -(q+27p) & p \\ q & -(p+27q) & 28p+27q & -(q+29p) \\ & q & -(p+27q) & 53p+28q \end{pmatrix}$$

$$D=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2p & 0 & 0 & 0 \\ -(2q+54p) & 2p & 0 & 0 \end{pmatrix}$$

$$A_{3}=\begin{pmatrix} 28p+53q & -(q+27p) & p & \\ -(p+29q) & 27p+28q & -(q+27p) & p \\ q & -(p+27q) & 26p+27q & -(q+25p) \\ & q & -(p+27q) & p+26q \end{pmatrix}$$

ISBN:978-988-18210-1-0

$$E = \begin{pmatrix} 0 & 0 & 2q & -(2p+54q) \\ 0 & 0 & 0 & 2q \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Applying Taylor's formula to (2.1)-(2.8) at (x_i, t_n), we can easily obtain that the truncation error is $\mathcal{O}(\tau^2+\tau h+\tau h^2+\tau h^3+h^4)$ respectively, and alternating use of (2.1)-(2.8) can lead to counteraction of the truncation error for the items containing τh, τh^2 and τh^3. Then we can denote the truncation error of (2.9) as $\mathcal{O}(\tau^2+h^4)$.

3 Stability Analysis

Kellogg Lemma[13] Let $r>0$, and G is nonnegative definite real matrix, then:

$$\begin{cases} \|(I+rG)^{-1}\|_2 \le 1 \\ \|(I-rG)(I+rG)^{-1}\|_2 \le 1 \end{cases} \tag{3.1}$$

Theorem The alternating group method defined by (2.9) is unconditionally stable.

Proof: From the construction of the matrices above we can see A and B are both diagonally dominant matrices, which shows A and B are both nonnegative definite real matrices. Then we have: $\|(I+rA)^{-1}\|_2 \le 1, \|(I-rA)(I+rA)^{-1}\|_2 \le 1, \|(I+rB)^{-1}\|_2 \le 1, \|(I-rB)(I+rB)^{-1}\|_2 \le 1$.

Let n be an even integer, from (2.9) it follows that $U^n = GU^{n-2} = G^{\frac{n}{2}}U^0$, $G=(I+rB)^{-1}(I-rA)(I+rA)^{-1}(I-rB)$. Let $\overline{G} = (I+rB)G(I+rB)^{-1} = (I-rA)(I+rA)^{-1}(I-rB)(I+rB)^{-1}$, then we can get $\rho(G)=\rho(\overline{G}) \le \|\overline{G}\|_2 \le 1$, which shows the method defined by (2.9) is unconditionally stable.

4 Numerical Experiments

We consider problem (1.1) with initial and boundary conditions:

$$\begin{cases} u(x,0) = sin(2\pi x), \\ u(x,t) = u(x+1,t). \end{cases} \tag{4.1}$$

The exact solution of the problem above is denoted as below:

$$u(x,t) = e^{-4\epsilon\pi^2 t} sin[2\pi(x-kt)]$$

Let $A.E. = |u_i^n - u(x_i,t_n)|$ and $P.E. = 100 \times \frac{|u_i^n - u(x_i,t_n)|}{u(x_i,t_n)}$ denote maximum absolute error and relevant error of the presented method respectively. we compare the numerical results of (2.9) with the results in [2, 9, 10].

Table 1: Results of comparisons at $m = 16$

	$\tau=10^{-4}, t=100\tau, \varepsilon=1$	$\tau=10^{-5}, t=100\tau, \varepsilon=1$
$A.E.$	8.586×10^{-6}	9.132×10^{-7}
$A.E.^{[2]}$	8.726×10^{-5}	9.596×10^{-6}
$A.E.^{[9]}$	6.517×10^{-5}	6.873×10^{-6}
$A.E.^{[10]}$	7.306×10^{-5}	8.062×10^{-6}
$P.E.$	1.312×10^{-2}	7.345×10^{-3}
$P.E.^{[2]}$	4.229×10^{-1}	9.456×10^{-2}
$P.E.^{[9]}$	2.958×10^{-1}	7.103×10^{-2}
$P.E.^{[10]}$	3.673×10^{-1}	8.218×10^{-2}

Table 2: Results of comparisons at $m = 24,\ \tau = 10^{-4}$

	$t=100\tau, \varepsilon=0.1$	$t=1000\tau, \varepsilon=0.01$
$A.E.$	6.731×10^{-6}	1.102×10^{-4}
$A.E.^{[2]}$	1.458×10^{-5}	2.647×10^{-3}
$A.E.^{[9]}$	1.074×10^{-5}	7.653×10^{-4}
$A.E.^{[10]}$	1.127×10^{-5}	8.914×10^{-4}
$P.E.$	2.016×10^{-2}	2.467×10^{-2}
$P.E.^{[2]}$	4.250×10^{-1}	5.364×10^{-1}
$P.E.^{[9]}$	1.156×10^{-1}	1.924×10^{-1}
$P.E.^{[10]}$	1.241×10^{-1}	2.758×10^{-1}

Then we let $m = 16, \varepsilon = 0.001, 0.0001$, and the results show that the method in [2] doesn't converge to the exact solution. The comparisons between the presented AGE method and the methods in [9, 10] are listed in Table 3.

Table 3: Results of comparisons at $m = 16,\ \tau = 10^{-4}$

	$t=100\tau, \varepsilon=0.001$	$t=100\tau, \varepsilon=0.0001$
$A.E.$	3.276×10^{-4}	4.105×10^{-4}
$A.E.^{[9]}$	2.137×10^{-3}	3.426×10^{-3}
$A.E.^{[10]}$	1.159×10^{-2}	1.194×10^{-2}
$P.E.$	4.891×10^{-2}	5.207×10^{-2}
$P.E.^{[9]}$	2.469×10^{-1}	2.683×10^{-1}
$P.E.^{[10]}$	3.689	3.723

Let t_1/t_2 denote the ratio of running CPU time between the AGE method and the implicit scheme (3). In Table 4, we compare the presented AGE method and the known Crank-Nicolson scheme in running time.

Table 4: Results of comparison in runningCPU time at $\tau = 10^{-3}$

	$m=16, t=100\tau$	$m=64, t=200\tau$	$m=96, t=500\tau$
t_1/t_2	0.246	0.141	0.0924

If we use a large $'m'$,for example, $m = 200, 600, 800$, then the implicit C-N method will be invalid. But considering the parallelism of the present method, the computing can be divided into many sub-domains independently, and the computing can be finished as usual.

In the end we let $\varepsilon = 0.00001$. We find the methods in [2, 9, 10] can't converge to the exact solution, but the presented method in (2.9) is also valid, and we have $A.E. = 5.691\times10^{-3}, P.E. = 0.814$.

5 Conclusions

In this paper, we present an alternating group explicit method, which is of absolute stability and parallelism. The results in Table 1-3 show that the method introduced is of higher accuracy than the methods in [2, 9, 10]. Furthermore, numerical results show the presented method in this paper won't lead to numerical vibration in the case of small ε such as $\varepsilon = 0.0001$. So the presented method is suitable for solving convection dominant problems, and is superior to the methods in [2, 9, 10].

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