τ^*-Generalized Closed Sets in Topological Spaces

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Abstract- In this paper, we introduce a new class of sets called τ^*-generalized closed sets and τ^*-generalized open sets in topological spaces and study some of their properties .

Keywords: τ^*-g-closed set, τ^*-g-open set.

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1. Introduction

In 1970, Levine[6] introduced the concept of generalized closed sets as a generalization of closed sets in topological spaces. Using generalized closed sets, Dunham[5] introduced the concept of the closure operator cl^* and a new topology τ^* and studied some of their properties. S.P.Arya[2], P.Bhattacharyya and B.K.Lahiri[3], J.Dontchev[4], H.Maki, R.Devi and K.Balachandran[9], [10], P.Sundaram and A.Pushpalatha[12], A .S.Mashhour, M.E.Abd El-Monsef and S.N.El-Deeb[11], D.Andrijevic[1] and S.N.Maheshwari and P.C.Jain[9] introduced and investigated generalized semi closed sets, semi generalized closed sets, generalized semi preclosed sets, α- generalized closed sets, generalized-α closed sets, strongly generalized closed sets, preclosed sets, semi-preclosed sets and α-closed sets respectively. In this paper, we obtain a new generalization of closed sets in the weaker topological space (X, τ^*).

Throughout this paper X and Y are topological spaces on which no separation axioms are assumed unless otherwise explicitly stated. For a subset A of a topological space X, int(A), cl(A), $cl^*(A)$, scl(A), spcl(A), $cl_\alpha(A)$ and A^c denote the interior, closure, closure*, semi-closure, semi-preclosure, α-closure and complement of A respectively.

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2. Preliminaries

We recall the following definitions:

Definition 2.1. A subset A of a topological space (X, τ) is called

(i) Generalized closed (briefly g-closed)[6] if $cl(A) \subseteq G$ whenever $A \subseteq G$ and G is open in X.
(ii) Semi-generalized closed (briefly sg-closed)[3] if $scl(A) \subseteq G$ whenever $A \subseteq G$ and G is semiopen in X.
(iii) Generalized semiclosed (briefly gs-closed)[2] if $scl(A) \subseteq G$ whenever $A \subseteq G$ and G is open in X.
(iv) α-closed[8] if $cl(int(cl(A))) \subseteq A$.
(v) α-generalized closed (briefly αg-closed)[9] if $cl_\alpha(A)\subseteq G$ whenever $A \subseteq G$ and G is open in X.
(vi) Generalized α-closed (briefly gα-closed)[10] if spcl (A) $\subseteq G$ whenever $A \subseteq G$ and G is open in X.
(vii) Generalized semi-preclosed (briefly gsp-closed)[2] if $scl(A) \subseteq G$ whenever $A \subseteq G$ and G is open in X.
(viii) Strongly generalized closed (briefly strongly g-closed) [12] if $cl(A) \subseteq G$ whenever $A \subseteq G$ and G is g-open in X.
(ix) Preclosed[11] if $cl(int(A)) \subseteq A$.
(x) Semi-closed[7] if $int(cl(A)) \subseteq A$.
(xi) Semi-preclosed (briefly sp-closed)[1] if $int(cl(int(A))) \subseteq A$.

The complements of the above mentioned sets are called their respective open sets.

Definition 2.2. For the subset A of a topological X, the generalized closure operator cl^*[5] is defined by the intersection of all g-closed sets containing A.

Definition 2.3. For the subset A of a topological X, the topology τ^* is defined by $\tau^* = \{G : cl^*(G^C) = G^C \}$

Definition 2.4. For the subset A of a topological X,

(i) the semi-closure of A (briefly scl(A))[7] is defined as the intersection of all semi-closed sets containing A.

(ii) the semi-preclosure of A (briefly spcl(A))[1] is defined as the intersection of all semi-preclosed sets containing A.

(iii) the α-closure of A (briefly $cl_\alpha(A)$)[8] is defined as the intersection of all α-closed sets containing A.

3. τ^*-Generalized Closed Sets in Topological Spaces

In this section, we introduce the concept of τ^*-generalized closed sets in topological spaces.

Definition 3.1. A subset A of a topological space X is called τ^*-generalized closed set (briefly τ^*-g-closed) if $cl^*(A) \subseteq G$ whenever $A \subseteq G$ and G is τ^*-open. The complement of τ^*-generalized closed set is called the τ^*-generalized open set (briefly τ^*-g-open).

Theorem 3.2. Every closed set in X is τ*-g-closed.

Proof. Let A be a closed set. Let $A \subseteq G$. Since A is closed, $cl(A) = A \subseteq G$. But $cl^*(A) \subseteq cl(A)$. Thus, we have $cl^*(A) \subseteq G$ whenever $A \subseteq G$ and G is τ^*-open. Therefore A is τ^*-g-closed.

Theorem 3.3. Every τ^*-closed set in X is τ*-g-closed.

Proof. Let A be a τ^*-closed set. Let $A \subseteq G$ where G is τ^*-open. Since A is τ^*-closed, $cl^*(A) = A \subseteq G$. Thus, we have $cl^*(A) \subseteq G$ whenever $A \subseteq G$ and G is τ^*-open. Therefore A is τ^*-g-closed.

Theorem 3.4. Every g-closed set in X is a τ^*-g-closed set but not conversely.

Proof : Let A be a g-closed set. Assume that $A \subseteq G$, G is τ^*-open in X. Then $cl(A) \subseteq G$, since A is g-closed. But $cl^*(A) \subseteq cl(A)$. Therefore $cl^*(A) \subseteq G$. Hence A is τ^*-g-closed.

The converse of the above theorem need not be true as seen from the following example.

Example 3.5. Consider the topological space X = {a, b, c} with topology τ = {X, ϕ, {a}}. Then the set {a} is τ*-g-closed but not g-closed.

Remark 3.6. The following example shows that τ*-g-closed sets are independent from sp-closed set, sg-closed set, α-closed set, preclosed set gs-closed set, gsp-closed set, αg-closed set and gα-closed set.

Example 3.7. Let X = {a, b, c} and Y = {a, b, c, d} be the topological spaces.
(i) Consider the topology τ = {X , ϕ, {a}}. Then the sets {a}, {a, b}and {a, c} are τ^*-g-closed but not sp-closed.
(ii) Consider the topology τ = {X , ϕ, {a, b}}. Then the sets {a}and {b}are sp-closed but not τ^*-g-closed.
(iii) Consider the topology τ = {X, ϕ}. Then the sets {a}, {b}, {c},{a, b}, {b, c}and {a, c} are τ^*-g-closed but not sg-closed.
(iv) Consider the topology τ = {X , ϕ, {a}, {b}, {a, b}}. Then the sets {a}and {b}are sg-closed but not τ^*-g-closed.
(v) Consider the topology τ = {X , ϕ, {a}}. Then the sets {a}, {b}, {c}, {a, b} and {a, c} are τ^*-g-closed but not α-closed.
(vi) Consider the topology τ = {X , ϕ, {a}, (a, b}}. Then the set {b} is α-closed but not τ^*-g-closed set.
(vii) Consider the topology τ = {X , ϕ, {a}}. Then the sets {a}, {a, b}and {a, c} are τ^*-g-closed but not pre-closed.
(viii) Consider the topology τ = {X , ϕ, {b}, {a, b}}. Then the set {a} is pre-closed but not τ^*-g-closed.
(xi) Consider the topology τ = {X , ϕ}}. Then the sets {a}, {b}, {c}, {a, b}, {b, c}and {a, c} are τ^*-g-closed but not gs-closed.
(x) Consider the topology τ = {Y , ϕ, {a}, {a, b, c}, {a, b, d}}. Then the sets {b}, {b, c} and {b, d} are gs-closed but not τ^*-g-closed.
(xi) Consider the topology τ = {X , ϕ, {a}, {b}, (a, b}}, where X = {a, b, c}. Then the sets {b}and {a, b} are gsp-closed but not τ^*-g-closed.
(xii) Consider the topology τ = {Y , ϕ, {a}}. Then the set {a} is τ^*-g-closed but not gsp-closed.
(xiii) Consider the topology τ = {X, ϕ, {a}}. Then the set {a} is τ*-g-closed but not αg-closed.
(xiv) Consider the topology τ = {Y , ϕ, {a},{a, b, c}, {a, b, d}}. Then the sets {b}, {b, c} and {b, d} are αg-closed but not τ^*-g-closed.
(xv) Consider the topology τ = {X, ϕ, {a}, {b}, (a, b},{a, c}}.Then the set {b}is τ*-g-closed but not gα-closed.
(xvi) Consider the topology τ = {Y, ϕ, {a},{a, b, c}, {a, b, d}}. Then the sets {b}, {b, c} and {b, d} are gα-closed but not τ^*-g-closed.

Theorem 3.8. For any two sets A and B, $\mathbf{cl}^*(A \cup B) = cl^*(A) \cup cl^*(B)$

Proof : Since $A \subseteq A \cup B$, we have $cl^*(A) \subseteq cl^*(A \cup B)$ and since $B \subseteq A \cup B$, we have $cl^*(B) \subseteq cl^*(A \cup B)$. Therefore $cl^*(A) \cup cl^*(B) \subseteq cl^*(A \cup B)$. Also , $cl^*(A)$ and $cl^*(B)$ are the closed sets Therefore $cl^*(A) \cup cl^*(B)$ is also a closed set. Again, $A \subseteq cl^*(A)$ and $B \subseteq cl^*(B)$ implies $A \cup B \subseteq cl^*(A) \cup cl^*(B)$.Thus, $cl^*(A) \cup cl^*(B)$ is a closed set containing $A \cup B$. Since $cl^*(A \cup B)$ is the smallest closed set containing $A \cup B$ we have $cl^*(A \cup B) \subseteq cl^*(A) \cup cl^*(B)$. Thus, $cl^*(A \cup B) = cl^*(A) \cup cl^*(B)$

Theorem 3.9. Union of two τ^* g-closed sets in X is a τ^*-g-closed set in X.

Proof : Let A and B be two τ^* g-closed sets. Let $A \cup B \subseteq G$, where G is τ^*-open. Since A and B are τ^*-g-closed sets, $cl^*(A) \cup cl^*(B) \subseteq G$. But by Theorem 3.8., $cl^*(A) \cup cl^*(B) = \mathbf{cl}^*(A \cup B)$. Therefore $\mathbf{cl}^*(A \cup B) \subseteq G$. Hence $A \cup B$ is a τ^*-g-closed set.

Theorem 3.10. A subset A of X is τ^*-g-closed if and only if $cl^*(A) – A$ contains no non-empty τ^* -closed set in X.

Proof: Let A be a τ^*-g-closed set. Suppose that F is a non-empty τ^*-closed subset of $cl^*(A) – A$. Now $F \subseteq cl^*(A) – A$.

Then $F \subseteq cl^*(A) \cap A^c$, since $cl^*(A) - A = cl^*(A) \cap A^c$. Therefore $F \subseteq cl^*(A)$ and $F \subseteq A^c$. Since F^c is a τ^*-open set and A is a τ^*-g-closed, $cl^*(A) \subseteq F^c$. That is $F \subseteq [cl^*(A)]^c$. Hence $F \subseteq cl^*(A) \cap [cl^*(A)]^c = \phi$. That is $F = \phi$, a contradiction. Thus $cl^*(A) - A$ contain no non-empty τ^*-closed set in X.

Conversely, assume that $cl^*(A) - A$ contains no non-empty τ^*-closed set. Let $A \subseteq G$, G is τ^*-open. Suppose that $cl^*(A)$ is not contained in G, then $cl^*(A) \cap G^c$ is a non-empty τ^*-closed set of $cl^*(A) - A$ which is a contradiction. Therefore $cl^*(A) \subseteq G$ and hence A is τ^*-g-closed.

Corollary 3.11. A subset A of X is τ^* g-closed if and only if $cl^*(A) - A$ contain no non-empty closed set in X.

Proof : The proof follows from the Theorem 3.10. and the fact that every closed set is τ^*- closed set in X.

Corollary 3.12. A subset A of X is τ^*-g-closed if and only if $cl^*(A) - A$ contain no non-empty open set in X.

Proof: The proof follows from the Theorem 3.10. and the fact that every open set is τ^*-open set in X.

Theorem 3.13. If a subset A of X is τ^*-g-closed and $A \subseteq B \subseteq cl^*(A)$, then B is τ^*-g-closed set in X.

Proof : Let A be a τ^*-g-closed set such that $A \subseteq B \subseteq cl^*(A)$. Let U be a τ^*-open set of X such that $B \subseteq U$. Since A is τ^*-g-closed, we have $cl^*(A) \subseteq U$. Now $cl^*(A) \subseteq cl^*(B) \subseteq cl^*[cl^*(A)] = cl^*(A) \subseteq U$. That is $cl^*(B) \subseteq U$, U is τ^*-open. Therefore B is τ^*-g-closed set in X.

The converse of the above theorem need not be true as seen from the following example..

Example 3.14. Consider the topological space (X, τ), where $X = \{a, b, c\}$ and the topology $\tau = \{X, \phi, \{a\}, \{a, b\}\}$ Let $A = \{c\}$ and $B = \{a, c\}$.Then A and B are τ^*-g-closed sets in (X, τ). But $A \subseteq B$ is not a subset of $cl^*(A)$.

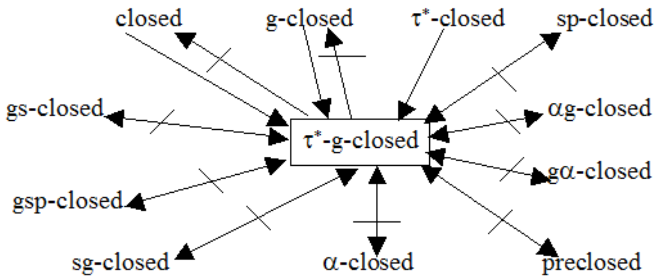
Theorem 3.15. Let A be a τ^*-g-closed in (X, τ). Then A is g-closed if and only if $cl^*(A) - A$ is τ^*-open.

Proof : Suppose A is g-closed in X. Then $cl^*(A) = A$ and so $cl^*(A) - A = \phi$ which is τ^*- open in X. Conversely, suppose $cl^*(A) - A$ is τ^*-open in X. Since A is τ^*-g-closed, by the Theorem 3.10, $cl^*(A) - A$ contains no non-empty τ^*-closed set in X. Then $cl^*(A) - A = \phi$ Hence A is g-closed.

Theorem 3.16. For $x \in X$, the set $X - \{x\}$ is τ^*-g-closed or τ^*-open.

Proof: Suppose $X - \{x\}$ is not τ^*-open. Then X is the only τ^*-open set containing $X - \{x\}$. This implies $cl^*(X - \{x\}) \subseteq X$. Hence $X - \{x\}$ is a τ^*-g-closed in X.

Remark 3.17. From the above discussion, we obtain the following implications.



A ⟶ B means A implies B, A ⟶̸ B means A does not imply B and A ⟷̸ B means A and B are independent.

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