A New Approach for Establishing the Energy Momentum Tensor in the Theory of Relativity

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Abstract — **The tensor calculus, used in a suitable manner, permits to establish the expression of the electromagnetic energy-momentum tensor (energy and quantity of motion tensor) in various cases interesting in the Theory of Relativity, and which have not been examined in the known works. In literature, in the works devoted to the Theory of Relativity, this problem has been especially treated for the vacuum medium. Here the author presents a new approach to the analysis of the case of linear but non-homogeneous electrically and magnetically polarized media. The problem of passing from a system of reference to another one in motion, and the selection of the volume density force formulae which are in accordance with the Theory of Relativity are also examined.**

Index Terms—**Tensor calculus, Energy-momentum tensor, Theory of relativity.**

I. Introduction

In Electrodynamics and in the Theory of Relativity, the energy-momentum tensor has a very important role [1-9]. Besides the widely accepted fact that this tensor allows a compact way of writing the conservation laws of linear momentum and energy in Electromagnetism, it permits to calculate the energy and stress, in any reference frame in terms of another reference frame, and especially in terms of the reference frame in which the substance is at rest.

The developments of the principles of the concerned mathematical methods, started from a relatively long time, are still examined nowadays [10-14].

Many works have been devoted to this subject. However, in the most treated case of empty space as well as in the case of a space filled with substance, the transition from a reference frame to another in motion has not been carefully analysed. In this paper, a new approach to the analysis of the tensor will be presented namely, the construction of the tensor, the case of non-homogeneous electrically and magnetically polarized substances and the transition from a reference frame to another one.

II. Volume Density of the Electromagnetic Force

An analysis of the electromagnetic forces in the frame of classical theories can be found in certain works among which ref. [15]. In the works concerning the Theory of Relativity the analysis of electromagnetic forces is achieved from the Lorentz formula of the force, e.g. [5, p. 133]. In the present paper, we shall start from the general formula of the electromagnetic force acting on a substance submitted to an electromagnetic field. It is derived from the principle of conservation of energy and the Theory of Relativity, through certain approximations, [8, p. 157]. The reasoning has led to the following formulae, both also deduced in various other manners and accepted by several authors:

$$\boldsymbol{f} = \rho_v \boldsymbol{E} - \frac{1}{2}\boldsymbol{E}^2 \operatorname{grad}\varepsilon - \frac{1}{2}\boldsymbol{H}^2 \operatorname{grad}\mu + \boldsymbol{J}\times\boldsymbol{B}, \tag{1 a}$$

and

$$\boldsymbol{f} = \rho_v \boldsymbol{E} - \frac{1}{2}\boldsymbol{E}^2 \operatorname{grad}\varepsilon - \frac{1}{2}\boldsymbol{H}^2 \operatorname{grad}\mu + \boldsymbol{J}\times\boldsymbol{B} + \frac{\partial}{\partial t}(\boldsymbol{D}\times\boldsymbol{B}), \tag{1 b}$$

where the symbols are the usual ones. In this case, the quantities ε and μ are considered as constant, but strongly depend on the point of the substance, hence varying in space. We shall denote the three axes of a Cartesian system of co-ordinates, by the indices *i*, *j*, *k*. In the further analysis, we shall consider formula (1), and we shall mention the modification occurring due to the supplementary term, if using formula (1 b). Relations (1 a) and (1 b) are considered as having three and four components, respectively:

$$f_k = (f_k)_1 + (f_k)_2 + (f_k)_3 + (f_k)_4, \tag{2}$$

given by the following expressions:

$$\begin{aligned} &(f_k)_1 = \rho_v E_k\,; \quad (f_k)_2 = [\boldsymbol{J}\times\boldsymbol{B}]_k\,; \\ &(f_k)_3 = \left[-\frac{1}{2}\boldsymbol{E}^2 \operatorname{grad}\varepsilon - \frac{1}{2}\boldsymbol{H}^2 \operatorname{grad}\mu\right]_k; \\ &(f_k)_4 = \left[\frac{\partial}{\partial t}(\boldsymbol{D}\times\boldsymbol{B})\right]_k . \end{aligned} \tag{3}$$

Henceforth, we shall write the expressions of the electromagnetic field state quantities by using the scalar and vector potentials *V* and ***A***, in the well-known form:

$$E_i = -\frac{\partial V}{\partial x^i} - \frac{\partial A_i}{\partial t}; \quad B_i = \frac{\partial A_k}{\partial x^j} - \frac{\partial A_j}{\partial x^k}; \tag{4 a}$$

$$\forall\, i, j, k \in [1, 3]. \tag{4 b}$$

The relations (4 a, b) may be written using a set of four quantities A_i as follows [10]:

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$$F_{i0} = \frac{1}{c}\left(\frac{\partial A_0}{\partial x^i} - c\frac{\partial A_i}{\partial x^0}\right), \quad B_i = \frac{\partial A_k}{\partial x^j} - \frac{\partial A_j}{\partial x^k}, \tag{5 a-d}$$
$$A_0 = -V, \quad x^0 = ct.$$

Further, we shall have in view the two sets of equations of the electromagnetic field (in the order used by H. A. Lorentz which differs from that of J. C. Maxwell) in a four-dimensional continuum space-time, where the symbols are those of [10, 11]. For the sake of facility, we shall recall these symbols in the case of empty space (vacuum).

The equations of the first group are given by the relationship:

$$\frac{\partial G^{ij}}{\partial x^j} = J^i, \quad \forall i, j \in [0, 3], \quad i \neq j; \quad J^0 = c\rho_v, \tag{6}$$

and:

$$\begin{aligned}
&E_i = cF_{i0}; \quad F_{ij} = e_{ii}e_{jj}F^{ij}; \quad F_{ij} = -F_{ji};\\
&G^{0j} = c^2\varepsilon_0 F^{0j};\\
&G_{i0} = c^2\varepsilon_0 F_{i0}; \quad \forall i, j \in [0, 3];\\
&c^2 = \frac{1}{\varepsilon_0\mu_0};\\
&B_{ij} = F_{ij}; \quad G^{ij} = \frac{1}{\mu_0}F^{ij};\\
&D^i = \frac{1}{c}G^{0i} = \varepsilon_0\delta^{ii}E_i;\\
&D_k = D^k; \quad H^{ij} = G^{ij}; \quad H_k = H^{ij};\\
&B_k = B_{ij}; \forall i, j \in [1, 3];
\end{aligned} \tag{6 a-m}$$

where the subscript index k in the relation (6 l) refers to the usual three-dimensional vectors, whereas indices i and j refer, as previously, to tensor components. All situations in which the index k has this role will be mentioned. It is to be noted that the components of the form F_{ii} and G^{ii} vanish.

Introducing the axis coefficients of the Galilean reference frame, e_{ii}, [10], we can write:

$$\begin{aligned}
&A_i = e_{ii}A^i = e_{is}A^s; \quad e_{00} = 1, \; e_{ii} = -1, \; \forall i \in [1, 3];\\
&e_{ij} = 0; \quad \forall \in i \neq j;\\
&A_i A^i = (A_0)^2 - (A_1)^2 - (A_2)^2 - (A_3)^2.
\end{aligned} \tag{6 n-r}$$

The equations of the second group are given by the relationship:

$$\frac{\partial F_{ij}}{\partial x^k} + \frac{\partial F_{jk}}{\partial x^i} + \frac{\partial F_{ki}}{\partial x^j} = 0, \quad \forall i, j, k \in [0, 3], \quad i \neq j \neq k. \tag{7}$$

In order to emphasize the tensors F_{ij} and G_{ij}, equation (3) can be written in the form below, taking into account relations (6), for instance, relation (8 d) has been written taking into account relations (6 a, b, h, l):

$$\begin{aligned}
&(f_k)_1 = \frac{\partial G^{0j}}{\partial x^j}F_{k0}; \quad (f_k)_2 = \frac{\partial G^{uj}}{\partial x^j}F_{ku};\\
&(f_k)_4 = \frac{\partial}{\partial x^0}\left(G^{0u}F_{ku}\right);\\
&(f_k)_3 = -\frac{1}{2}c^2 F_{u0}F^{0u}\frac{\partial\varepsilon}{\partial x^k} - \frac{1}{2}G_{uv}G^{uv}\frac{\partial\mu}{\partial x^k};
\end{aligned} \tag{8 a-d}$$

$$\begin{aligned}
&G^{0j} = c^2\varepsilon_0\varepsilon_r F^{0j}; \quad G^{uv} = \frac{1}{\mu_0\mu_r}F^{uv};\\
&c^2 = \frac{1}{\varepsilon_0\mu_0}; \quad \varepsilon = \varepsilon_0\varepsilon_r, \quad \mu = \mu_0\mu_r,\\
&\forall i, j, k \in [0, 3], \quad u, v \in [1, 3]; \quad u < v.
\end{aligned} \tag{8 e-i}$$

Summing up, side by side relations (8 a) and (8 b), we shall get:

$$(f_k)_{12} = \frac{\partial G^{0j}}{\partial x^j}F_{k0} + \frac{\partial G^{uj}}{\partial x^j}F_{ku}, \quad \forall j, k \in [0, 3], \quad u \in [1, 3] \tag{9}$$

or equivalently:

$$(f_k)_{12} = \frac{\partial G^{ij}}{\partial x^j}F_{ki}, \quad \forall i, j, k \in [0, 3]. \tag{10}$$

III. Expression of the Energy-Momentum Tensor

We shall now consider the case of a linear isotropic electric and magnetic polarization of the considered medium, with the relative permittivity ε_r and the relative permeability μ_r, which vary with the position of the considered point.

In order to express the force component as the derivative of an expression, we shall write relation (10) in the form:

$$(f_k)_{12} = \frac{\partial G^{ij}}{\partial x^j}F_{ki} = \frac{\partial}{\partial x^j}\left(G^{ij}F_{ki}\right) - G^{ij}\frac{\partial F_{ki}}{\partial x^j}, \quad \forall i, j, k \in [0, 3]. \tag{11}$$

Now, we shall modify the second term of the right-hand side as follows:

$$G^{ij}\frac{\partial F_{ki}}{\partial x^j} = G^{ij}\frac{\partial F_{ki}}{\partial x^j}, \tag{12 a}$$

$$G^{ij}\frac{\partial F_{ki}}{\partial x^j} = G^{ji}\frac{\partial F_{kj}}{\partial x^i} = -G^{ij}\frac{\partial F_{kj}}{\partial x^i} = G^{ij}\frac{\partial F_{jk}}{\partial x^i}. \tag{12 b}$$

Summing up the left-hand and the last right-hand sides of the two expressions (12 a, b), and taking into account (7), we get:

$$2G^{ij}\frac{\partial F_{ki}}{\partial x^j}=G^{ij}\left(\frac{\partial F_{ki}}{\partial x^j}+\frac{\partial F_{jk}}{\partial x^i}\right)=-G^{ij}\frac{\partial F_{ij}}{\partial x^k}. \tag{13}$$

Replacing (13) into (11), we shall obtain:

$$\frac{\partial G^{ij}}{\partial x^j}F_{ki}=\frac{\partial}{\partial x^j}\left(G^{ij}F_{ki}\right)+\frac{1}{2}G^{ij}\frac{\partial F_{ij}}{\partial x^k}, \tag{14}$$

and by expanding the second term:

$$\begin{aligned}(f_k)_{12}&=\frac{\partial}{\partial x^j}\left(G^{ij}F_{ki}\right)+\frac{1}{2}G^{0j}\frac{\partial F_{0j}}{\partial x^k}+\frac{1}{2}G^{i0}\frac{\partial F_{i0}}{\partial x^k}\\&+\frac{1}{2}G^{uv}\frac{\partial}{\partial x^k}\left(\mu G_{uv}\right),\forall i,\ j,k\in[0,3],u,v\in[1,3].\end{aligned} \tag{15}$$

Replacing the symbols of (8 e, f) into (15), we shall get:

$$\begin{aligned}(f_k)_{12}&=\frac{\partial}{\partial x^j}\left(G^{ij}F_{ki}\right)+\frac{1}{2}c^2\varepsilon_0\varepsilon_r F^{0j}\frac{\partial F_{0j}}{\partial x^k}\\&+\frac{1}{2}c^2\varepsilon_0\varepsilon_r F^{i0}\frac{\partial F_{i0}}{\partial x^k}+\frac{1}{2}G^{uv}\frac{\partial}{\partial x^k}\left(\mu G_{uv}\right),\\&\forall i,\ j,k\in[0,3],u,v\in[1,3].\end{aligned} \tag{16}$$

We are now going to calculate the components of f_k which, according to the types of the included electromagnetic field state quantities, can be of the following types: electric, magnetic, mixed.

In order to facilitate the understanding of the formulae, we shall successively use the tensor notation and the vector notation. We shall use for indices numbers, instead of letters, because it is easier to perform the computation and to avoid the use of the summation convention when not allowed. Then, the indices will be subscripts. We adopt $k=3$. We shall not write the terms containing factors of the form F_{uu} or G^{uu}, because these factors are equal to zero.

We shall express the electric component considering expression (15). We take into account the relation:

$$-\varepsilon F^{0\mathrm{u}}\frac{\partial F_{\mathrm{u}0}}{\partial x}=-\frac{\partial}{\partial x}\left(\frac{1}{2}\varepsilon F_{\mathrm{u}0}^2\right)+\frac{1}{2}F_{u0}^2\frac{\partial\varepsilon}{\partial x}. \tag{17 a}$$

The electric component will be:

$$\begin{aligned}(f_k)_{\mathrm{elec}}&=\frac{\partial}{\partial x^v}\left(G^{0v}F_{k0}\right)+\frac{1}{2}c^2\varepsilon F^{0v}\frac{\partial F_{0v}}{\partial x^k}\\&+\frac{1}{2}c^2\varepsilon F^{u0}\frac{\partial F_{u0}}{\partial x^k}=\frac{\partial}{\partial x_1}\left(G^{01}F_{30}\right)\\&+\frac{\partial}{\partial x_2}\left(G^{02}F_{30}\right)+\frac{\partial}{\partial x_3}\left(G^{03}F_{30}\right)-c^2\varepsilon F^{0v}\frac{\partial F_{v0}}{\partial x^k}\\&=\frac{\partial}{\partial x_1}\left(\varepsilon E_1E_3\right)+\frac{\partial}{\partial x_2}\left(\varepsilon E_2E_3\right)+\frac{\partial}{\partial x_3}\left(\varepsilon E_3E_3\right)\\&-\frac{1}{2}\frac{\partial}{\partial x_3}\left(\varepsilon E^2\right)+\frac{1}{2}E^2\frac{\partial\varepsilon}{\partial x_3},\quad\forall u,v\in[1,3],\end{aligned} \tag{17 b}$$

and:

$$\begin{aligned}(f_k)_{\mathrm{elec}}&=\frac{\partial}{\partial x^v}\left(G^{0v}F_{k0}\right)-\frac{1}{2}\frac{\partial}{\partial x^k}\left(G^{0v}F_{v0}\right)\\&+\frac{1}{2}\left(G^{0v}F_{v0}\right)\frac{\partial\varepsilon}{\partial x^k},\quad\forall v\in[1,3].\end{aligned} \tag{17 c}$$

Then, we shall express the magnetic component considering expression (15). In this respect, we shall take into consideration the following relation:

$$\begin{aligned}&\frac{1}{2}G^{uv}\frac{\partial}{\partial x}\left(\mu G_{uv}\right)=\frac{1}{4}\frac{\partial}{\partial x}\left(\mu G^{uv}G_{uv}\right)\\&\quad+\frac{1}{4}\left(G^{uv}G_{uv}\right)\frac{\partial\mu}{\partial x},\\&\frac{1}{2}G^{uv}\frac{\partial}{\partial x}\left(F_{uv}\right)=\frac{1}{4}\frac{\partial}{\partial x}\left(G^{uv}F_{uv}\right)\\&\quad+\frac{1}{4}\left(G^{uv}F_{uv}\right)\frac{\partial\mu}{\partial x},\quad\forall u,v\in[1,3].\end{aligned} \tag{18 a}$$

The magnetic component is:

$$\begin{aligned}(f_k)_{\mathrm{mag}}&=\frac{\partial}{\partial x^v}\left(G^{uv}F_{ku}\right)+\frac{1}{2}\frac{\partial}{\partial x^k}\left(\mu G^{uv}G_{uv}\right)\\&+\frac{1}{2}\left(G^{uv}G_{uv}\right)\frac{\partial\mu}{\partial x^k}\\&=\frac{\partial}{\partial x_1}\left(G^{u1}F_{3u}\right)+\frac{\partial}{\partial x_2}\left(G^{u2}F_{3u}\right)+\frac{\partial}{\partial x_3}\left(G^{u3}F_{3u}\right)\\&+\frac{1}{2}\frac{\partial}{\partial x^k}\left(G^{uv}F_{uv}\right)+\frac{1}{2}\left(G^{uv}G_{uv}\right)\frac{\partial\mu}{\partial x^k}\\&=\frac{\partial}{\partial x_1}\left(\mu H_3H_1\right)+\frac{\partial}{\partial x_2}\left(\mu H_3H_2\right)\\&-\frac{\partial}{\partial x_3}\left[\mu\left(H_2H_2+H_1H_1\right)\right]\\&+\frac{1}{2}\frac{\partial}{\partial x_3}\left(\mu H^2\right)+\frac{1}{2}\left(H^2\right)\frac{\partial\mu}{\partial x_3},\end{aligned} \tag{18 b}$$

$\forall u,v\in[1,3]$, and $u<v$ in the products of the form $G^{uv}G_{uv}$ or $G^{uv}F_{uv}$. There follows:

$$\begin{aligned}(f_k)_{\mathrm{mag}}&=\frac{\partial}{\partial x_1}\left(\mu H_3H_1\right)+\frac{\partial}{\partial x_2}\left(\mu H_3H_2\right)\\&-\frac{\partial}{\partial x_3}\left[\mu(H_2^2+H_1^2)\right]+\frac{1}{2}\frac{\partial}{\partial x_3}\left(\mu H^2\right)+\frac{1}{2}H^2\frac{\partial\mu}{\partial x^3}\\&=\frac{\partial}{\partial x_1}\left(\mu H_3H_1\right)+\frac{\partial}{\partial x_2}\left(\mu H_3H_2\right)+\frac{\partial}{\partial x_3}\left(\mu H_3H_3\right)\\&-\frac{\partial}{\partial x_3}\left[\mu(H_1^2+H_2^2+H_3^2)\right]\\&+\frac{1}{2}\frac{\partial}{\partial x_3}\left(\mu H^2\right)+\frac{1}{2}H^2\frac{\partial\mu}{\partial x_3},\end{aligned} \tag{18 c}$$

and

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$$\begin{aligned}\left(f_k\right)_{\mathrm{mag}} &= \frac{\partial}{\partial x_1}\left(\mu H_3 H_1\right) + \frac{\partial}{\partial x_2}\left(\mu H_3 H_2\right)\\ &+ \frac{\partial}{\partial x_3}\left(\mu H_3 H_3\right) - \frac{1}{2}\frac{\partial}{\partial x_3}\left(\mu H^2\right) + \frac{1}{2}H^2\frac{\partial \mu}{\partial x_3}\end{aligned} \tag{18 d}$$

or in tensor form:

$$\begin{aligned}\left(f_k\right)_{\mathrm{mag}} &= \frac{\partial}{\partial x^v}\left(\mu G^{uv} G_{ku}\right) + \frac{1}{2}\frac{\partial}{\partial x^k}\left(\mu G^{uv} G_{uv}\right)\\ &+ \frac{1}{2}\left(G^{uv} G_{uv}\right)\frac{\partial \mu}{\partial x^k},\end{aligned} \tag{18 e}$$

$\forall u, v \in [1, 3]$, and $u < v$ in the products of the form $G^{uv} G_{uv}$.

Returning to previous letter indices, and summing side by side relations (17 b), (18 d) and (8 d), we get:

$$\begin{aligned}\left(f_k\right)_{\mathrm{elmag}} &= \frac{\partial}{\partial x_v}\left(\varepsilon E_k E_v\right) - \frac{1}{2}\frac{\partial}{\partial x_k}\left(\varepsilon E^2\right)\\ &+ \frac{1}{2}E^2\frac{\partial \varepsilon}{\partial x_k} + \frac{\partial}{\partial x_v}\left(\mu H_k H_v\right)\\ &- \frac{1}{2}\frac{\partial}{\partial x_k}\left(\mu H^2\right) + \frac{1}{2}H^2\frac{\partial \mu}{\partial x_k}\\ &- \frac{1}{2}E^2\frac{\partial \varepsilon}{\partial x_k} - \frac{1}{2}H^2\frac{\partial \mu}{\partial x_k}.\end{aligned} \tag{19}$$

We now express the mixed components considering the first term of expression (15) and the expression (8 c). The mixed components are given by:

$$\begin{aligned}\left(f_k\right)_{\mathrm{mix1}} &= \frac{\partial}{\partial x^0}\left(G^{u0} F_{ku}\right)\\ &= \frac{\partial}{\partial x^0}\left(G^{10} F_{31}\right) + \frac{\partial}{\partial x^0}\left(G^{20} F_{32}\right)\\ &= \frac{\partial}{\partial x^0}\left(-cD^1 B_2\right) + \frac{\partial}{\partial x^0}\left(cD^2 B_1\right)\\ &= -\frac{\partial}{\partial t}\left(D_1 B_2 - D_2 B_1\right)\\ &= -\frac{\partial}{\partial t}\left(\boldsymbol{D}\times\boldsymbol{B}\right)_3, \quad \forall u \in [1, 3],\end{aligned} \tag{20 a}$$

and similarly:

$$\left(f_k\right)_{\mathrm{mix2}} = \frac{\partial}{\partial x^0}\left(G^{0u} F_{ku}\right) = \frac{\partial}{\partial t}\left(\boldsymbol{D}\times\boldsymbol{B}\right)_3, \quad \forall u \in [1, 3]. \tag{20 b}$$

Returning to previous letter indices, we get:

$$\left(f_k\right)_{\mathrm{mix2}} = \frac{\partial}{\partial x^0}\left(D^i B_j - D^j B_i\right). \tag{21}$$

Adding up relations (19), (20 a) as well as (8 d), which has still not been used, we obtain:

$$\begin{aligned}f_k &= \frac{\partial}{\partial x_v}\left(\varepsilon E_k E_v\right) - \frac{1}{2}\frac{\partial}{\partial x_k}\left(\varepsilon E^2\right)\\ &+ \frac{\partial}{\partial x_v}\left(\mu H_k H_v\right) - \frac{1}{2}\frac{\partial}{\partial x_k}\left(\mu H^2\right)\\ &- \frac{\partial}{\partial x_0}\left(D_i B_j - D_j B_i\right), \quad \forall i, j, k \in [1, 3].\end{aligned} \tag{22}$$

By summing up, side by side relations (17 c), (18 e), (20 a), we can write:

$$\begin{aligned}\left(f_k\right)_{\mathrm{elmag}} &= \frac{\partial}{\partial x^v}\left(G^{0v} F_{k0}\right) + \frac{1}{2}\frac{\partial}{\partial x^k}\left(G^{v0} F_{v0}\right)\\ &+ \frac{1}{2}\left(G^{0v} F_{v0}\right)\frac{\partial \varepsilon}{\partial x^k} + \frac{\partial}{\partial x^v}\left(G^{uv} F_{ku}\right)\\ &+ \frac{1}{2}\frac{\partial}{\partial x^k}\left(G^{uv} F_{uv}\right) + \frac{1}{2}\left(G^{uv} F_{uv}\right)\frac{\partial \mu}{\partial x^k}\\ &+ \frac{\partial}{\partial x^0}\left(G^{u0} F_{ku}\right), \quad \forall u, v \in [1, 3],\end{aligned} \tag{23 a}$$

and after summing up with relation (8 d), we get:

$$\begin{aligned}\left(f_k\right)_{\mathrm{sum}} &= \frac{\partial}{\partial x^v}\left(G^{0v} F_{k0}\right) + \frac{1}{2}\frac{\partial}{\partial x^k}\left(G^{u0} F_{u0}\right)\\ &+ \frac{\partial}{\partial x^v}\left(G^{uv} F_{ku}\right) + \frac{1}{2}\frac{\partial}{\partial x^k}\left(G^{uv} F_{uv}\right)\\ &+ \frac{\partial}{\partial x^0}\left(G^{v0} F_{kv}\right), \quad \forall u, v \in [1, 3], \quad u < v.\end{aligned} \tag{23 b}$$

If we started from formula (1 b) we would have added also in the right hand side of relation (23 b) the expression (20 b) and then the final relation would differ.

The force relation may be written in a compact form as follows:

$$\begin{aligned}f_k &= \frac{\partial}{\partial x^j}\left(G^{ij} F_{ki}\right) + \frac{1}{2}\frac{\partial}{\partial x^k}\left(G^{uv} F_{uv}\right),\\ &\forall i, j, k \in [0, 3], \forall u, v \in [0, 3], u < v\end{aligned} \tag{24 a}$$

or in a more compact form as follows:

$$\begin{aligned}f_k &= \frac{\partial}{\partial x^j}\left(G^{ij} F_{ki} + \frac{1}{2}\delta_k^j G^{uv} F_{uv}\right),\\ &\forall i, j, k \in [0, 3], \forall u, v \in [0, 3], u < v.\end{aligned} \tag{24 b}$$

Finally, the component of the volume density of the force along the *k*-axis can be expressed as:

$$f_k = \frac{\partial}{\partial x^j} W_k^j, \tag{25}$$

where the expression:

$$W_k^j = \left(G^{ij} F_{ki}\right) + \delta_k^j \frac{1}{2}\left(G^{uv} F_{uv}\right), \quad \forall i, j, k \in [0, 3], u, v \in [0, 3], u < v, \tag{26}$$

represents *the energy-momentum tensor*, also called *tensor of energy and quantity of motion*.

Remarks

1° If the media were not assumed isotropic and had not linear electric and magnetic polarization, the transformation from relations (17 a) and (18 a), respectively, will be no longer possible.

2° Having established the expression of the tensor in one reference frame, we can obtain its expression in any other one. The calculation is to be performed by using the group of co-ordinate transformations, for instance the Lorentz transformations. We consider useful to make the following remark. The Lorentz transformation group has been established for empty space (vacuum), and the involved light velocity is that in vacuo. In the present case, we consider that polarization exists, and in this case, also all transformations of the quantities are like those established by Minkowski. But a doubt appears, namely if the transformations are still valid because in any media the velocity of light is different. *For this reason, the Lorentz transformation group may be considered as an assumption that is so better the smaller will be the space regions filled with substance.*

IV. Expression of the Energy-Momentum Tensor

1° Component W_0^0. Using formula (26), and after performing the calculation, passing from tensor notation to vector notation, we get:

$$\begin{aligned}
W_0^0 &= G^{i0} F_{0i} + \frac{1}{2}\delta_0^0 G^{uv} F_{uv} = c^2 \varepsilon_r \varepsilon_0 F_{q0} F_{q0} \\
&+ \frac{1}{2} G^{0s} F_{0s} + \frac{1}{2} \cdot \frac{1}{\mu_0 \mu_r} F^{qs} F_{qs}; \\
&\varepsilon_r \varepsilon_0\, c F_{q0} c F_{q0} = D_q E_q; \\
&\frac{1}{2} G^{0s} F_{0s} + \frac{1}{2} \cdot \frac{1}{\mu_0 \mu_r} F^{qs} F_{qs} \\
&= -\frac{1}{2} \sum_{s=1}^{s=3} \left(D_s E_s - H_s B_s\right); \\
&\forall q, s \in [1, 3], q < s; \\
W_0^0 &= W^{00} = \frac{1}{2}\left(\varepsilon \boldsymbol{E}^2 + \mu \boldsymbol{H}^2\right),
\end{aligned} \tag{27}$$

which represents the volume density of the electromagnetic energy, and the quantities E_i, D_i, H_i, B_i are considered as three-dimensional vector components.

2° Component W_k^j for both cases $k \neq j$ and $k = j$. We use, as above, formula (26), and after performing the calculation, we shall pass from tensor notation to vector notation. In the first case, remarking that *j* and *k* are different, we should keep only the first term of expression (26). We get:

$$\begin{aligned}
W_k^j &= G^{0j} F_{k0} + G^{ij} F_{ki} \\
&= c^2 \varepsilon_0 \varepsilon_r F_{j0} F_{k0} + G^{ij} F_{ki}, \quad \forall i \in [1, 3]; \\
&\varepsilon_0 \varepsilon_r\, c F_{j0} c F_{k0} = E_j D_k = E_k D_j; \\
&G^{ij} F_{ki} = H_k B_j = H_j B_k; \\
W_k^j &= -W^{kj} = E_j D_k + H_j B_k.
\end{aligned} \tag{28 a}$$

In the second case, for more clarity, instead of letter indices, we shall use number indices, considering a certain case, namely for $j = k = 2$:

$$\begin{aligned}
W_2^2 &= G^{02} F_{20} + G^{12} F_{21} + G^{32} F_{23} \\
&+ \frac{1}{2}\left(G^{0s} F_{0s} + G^{qs} G_{qs}\right) \\
&= c^2 \varepsilon_0 \varepsilon_r F^{02} F_{20} + G^{12} F_{21} + G^{32} F_{23} \\
&+ G^{qs} F_{qs}, \quad \forall q, s \in [1, 3]; q < s; \\
&\varepsilon_r \varepsilon_0\, c F^{02} c F_{20} = E_2 D_2; \quad G^{12} F_{21} + G^{32} F_{23} \\
&= -H_3 B_3 - H_1 B_1; \\
&G^{0s} F_{0s} + G^{qs} G_{qs} = -E_1 D_1 - E_2 D_2 - E_3 D_3 \\
&+ H_1 B_1 + H_2 B_2 + H_3 B_3; \\
W_2^2 &= -W^{22} \\
&= E_2 D_2 - H_3 B_3 - H_1 B_1 + H_2 B_2 - H_2 B_2 \\
&+ \frac{1}{2}(-E_1 D_1 - E_2 D_2 - E_3 D_3) \\
&+ \frac{1}{2}(+H_1 B_1 + H_2 B_2 + H_3 B_3);
\end{aligned} \tag{28 b}$$

and:

$$W_2^2 = -W^{22} = E_2 D_2 + H_2 B_2 - \frac{1}{2}\left(\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{H} \cdot \boldsymbol{B}\right). \tag{28 c}$$

The reason, for which we put numbers instead of letter indices, has been to avoid the usage of summation convention when not allowable. The results above, expressed by relations (28 a) and (28 c), represent the Maxwell stress tensors.

3° Component W_0^j. As previously we shall use, formula (26), and after performing the calculation, we pass from tensor notation to vector notation. We begin with one example and then express the general form. There follows:

$$\begin{aligned}
W_0^2 &= G^{i2} F_{0i} = G^{12} F_{01} + G^{32} F_{03} \\
&= H^{12}\left(-\frac{1}{c} E_1\right) + \left(-H^{23}\right)\left(-\frac{1}{c} E_3\right); \\
W_0^2 &= \frac{1}{c}\left(E_3 H_1 - E_1 H_3\right); \quad W_0^j = W^{j0} \\
&= \frac{1}{c}\left(E_k H_i - E_i H_k\right);
\end{aligned} \tag{29}$$

and the general form is as expected:

$$W_0^j = W^{j0} = \frac{1}{c}\left(E_k H_i - E_i H_k\right) = \frac{1}{c}\left(\boldsymbol{E}\times\boldsymbol{H}\right)_j \tag{30}$$

which apart the denominator c, represents the j-component of Poynting vector, i.e., the rate of the radiated flux of energy per unit of surface and unit of time.

4° The force along the time axis. We use formula (24 a) or (24 b), and after performing the calculation, we pass from tensor notation to vector notation. There follows:

$$\begin{aligned} f_0 &= \frac{\partial}{\partial x^j}\left(G^{ij}F_{0i}\right) + \frac{1}{2}\frac{\partial}{\partial x^k}\left(G^{uv}F_{uv}\right), \\ &\forall i,\ j \in [0, 3],\ k = 0, \quad u, v \in [0, 3],\ u < v, \end{aligned} \tag{31}$$

and:

$$\begin{aligned} f_0 &= \frac{\partial}{\partial x^0}\left(G^{i0}F_{0i}\right) + \frac{\partial}{\partial x^1}\left(G^{i1}F_{0i}\right) \\ &+ \frac{\partial}{\partial x^2}\left(G^{i2}F_{0i}\right) + \frac{\partial}{\partial x^3}\left(G^{i3}F_{0i}\right) \\ &+ \frac{1}{2}\frac{\partial}{\partial x^k}\left(G^{uv}F_{uv}\right), \\ &\forall i \in [0, 3],\ k = 0, \quad u, v \in [0, 3],\ u < v. \end{aligned} \tag{32}$$

Calculating the first and second parentheses, we get:

$$\begin{aligned} G^{i0}F_{0i} &= c^2\varepsilon_r\varepsilon_0 F^{i0}F_{0i} \\ &= -c^2\varepsilon_r\varepsilon_0\left(-F_{i0}F_{i0}\right) = \boldsymbol{E}\cdot\boldsymbol{D}; \\ G^{i1}F_{0i} &= G^{21}F_{02} + G^{31}F_{03} = -H^{12}\left(-\frac{1}{c}E_2\right) \\ &+ H^{31}\left(-\frac{1}{c}E_3\right) \\ &= \frac{1}{c}\left(E_2H_3 - E_3H_2\right) = \frac{1}{c}\left(\boldsymbol{E}\times\boldsymbol{H}\right)_1; \end{aligned} \tag{33 a}$$

where, for $i = 2$, $G^{i1} = G^{21} = H^{21}$, the vector $\boldsymbol{H}$ being contravariant, and according to relation (6 *l*), $H^{21} = -H^{12} = -H_3$, since $H_1 = H^{23}$. Calculating the derivatives of the first two parentheses, we get:

$$\begin{aligned} \frac{\partial}{\partial x^0}\left(G^{i1}F_{0i}\right) &= \frac{\partial}{\partial x^0}\left(\boldsymbol{E}\cdot\boldsymbol{D}\right); \\ \frac{\partial}{\partial x^1}\left(G^{i1}F_{0i}\right) &= \frac{\partial}{\partial x^1}\frac{1}{c}\left(\boldsymbol{E}\times\boldsymbol{H}\right)_1. \end{aligned} \tag{33 b}$$

Handling similarly the next two parentheses and summing up all terms, there follows:

$$\begin{aligned} f_0 &= \frac{1}{c}\operatorname{div}\left(\boldsymbol{E}\times\boldsymbol{H}\right) + \frac{\partial}{\partial x^0}\left(\boldsymbol{E}\cdot\boldsymbol{D}\right) + \frac{1}{2}\frac{\partial}{\partial x^0}\left(G^{uv}F_{uv}\right), \\ &\forall u, v \in [0, 3],\ u < v. \end{aligned} \tag{34}$$

For the last parenthesis we obtain:

$$\begin{aligned} G^{uv}F_{uv} &= G^{0r}F_{0r} + G^{rs}F_{rs} = -D_r E_r + H_q B_q \\ &= -\boldsymbol{D}\cdot\boldsymbol{E} + \boldsymbol{H}\cdot\boldsymbol{B}, \\ &\forall u, v \in [0, 3],\ u < v, \quad r, s, q \in [1, 3],\ r < s. \end{aligned} \tag{35}$$

Replacing (35) into (34), we get:

$$\begin{aligned} f_0 &= \frac{1}{c}\operatorname{div}\left(\boldsymbol{E}\times\boldsymbol{H}\right) + \frac{\partial}{\partial x_0}\left(\boldsymbol{E}\cdot\boldsymbol{D}\right) \\ &- \frac{1}{2}\frac{\partial}{\partial x_0}\left(\boldsymbol{E}\cdot\boldsymbol{D}\right) + \frac{1}{2}\frac{\partial}{\partial x_0}\left(\boldsymbol{H}\cdot\boldsymbol{B}\right), \end{aligned} \tag{36}$$

and hence:

$$f_0 = \frac{1}{c}\left[\operatorname{div}\left(\boldsymbol{E}\times\boldsymbol{H}\right) + \boldsymbol{E}\cdot\frac{\partial\boldsymbol{D}}{\partial t} + \boldsymbol{H}\cdot\frac{\partial\boldsymbol{B}}{\partial t}\right], \tag{37}$$

where we replace the derivative with respect to time with the known Maxwell relations, as follows:

$$f_0 = \frac{1}{c}\left[\operatorname{div}\left(\boldsymbol{E}\times\boldsymbol{H}\right) + \boldsymbol{E}\cdot\left(\operatorname{curl}\boldsymbol{H} - \boldsymbol{J}\right) - \boldsymbol{H}\cdot\operatorname{curl}\boldsymbol{E}\right]. \tag{38}$$

We get:

$$\begin{aligned} f_0 &= \frac{1}{c}\left[\operatorname{div}\left(\boldsymbol{E}\times\boldsymbol{H}\right) - \operatorname{div}\left(\boldsymbol{E}\times\boldsymbol{H}\right) - \boldsymbol{J}\cdot\boldsymbol{E}\right] \\ &= -\frac{1}{c}\boldsymbol{J}\cdot\boldsymbol{E}, \end{aligned} \tag{39}$$

which represents the component of the force along the time axis.

The set f_i represents a four-vector, according to formulae (24)-(26), or (1a) and (39), indeed the product of the set f_i and the four-vector velocity yields a scalar.

V. Conclusion

The aim of this paper has been to establish the expression of the energy-momentum tensor within the frame of the Theory of Relativity, starting from the general formula of the electromagnetic force acting on a substance submitted to an electromagnetic field. The case of linear non-homogeneous media has been examined.

This subject has not been treated in the known papers or works published so far. Meanwhile, the analysis carried out has shown that no all-general known formulae are in agreement with the tensor energy momentum expression when passing from a system of reference to another one. If the media were not assumed as isotropic and had not linear electric and magnetic polarization the deduction carried out for obtaining the tensor would not be possible.

The expression of the tensor established in one system of reference can be obtained in any other system of reference owing to the group of Lorentz transformation and the Minkowski transformation formulae using this group. However, a doubt appears because the velocity of light in any media is different, and the Lorentz transformation has been established for this case.

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List of Symbols

- A_i – component of the four-vector potential;
- B_{ij} – twice covariant tensor component of magnetic induction, yielding B_k;
- B_k – component of the magnetic induction along axis k, considered as a usual three-dimensional vector;
- c – velocity of light in empty space, supposed to be constant;
- D_i – component of the electric displacement, considered as a usual three-dimensional vector;
- D^i – contravariant component of the electric displacement yielding D_k or D_i considered as a usual three-dimensional vector;
- E_i – covariant component of the electric field strength, as well as component of the electric field strength along axis i as a usual three-dimensional vector;
- E^i – contravariant component of the electric field strength;
- e_{ii} – axis coefficient, for the axis ***i*** of the Galilean reference frame;
- F_{ij} – component of the covariant tensor of rank 2, yielding B_{ij} for i and j non-zero;
- F_{i0} – component of a covariant tensor, deriving from the previous one, and yielding the component E_i of the electric field strength, considered as a usual three-dimensional vector;
- f_i – four-vector of the volume density of the electromagnetic field;
- G^{ij} – contravariant tensor of rank 2, yielding H^{ij};
- G_{i0}, G^{i0} – covariant and contravariant tensor, deriving from the previous one, and yielding the component D^i;
- H_k – component of the magnetic field strength along axis k, considered as a usual three-dimensional vector;
- J^i – component of a contravariant four-vector represents the density of the conduction electric current;
- V – electric potential;
- x^i – co-ordinate along axis i;
- δ_{ij} – symbol equal to unity for equal indices, and equal to zero for different ones (Kronecker symbol);
- ε – electric permittivity, *in vacuo* it is ε_0;
- μ – magnetic permeability, *in vacuo* it is μ_0;
- ρ_v – volume density of the electric charge.

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