Proceedings of the World Congress on Engineering 2009 Vol II
WCE 2009, July 1 - 3, 2009, London, U.K.

Parallel Alternating Group Explicit Iterative Method For Hyperbolic Equations

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Abstract—**In this paper, we present a four order unconditionally stable implicit scheme for hyperbolic equations. Based on the scheme and the concept of decomposition a class of parallel alternating group explicit (AGE) iterative method is derived, and convergence analysis is given. In order to verify the AGE iterative method, we give an example at the end of the paper.**

Keywords: hyperbolic equations, parallel computation, iterative method, alternating group

1 Preface

In scientific and engineering computing, we need to solve large equation set by numerical methods. Considering the stability and accuracy of explicit schemes and the computation difficulty of implicit schemes, it is necessary to construct methods with the advantages of explicit methods and implicit methods, that is, simple for computation and good stability. Recently with the development of parallel computer many scientists payed much attention to the finite difference methods with the property of parallelism. D. J. Evans presented an AGE method in [1] originally. The AGE method is used in computing by applying the special combination of several asymmetry schemes to a group of grid points, and then the numerical solutions at the group of points can be denoted explicitly. Furthermore, by alternating use of asymmetry schemes at adherent grid points and different time levels, the AGE method can lead to the property of unconditional stability. The AGE method is soon applied to hyperbolic equations [2] and other problems [3-5]. But most of the developed AGE method has only two order accurate for spatial step.

In this paper, we will consider the initial boundary value problem of hyperbolic equations, and organize the rest of this paper as follows:

In section 2, we present a four order accurate unconditionally stable implicit scheme for hyperbolic equations. Based on the scheme we construct a class of parallel AGE iterative method. Convergence analysis is given in section 3. Comparison of numerical examples with the original AGE method in [2] and the full implicit scheme are presented in section 4.

2 The AGE Iterative Method

We consider the following initial boundary value problem of 1D hyperbolic equations:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2},\ 0 \le x \le 1,\ 0 \le t \le T \\ u(x,0) = f(x), 0 \le x \le 1 \\ \dfrac{\partial u(x,0)}{\partial t} = \phi(x), 0 \le x \le 1 \\ u(0,t) = e_1(t),\ u(1,t) = e_2(t). \end{cases} \tag{1}$$

The domain $\Omega : [0,1] \times [0,T]$ will be divided into $(m \times N)$ meshes with spatial step size h=$\frac{1}{m}$ in x direction and the time step size τ=$\frac{T}{N}$. Grid points are denoted by (x_i, t_n) or $(i,\ n)$, $x_i = ih(i = 0,\ 1, \cdots, m)$, $t_n = n\tau(n = 0,\ 1, \cdots, \frac{T}{\tau})$. The numerical solution of (1) is denoted by u_i^n, while the exact solution $u(x_i,\ t_n)$. Let $r = \frac{\tau^2}{h^2}$.

We present an implicit finite difference scheme with parameters for solving (1) as below:

$$\xi_1 \frac{u_{i-1}^{n+1} - 2u_{i-1}^n + u_{i-1}^{n-1}}{\tau^2} + \xi_2 \frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\tau^2}$$

$$+\xi_3 \frac{u_{i+1}^{n+1} - 2u_{i+1}^n + u_{i+1}^{n-1}}{\tau^2} = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2} \tag{2}$$

Considering $\frac{\partial^{2k} u}{\partial t^{2k}} = \frac{\partial^{2k} u}{\partial x^{2k}}$, applying Taylor formula to the scheme at (x_i, t_n) we have

$$(\xi_1 + \xi_2 + \xi_3)\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + (\xi_1 - \xi_3)h\frac{\partial^3 u}{\partial x^3}$$

$$+(\frac{\xi_1}{2} + \frac{\xi_3}{2} - \frac{1}{12})h^2 \frac{\partial^4 u}{\partial x^4} = O(\tau^2 + h^4)$$

Let

$$\begin{cases} \xi_1 + \xi_2 + \xi_3 = 1 \\ \xi_1 - \xi_3 = 0 \\ \frac{\xi_1}{2} + \frac{\xi_3}{2} - \frac{1}{12} = 0 \end{cases}$$

that is $\xi_1 = \xi_3 = \frac{1}{12},\ \xi_2 = \frac{5}{6}$.

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Then we denote (2) as the fourth order center-difference scheme:

$$\frac{1}{12}\delta_t^2 u_{i-1}^n + \frac{5}{6}\delta_t^2 u_i^n + \frac{1}{12}\delta_t^2 u_{i+1}^n = \delta_x^2 u_i^n \qquad (3)$$

here

$$\delta_t^2 u_i^n = \frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\tau^2}$$

$$\delta_x^2 u_i^n = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2}$$

It is obvious that the truncation error of the scheme (3) is $O(\tau^2 + h^4)$.

Let $U^n = (u_1^n, u_2^n, \cdots, u_{m-1}^n)^T$, then we rewrite (3) as:

$$KU^{n+1} = E^n$$

Here $E^n = E_1 U^n + E_2 U^{n-1} + [-u_0^{n-1} + (2+12r)u_0^n, 0, \cdots, 0, -u_m^{n-1} + (2+12r)u_m^n]^T$.

$$K = \begin{pmatrix} 10 & 1 & & & \\ 1 & 10 & 1 & & \\ & \cdots & \cdots & \cdots & \\ & & 1 & 10 & 1 \\ & & & 1 & 10 \end{pmatrix}$$

$$E_2 = \begin{pmatrix} -10 & -1 & & & \\ -1 & -10 & -1 & & \\ & \cdots & \cdots & \cdots & \\ & & -1 & -10 & -1 \\ & & & -1 & -10 \end{pmatrix}$$

$$E_1 = \begin{pmatrix} 20-24r & 2+12r & & & \\ 2+12r & 20-24r & 2+12r & & \\ & \cdots & \cdots & \cdots & \\ & & 2+12r & 20-24r & 2+12r \\ & & & 2+12r & 20-24r \end{pmatrix}$$

K, E_1, E_2 are all $(m-1)\times(m-1)$ matrices.

In order to solve U^{n+1} with U^n and U^{n-1} known, we will try to construct an AGE iterative method so as to avoid solving an implicit equation set.

Let k denotes the iterative number. First we present two iterative computing groups, in which computation can be finished independently.

"$\kappa 1$"group: four inner grid points are involved.

Let $\widetilde{U}_i^n = (u_i^n, u_{i+1}^n, u_{i+2}^n, u_{i+3}^n)^T$,
$\widetilde{U}_{i(k)}^n = (u_i^{n(k)}, u_{i+1}^{n(k)}, u_{i+2}^{n(k)}, u_{i+3}^{n(k)})^T$, then we have

$$(\rho I + H_{11})\widetilde{U}_{i(k+1)}^{n+1} = (\rho I - H_{22})\widetilde{U}_{i(k)}^{n+1} + B_1\widetilde{U}_i^n + C_1\widetilde{U}_i^{n-1} + \widehat{E}_i^n \qquad (5)$$

here

$$\widehat{E}_i^n = (-u_{i-1}^{n-1} + (2+12r)u_{i-1}^n, 0, 0, -u_{i+4}^{n-1} + (2+12r)u_{i+4}^n)^T$$

$$H_{11} = \begin{pmatrix} 10 & 1 & 0 & 0 \\ 1 & 10 & 2 & 0 \\ 0 & 2 & 10 & 1 \\ 0 & 0 & 1 & 10 \end{pmatrix}$$

$$H_{22} = \begin{pmatrix} H_{21} & \\ & H_{21} \end{pmatrix}, \ H_{21} = \begin{pmatrix} 10 & 1 \\ 1 & 10 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} 20-24r & 2+12r & 0 & 0 \\ 2+12r & 20-24r & 2+12r & 0 \\ 0 & 2+12r & 20-24r & 2+12r \\ 0 & 0 & 2+12r & 20-24r \end{pmatrix}$$

$$C_1 = \begin{pmatrix} -10 & -1 & 0 & 0 \\ -1 & -10 & -1 & 0 \\ 0 & -1 & -10 & -1 \\ 0 & 0 & -1 & -10 \end{pmatrix}$$

Then the numerical solution of $\widetilde{U}_{i(k+1)}^{n+1}$ at grid nodes $(i, n+1)$, $(i+1, n+1)$, $(i+2, n+1)$, $(i+3, n+1)$ can be obtained in "$\kappa 1$"group as below:

$$\widetilde{U}_{i(k+1)}^{n+1} = (\rho I + H_{11})^{-1}[(\rho I - H_{22})\widetilde{U}_{i(k)}^{n+1} + B_1\widetilde{U}_i^n + C_1\widetilde{U}_i^{n-1} + \widehat{E}_i^n] \qquad (6)$$

"$\kappa 2$"group: two inner grid points are involved. Let $\overline{U}_i^n = (u_i^n, u_{i+1}^n)^T$, $\overline{U}_{i(k)}^n = (u_i^{n(k)}, u_{i+1}^{n(k)})^T$, then we have

$$(\rho I + H_{21})\overline{U}_{i(k+1)}^{n+1} = (\rho I - H_{21})\overline{U}_{i(k)}^{n+1} + B_2\overline{U}_i^n + C_2\overline{U}_i^{n-1} + \overline{E}_i^n \qquad (7)$$

here

$$\overline{E}_i^n = (-u_{i-1}^{n-1} + (2+12r)u_{i-1}^n, -u_{i+2}^{n-1} + (2+12r)u_{i+2}^n)^T$$

$$B_2 = \begin{pmatrix} 20-24r & 2+12r \\ 2+12r & 20-24r \end{pmatrix}, \ C_2 = \begin{pmatrix} -10 & -1 \\ -1 & 10 \end{pmatrix}$$

The numerical solution of $\overline{U}_{i(k+1)}^{n+1}$ at grid nodes $(i, n+1), (i+1, n+1)$ can be denoted as below:

$$\overline{U}_{i(k+1)}^{n+1} = (\rho I + H_{21})^{-1}[(\rho I - H_{21})\overline{U}_{i(k)}^{n+1} + B_2\overline{U}_i^n + C_2\overline{U}_i^{n-1} + \overline{E}_i^n] \qquad (8)$$

Let $m = 4a+1$, a is an integer, $U_k^{n+1} = (u_1^{n+1(k)}, u_2^{n+1(k)}, \cdots, u_{m-1}^{n+1(k)})^T$, then we construct the alternating group explicit (AGE) iterative method as below:

$$\begin{cases} (\rho I + H_1)U_{k+\frac{1}{2}}^{n+1} = (\rho I - H_2)U_k^{n+1} + \widetilde{E}^n \\ (\rho I + H_2)U_{k+1}^{n+1} = (\rho I - H_1)U_{k+\frac{1}{2}}^{n+1} + \widetilde{E}^n \end{cases} \quad k = 0, 1, \cdots \qquad (4)$$

$$H_1 = diag(H_{11}, \cdots, H_{11})_{(m-1)\times(m-1)}$$

$$H_2 = diag(H_{21}, H_{11}, \cdots, H_{11}, H_{21})_{(m-1)\times(m-1)}$$

ISBN:978-988-18210-1-0

$\widetilde{E}^n = 2E^n$, k is the iterative number, ρ is the iterative parameter.

From the construction of H_1 and H_2 in (4), we arrive to the conclusion that all the computation can be fulfilled in two separate groups "$\kappa 1$" and "$\kappa 2$". By alternating use of $\kappa 1$ group and $\kappa 2$ group, we can divide the computation in the whole domain into many sub-domains, and computation in each sub-domain can be finished independently and simultaneously. So the AGE iterative method (4) is suitable for parallel computation.

3 Convergence Analysis For The AGE Iterative Method

Lemma 1[6] Let $\theta > 0$, and $G + G^T$ is nonnegative, then $(\theta I + G)^{-1}$exists, and

$$\|(\theta I + G)^{-1}\|_2 \leq \theta^{-1} \quad (9)$$

Lemma 2[6] On the conditions of Lemma 1, we have:

$$\|(\theta I - G)(\theta I + G)^{-1}\|_2 \leq 1 \quad (10)$$

Theorem 1 The AGE iterative method (4) is convergent for any $\rho > 0$.

Proof: From the construction of the matrices we can see that $H_1,\ H_2,\ (H_1 + H_1^T),\ (H_2 + H_2^T)$ are all nonnegative matrixes. Then from lemma 3 we have

$$\|(\rho I - H_1)(\rho I + H_1)^{-1}\|_2 \leq 1,\ \|(\rho I - H_2)(\rho I + H_2)^{-1}\|_2 \leq 1$$

From (4), we obtain $U_{k+1}^{n+1} = HU_k^{n+1} + (\rho I + H_2)^{-1}[(\rho I - H_1)(\rho I + H_1)^{-1}\widetilde{E}^n + \widetilde{E}^n]$. Here $H = (\rho I + H_2)^{-1}(\rho I - H_1)(\rho I + H_1)^{-1}(\rho I - H_2)$ is the growth matrix.

Let $\widetilde{H} = (\rho I + H_2)H(\rho I + H_2)^{-1} = (\rho I - H_1)(\rho I + H_1)^{-1}(\rho I - H_2)(\rho I + H_2)^{-1}$, then $\rho(H) = \rho(\widetilde{H}) \leq \|\widetilde{H}\|_2 \leq 1$, which shows the AGE iterative method given by (4) is convergent.

4 Numerical Experiments

Example :

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2},\ 0 \leq x \leq 1,\ 0 \leq t \leq T \\ u(x,0) = 0, \\ \dfrac{\partial u(x,0)}{\partial t} = sin(\pi x), 0 \leq x \leq 1 \\ u(0,t) = 0,\ u(1,t) = 0. \end{cases} \quad (11)$$

The exact solution of (11) is denoted as $u(x,t) = sin\pi t sin(\pi x)$.

Considering the scheme (2) is a three time level method, we can approach U^1 with the following difference scheme so as to avoid loss of accuracy.

$$\begin{cases} \dfrac{1}{12}\dfrac{u_{i-1,j}^1 - 2u_{i-1,j}^0 + u_{i-1,j}^{-1}}{\tau} + \dfrac{5}{6}\dfrac{u_i^1 - 2u_i^0 + u_i^{-1}}{\tau} \\ +\dfrac{1}{12}\dfrac{u_{i+1}^1 - 2u_{i+1}^0 + u_{i+1}^{-1}}{\tau} = \dfrac{u_{i+1}^0 - 2u_i^0 + u_{i-1}^0}{h^2} \\ \qquad \dfrac{u_i^1 - u_i^{-1}}{2\tau} = \phi(x_i), i = 1, 2, \cdots, m-1 \end{cases}$$

Let $||E_1||_\infty = max|u_i^n - u(x_i,t_n)|$, $||E_2||_\infty = max|(u_i^n - u(x_i,t_n))/u(x_i,t_n)|$, $i = 1, 2, \cdots, m-1$. We use the iterative error 1×10^{-10} to control the process of iterativeness. Let A.I.T. denotes the average iterative times, then we compare the numerical results of the proposed AGE iterative method in this paper with the alternating group explicit iterative method by Evans [2] and the results from the fourth order center-difference scheme (3) in Table 1 and Table 2. Let t_1/t_2 denote the ratio of execution time between the AGE method and the fourth order center-difference scheme (3).

Table 1: Results of comparison $m = 17, \tau = 10^{-3},\ \rho = 1$

	$t = 100\tau$	$t = 200\tau$	$t = 500\tau$
A.I.T.	50.49	50.54	48.68
A.I.T.[2]	81.52	81.75	80.63
$\|\|E_1\|\|_\infty$	2.978×10^{-4}	4.683×10^{-4}	2.475×10^{-5}
$\|\|E_1\|\|_\infty$[2]	3.104×10^{-2}	1.243×10^{-2}	7.638×10^{-3}
$\|\|E_2\|\|_\infty$	9.677×10^{-2}	4.336×10^{-2}	2.527×10^{-3}
$\|\|E_2\|\|_\infty$[2]	4.258	9.491×10^{-1}	6.573×10^{-1}
t_1/t_2	0.246	0.253	0.261

Table 2: Results of comparison $m = 17, \tau = 10^{-4},\ \rho = 1$

	$t = 1000\tau$	$t = 2000\tau$	$t = 5000\tau$
A.I.T.	43.96	43.76	41.83
A.I.T.[2]	74.58	74.67	72.32
$\|\|E_1\|\|_\infty$	3.999×10^{-5}	6.689×10^{-5}	2.279×10^{-5}
$\|\|E_1\|\|_\infty$[2]	1.426×10^{-3}	2.371×10^{-3}	9.896×10^{-4}
$\|\|E_2\|\|_\infty$	1.348×10^{-2}	1.186×10^{-2}	2.301×10^{-3}
$\|\|E_2\|\|_\infty$[2]	8.734×10^{-1}	8.518×10^{-1}	7.964×10^{-2}
t_1/t_2	0.255	0.267	0.315

From above we can see that the numerical solution for the proposed AGE method can converge to the exact solution faster and are of higher accuracy than the alternating group explicit iterative method by Evans [2]. Furthermore, for its intrinsic parallelism, the presented AGE method can shorten the running CPU time in comparison with the fully implicit scheme, and the effect becomes obvious when the amount of grid points increases.

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