A Padé-Chebyshev Reconstruction of Functions with Jump Discontinuities

Arnel L. Tampos * Jose Ernie C. Lope †

Abstract—**The Gibbs phenomenon poses a major obstruction in reconstructing a function with finite jumps by a partial Fourier sum or truncated orthogonal series expansion. In this paper, we present a Padé-type approach to treat the problem. To better reconstruct a function, we utilize an amplified Padé-type approximation that takes into account the jump discontinuities of the function. The effectiveness of the proposed method is measured in terms of the Gibbs constant and steepness of the approximant at the location of the discontinuity. We present numerical examples to demonstrate the reconstruction.**

Keywords: Gibbs phenomenon, function reconstruction, Padé-type approximation

1 Introduction

It is well-known that the use of a partial Fourier sum or a truncated orthogonal polynomial expansion in general provides a convergent and highly accurate approximation for smooth functions [1, 3]. Sadly their suitability for nonsmooth functions is adversely challenged by the presence of any discontinuity. It is shown in [3] that for a piecewise continuous function f, the convergence of the Nth partial sum of the Fourier or an orthogonal series is non-uniform with a rate of $\mathcal{O}(1)$ in the vicinity the jump locations; elsewhere in the domain, convergence is slow with a rate of $\mathcal{O}(\frac{1}{N})$. This convergence problem has come to be known as the *Gibbs phenomenon.* Graphically, this phenomenon is characterized by overshoots and undershoots (oscillatory behavior) of the approximant at the vicinity of the jumps. Figure 1 describes this scenario for the case of the sawtooth function defined by

$$S(x) = \begin{cases} x-\pi, & 0 \le x \le \pi \\ x+\pi, & -\pi \le x < 0 \end{cases}$$

approximated by the 50th partial Fourier sum.

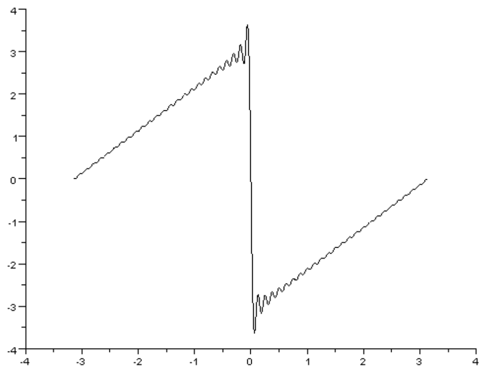

Figure 1: Partial Fourier sum approximation of $S(x)$.

The *Gibbs constant* is defined as the maximum overshoot or undershoot of the approximant. The *steepness*, given by the approximant's derivative at the point of discontinuity, measures the approximant's ability to reproduce the discontinuity. Generally, for an Nth partial Fourier sum or a truncated series approximant, the Gibbs constant is about 9% of the magnitude of the jump and its steepness is $\frac{4}{\pi}(N+1)$ [4]. Strangely, however, increasing the number of terms in the series does not diminish the amplitude of the overshoot although its interval of occurrence gets smaller.

The task of remediating the Gibbs phenomenon has been the subject of many studies (e.g., [1, 3, 5, 6]). The method ranges from modifying the truncated series through filtering [2], to using a rational function (Padé-type) approximant [1, 5, 6]. Some Padé-type methods operate in the absence of the knowledge of jump locations. These methods argue that poles of the approximants are close enough to the singularities of the function. However, realizing that poles do not adequately reproduce the jump behavior, Driscoll and Fornberg [1] developed the Singular Fourier-Padé (SFP) method of correcting the Gibbs phenomenon which incorporate the singularities of the function into the process. Following their lead, we derive a similar approach using a transformed Chebyshev series and assimilate their concept into a Padé-Chebyshev approximation.

*Division of Physical Sciences and Mathematics, University of the Philippines in the Visayas, Miag-ao, Iloilo, Philippines. Email: altampos@yahoo.com
†Institute of Mathematics, University of the Philippines Diliman, Quezon City, Philippines. Email: ernie@math.upd.edu.ph

ISBN:978-988-18210-1-0

2 Singular Fourier-Padé Correction of the Gibbs phenomenon

Consider a function f with formal power series $\sum_{k=0}^{\infty} c_k x^k$ and a rational function defined by $R_{(N,M)} = P_N/Q_M$, where P_N and Q_M are the polynomials

$$P_N(x) = \sum_{i=0}^{N} p_i x^i, \quad Q_M(x) = \sum_{j=0}^{M} q_j x^j \neq 0.$$

We say that $R_{(N,M)}$ is the *(linear) Padé approximant* of order (N, M) to the formal series if

$$Q_M(x)f(x) - P_N(x) = \mathcal{O}\left(x^{N+M+1}\right) \quad (\text{as } x \to 0).$$

Finding the approximant $R_{(N,M)}$ involves determining the coefficients of polynomials P_N and Q_M through the following linear system:

$$\sum_{j=0}^{M} c_{N-j+k} q_j = 0, \quad k = 1, \ldots, M \tag{2.1}$$

$$\sum_{j=0}^{k} c_{k-j} q_j = p_k, \quad k = 1, \ldots, N. \tag{2.2}$$

For this system to be well-determined, we usually employ a normalization by setting, say, $q_0 = 1$.

Let f be a piecewise analytic function defined over $[-\pi, \pi)$ with s jump locations at $x = \xi_k \in [-\pi, \pi)$, $k = 1, \ldots, s$. The complex Fourier series of f is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

The transformation $z = e^{ix}$ which maps the interval $[-\pi, \pi)$ into the unit circle in the complex plane transforms the Fourier series into the following Laurent series in z which can split into

$$\begin{aligned} f(z) &= \sum_{n=-\infty}^{\infty} c_n z^n = {\sum_{n=0}^{\infty}}' c_n z^n + {\sum_{n=0}^{\infty}}' c_{-n} z^{-n} \\ &= f^+(z) + f^-\left(z^{-1}\right), \end{aligned}$$

where the primed sums indicate that the zeroth term should be halved.

The Fourier-Padé (FP) approximation of $f^{\pm}$ consists in looking for polynomials $P_N^{\pm}(z)$ and $Q_M^{\pm}(z)$ such that

$$P_N^{\pm}(z) - Q_M^{\pm}(z) f^{\pm}(z) = \mathcal{O}\left(z^{N+M+1}\right) \quad (\text{as } z \to 0).$$

The resulting approximant is then defined as

$$\frac{P_N^+(z)}{Q_M^+(z)} + \frac{P_N^-(z^{-1})}{Q_M^-(z^{-1})}.$$

However, Driscoll and Fornberg noted that this approximant does not reproduce very well the jumps of the function. They then suggested that every jump in value of f at $x = \xi$ can be attributed to a logarithm of the form

$$\log\left(1 - \frac{z}{e^{i\xi}}\right). \tag{2.3}$$

This logarithmic singularity in $f^{\pm}$, which is difficult for the Padé approximant to simulate, can be exploited to enhance the approximation process. This is the rationale behind the Singular Fourier-Padé (SFP) method, introduced in [1], which modifies the FP approach to satisfy the following condition

$$P_N^{\pm}(z) + L = f^{\pm}(z) Q_M^{\pm}(z) + \mathcal{O}\left(z^{\eta+1}\right). \tag{2.4}$$

Here,

$$L = \sum_{k=1}^{s} R_k^{\pm}(z) \log\left(1 - \frac{z}{e^{i\xi_k}}\right)$$

for some polynomials R_k, $k = 1, \ldots, s$, and η is determined by s and the degrees of P_N, Q_M, and the R_k's. The unknown coefficients of these polynomials are obtained through the linear system arising from (2.4) which is similar in structure to the system (2.1)–(2.2).

3 Padé-Chebyshev Reconstruction

Let $x \in [-1, 1]$ and n be a nonnegative integer. The Chebyshev polynomials of the first kind, denoted by T_n, are defined as $T_n(x) = \cos(n\theta)$, $\theta \in [0, \pi]$ and $\theta = \arccos x$. These polynomials are orthogonal in the interval $[-1, 1]$ with respect to the weight function $\omega(x) = \frac{1}{\sqrt{1-x^2}}$, that is,

$$\int_{-1}^{1} T_m(x) T_n(x) \frac{1}{\sqrt{1-x^2}} dx = \begin{cases} 0, & n \neq m \\ \frac{\pi}{2}, & n = m \neq 0 \\ \pi, & n = m = 0 \end{cases}.$$

Define the orthogonal expansion of f in Chebyshev by

$$f(x) = {\sum_{n=0}^{\infty}}' c_n T_n(x),$$

where the coefficients c_n are given by the formula

$$c_n = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x) T_n(x)}{\sqrt{1-x^2}} dx. \tag{3.1}$$

When deriving c_n analytically using this formula becomes unwieldy, inexact expansion coefficients may be obtained by estimating the integral through the following Gauss-Chebyshev quadrature rule

$$\int_{-1}^{1} h(x)\omega(x) dx \cong \sum_{k=1}^{m} A_k h\left(x_k\right), \tag{3.2}$$

where $\{x_k\}$ are the zeros of the Chebyshev polynomials $T_m(x)$, $h(x) = f(x)T_n(x)$, $\omega(x) = \frac{1}{\sqrt{1-x^2}}$, and $A_k = \frac{\pi}{m}$ for all k.

ISBN:978-988-18210-1-0

By the definition of the Chebyshev polynomials, we may express the Chebyshev expansion of f as

$$\begin{aligned} f(x) &= \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\theta), \qquad \theta = \cos^{-1}(x) \\ &= \frac{c_0}{2} + \frac{1}{2}\sum_{n=1}^{\infty} c_n \left(e^{in\theta} + e^{-in\theta}\right), \end{aligned}$$

since $\cos(n\theta) = \frac{1}{2}\left(e^{in\theta} + e^{-in\theta}\right)$. Via the tranformation $z = e^{i\theta}$, the Chebyshev expansion of f is transformed into the following Laurent expansion in z

$$f(z) = \frac{c_0}{2} + \frac{1}{2}\sum_{n=1}^{\infty} c_n \left(z^n + z^{-n}\right)$$

which can be rewritten as

$$f(z) = \frac{1}{2}\left(\sum_{n=0}^{\infty}{}' c_n z^n + \sum_{n=0}^{\infty}{}' c_n z^{-n}\right).$$

Letting

$$g(z) = \sum_{n=0}^{\infty}{}' c_n z^n, \tag{3.3}$$

we have $f(z) = \frac{1}{2}\big(g(z) + g(z^{-1})\big)$. As the Chebyshev coefficients c_n's are real, $g\left(z^{-1}\right) = \overline{g(z)}$. Thus,

$$f(z) = \frac{1}{2}\left(g(z) + \overline{g(z)}\right) = \Re e\,(g(z)). \tag{3.4}$$

We refer to $g(z)$ in (3.3) as the *transformed Chebyshev series associated with* $f(x)$.

Let $f(x)$ be piecewise analytic over $[-1,1]$ with s jump locations at $x = \xi_k \in [-1,1]$, $k = 1,\dots,s$. Consider the transformed Chebyshev series $g(z)$ associated with $f(x)$. The transformation $z = e^{i\theta}$ suggests that the jump representation in (2.3) translates into the form

$$\log\left(1 - \frac{z}{e^{i\theta_k}}\right), \tag{3.5}$$

where $0 \le \theta_k = \cos^{-1}(\xi_k) \le \pi$. Consequently, we may modify (2.4) into

$$\frac{P_N(z) + \sum_{k=1}^{s} R_{V_k}(z)\log\left(1 - \frac{z}{e^{i\theta_k}}\right)}{Q_M(z)} = g(z) + \mathcal{O}\left(z^{\eta+1}\right),$$

where

$$\begin{aligned} P_N(z) &= \sum_{j=0}^{N} p_j z^j, \quad Q_M(z) = \sum_{j=0}^{M} q_j z^j \ne 0, \\ R_{V_k}(z) &= \sum_{j=0}^{V_k} r_j^{(k)} z^j, \quad k = 1,\dots,s, \\ \eta &= N + M + s + \sum_{k=1}^{s} V_k. \end{aligned}$$

We call the modified SFP approximant just defined the **Singular Padé-Chebyshev (SPC) approximant** to $g(z)$ whose real part is the SPC approximant to $f(x)$. Denote this approximant by SPC$(N, M, V_1, \dots, V_k)$. The unknown coefficients of polynomials P,Q, and R_{V_k}, $k = 1,\dots,s$, are then computed through the following linear system:

$$\sum_{j=0}^{M}{}^{*} c_{N-j+t} q_j - \sum_{j=0}^{V_1} a^{(1)}_{N-j+t} r^{(1)}_j - \cdots - \sum_{j=0}^{V_s} a^{(s)}_{N-j+t} r^{(s)}_j = 0$$

$$\sum_{j=0}^{M}{}^{*} c_{l-j} q_j - \sum_{j=0}^{V_1} a^{(1)}_{l-j} r^{(1)}_j - \cdots - \sum_{j=0}^{V_s} a^{(s)}_{l-j} r^{(s)}_j = p_l.$$

In the above, $t = 1,\dots,\eta - N$, $l = 0,\dots,N$, and the asterisk-marked summation indicates that the term with c_0 is halved, and for all $n < 0$, $c_n = 0$. It should be noted that the $a_n^{(k)}$'s are the coefficients in the Taylor's expansion of (3.5), and for all $n \le 0$, $a_n^{(k)} = 0$.

4 Numerical Results

We demonstrate the SPC reconstruction for these two test functions:

1. signum function $f_1(x) = \begin{cases} 1, & 0 \le x \le 1 \\ -1, & -1 \le x < 0 \end{cases}$
2. absolute value function $f_2(x) = |x|, \quad x \in [-1,1]$

4.1 Reconstructing the Signum Function

The exact Chebyshev expansion coefficients of f_1 are

$$c_n = \begin{cases} \frac{4}{n\pi}(-1)^{\frac{n-1}{2}}, & n = 2k+1,\ k \ge 0 \\ 0, & \text{otherwise.} \end{cases} \tag{4.1}$$

Since f_1 has a jump discontinuity at $x = 0$ that corresponds to $\theta = \frac{\pi}{2}$, its SPC approximant is determined by

$$\frac{P(z) + R(z)\log\left(1 - \frac{z}{i}\right)}{Q(z)}.$$

Figure 2 exhibits a remarkable reconstruction of f_1 by the SPC (5,5,7) approximant where the Gibbs phenomenon is practically eliminated. This reflects the significant effect of incorporating the singularity of f_1 into the process. A closer look at the approximation near the jump is shown in Figure 3.

For comparison purposes, let us also plot the Padé-Chebyshev (PC) approximant $\frac{P_N}{Q_M}$ denoted by PC (N, M) and a truncated Chebyshev (Cheb) approximant P_N denoted by Cheb (N). Among the three, SPC approximant provides the best fit for f_1 as shown in Figure 4. Furthermore, Figure 5 shows that there is a drastic drop in the

Table 1: The Gibbs constant of PC and SPC approximants to f_1 using exact and inexact expansion coefficients.

	Gibbs Constant	
Approximant	Exact	Inexact
PC (5,5)	0.0585387	0.0606156
SPC (5,5,0)	0.0158730	0.0268526
SPC (5,5,2)	0.0009775	0.0269967
SPC (5,5,5)	0.0000027	0.0717451
SPC (5,5,7)	0.0000494	0.0051746
SPC (5,5,10)	0.0000511	0.0006276

Table 2: The steepness of PC and SPC approximants to f_1 using exact and inexact expansion coefficients.

	Steepness	
Approximant	Exact	Inexact
PC (5,5)	19.096826	19.316588
SPC (5,5,0)	2030.9154	2052.5977
SPC (5,5,2)	2001.8786	2052.5868
SPC (5,5,5)	1999.8222	2138.5651
SPC (5,5,7)	2000.0926	2009.2583
SPC (5,5,10)	2000.0963	1789.9199

pointwise error for the SPC case especially at the vicinity of the jump.

Shown partially in Figure 6 is the SPC (5,5,7) approximant to f_1 with inexact expansion coefficients obtained using the Gauss-Chebyshev quadrature with $m = 100$. The gap reveals a considerable difference between the SPC (5,5,7) approximants generated using exact versus inexact coefficients. Further clarification is provided by their pointwise error plots in Figure 7.

The fitness of the approximant may be gauged by its Gibbs constant and steepness; it is desirable for the Gibbs constant to be as small as possible and for the steepness to be as high as possible. Although specific to the approximants mentioned, Tables 1 and 2 reveal a sense of superiority of the SPC over the PC approximants in terms of their Gibbs constant and steepness. Between the two cases of exact and inexact coefficients, however, results are expectedly favorable in the exact case. But as shown in Figure 6 for instance, under the restriction of having just the estimated coefficients, those approximants in the inexact case may be good enough. Generally, our results, particularly that of the exact case, are far better in comparison with those obtained in [4] for the sign function wherein a Padé-type approximant of order (N, M) has the Gibbs constant of only 0.008149 with a steepness of about 35.682482 for the approximant (5,5).

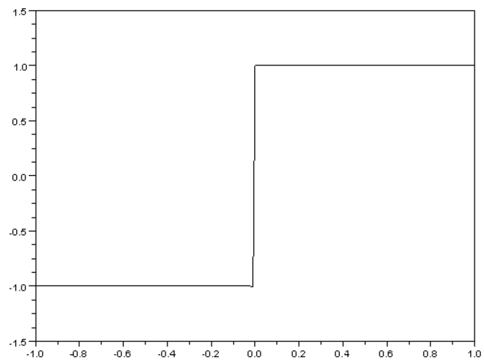

Figure 2: The SPC (5,5,7) approximant of f_1.

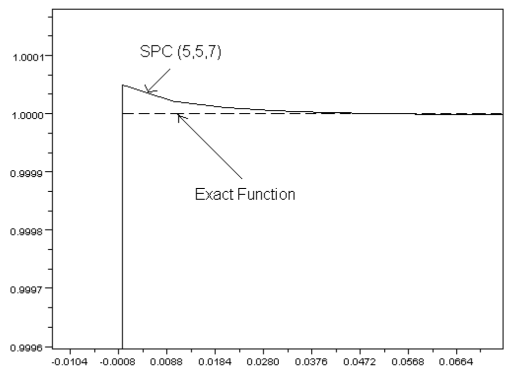

Figure 3: The graph of SPC (5,5,7) near the jump.

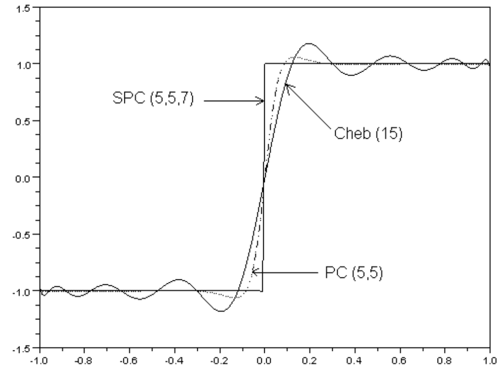

Figure 4: Cheb (15), PC (5,5), and SPC (5,5,7) approximants to f_1.

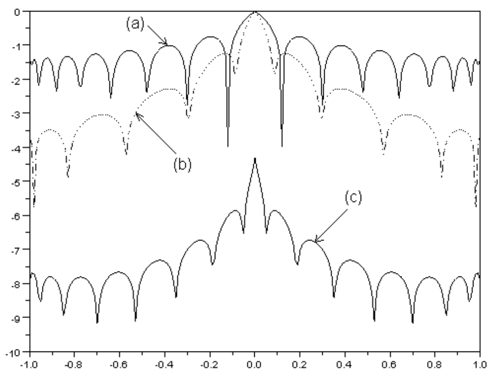

Figure 5: Convergence of pointwise error, in logarithmic scale, of (a) Cheb (15), (b) PC (5,5), and (c) SPC (5,5,7) approximants to f_1.

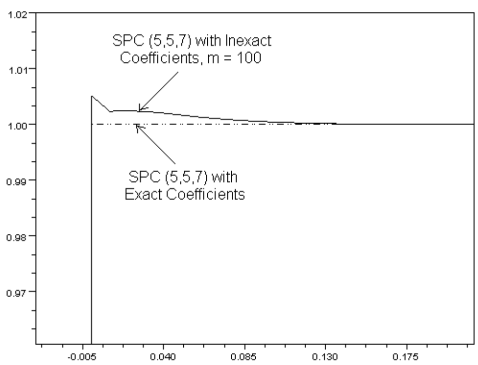

Figure 6: The difference between f_1's SPC (5,5,7) approximants with exact and inexact coefficients near the jump location.

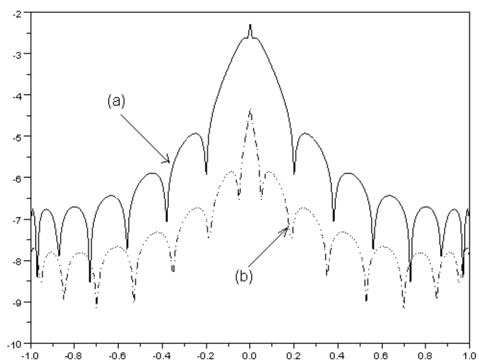

Figure 7: Convergence of pointwise error, in logarithmic scale, of SPC (5,5,7) approximants to f_1 using (a) inexact and (b) exact coefficients.

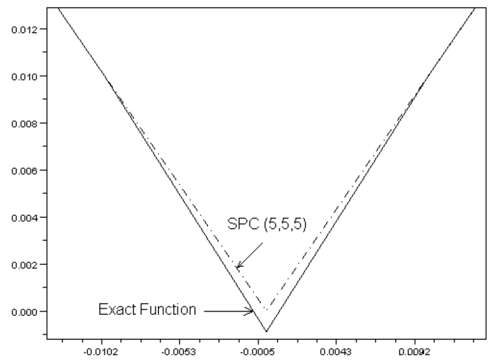

Figure 8: Graphs of f_2 and its SPC (5,5,5) approximant near the jump.

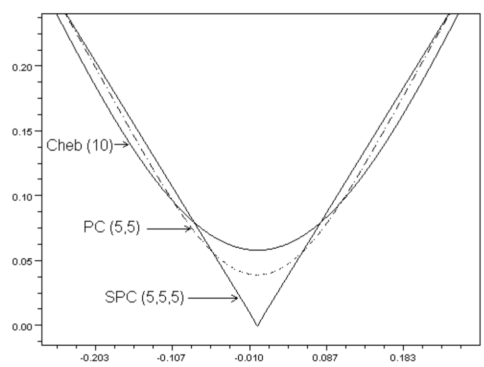

Figure 9: C(10), PC(5,5), and SPC(5,5,5) approximants to f_2 near the jump.

4.2 Reconstructing f_2

Although f_2 is continuous at $x = 0$, it has a first order jump at that point. Its SPC approximant has the same form as that of f_1. Experiments show that the SPC method reconstructs f_2 very well over the entire domain. We exhibit in Figure 8 a portion of the SPC (5,5,5) approximant near the jump location that gives a better view of its slight deviation from the actual function. In Figure 9, we display a comparison of Cheb (10), PC (5,5), and SPC (5,5,5) approximations and note how the pointwise errors in Figure 10 highlight the remarkable performance of the SPC approximant.

The maximum overshoot of the SPC (5,5,5) approximant as seen in Figure 8 measures to 0.0008842 which is much better than that of PC (5,5) approximant which is just 0.0107996.

A comparison near the jump location of the SPC (5,5,5) approximants using exact versus inexact expansion coef-

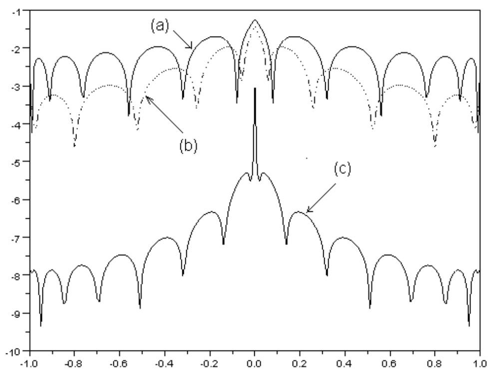

Figure 10: Convergence of pointwise error, in logarithmic scale, of (a) Cheb (10), (b) PC (5,5), and (c) SPC (5,5,5) approximants to f_2.

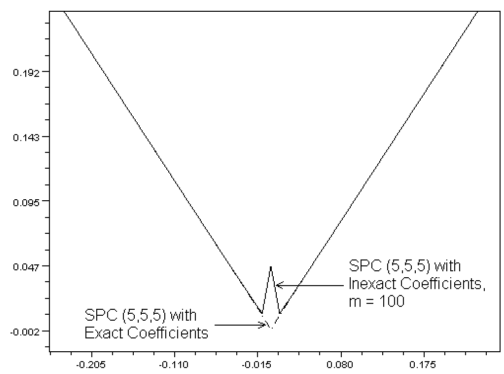

Figure 11: SPC (5,5,5) approximant to f_2 with inexact coefficients shoots up at the jump.

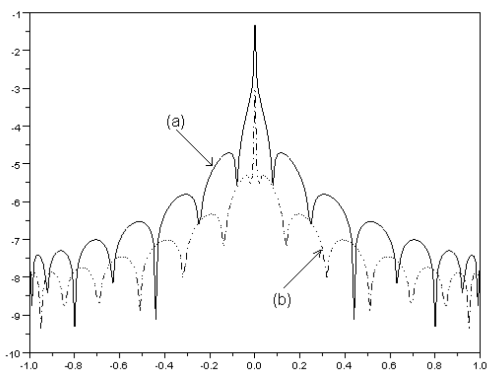

Figure 12: Convergence of pointwise error, in logarithmic scale, of the SPC (5,5,5) approximants to f_2 using (a) inexact and (b) exact coefficients.

ficients is shown in Figure 11. The maximum overshoot here is 0.0466143. The error plots displayed in Figure 12 clearly show the differences of the two approximants in terms of accuracy.

5 Conclusion

The Singular Padé-Chebyshev (SPC) method presented in this paper is shown to remarkably reduce the Gibbs phenomenon and effectively reconstruct functions with jump discontinuity. As reflected in the SPC approximant's Gibbs constant and steepness, our method significantly improves convergence particularly at the vicinity of the jump. While the use of SPC approximants computed with exact expansion coefficients results in outstanding function reconstruction, we remark that approximants computed with inexact coefficients offer a potentially good alternative in situations where exact coefficients are unavailable.

References

[1] Driscoll, T.A., Fornberg, B., *A Pade-based algorithm for overcoming the Gibbs phenomenon*, Numerical Algorithms, 26 (2001), no.1, pp.77 - 92

[2] Jerri, A.J., *The Gibbs Phenomenon in Fourier Analysis, Splines and Wavelet Approximation*, Kluwer Dordrecht, 1998

[3] Hesthaven, J.S., Gottlieb, S., Gottlieb, D., *Spectral Methods for Time-Dependent Problems*, Cambridge University Press, U.K., 2007

[4] Németh, G., Páris, G., *The Gibbs phenomenon in generalized Padé approximation*, Journal of Mathematical Physics, 26 (1985)

[5] Don, W.S., Kaber, S.M., Min, M.S., *Fourier-Padé approximations and filtering for the spectral simulations of incompressible Boussinesq convection problem*, Mathematics of Computation, 76 (2007), pp. 1275-1290

[6] Hesthaven, J.S., Kaber, S.M., Lurati, L., *Padé-Legendre interpolants for Gibbs reconstruction*, Journal of Scientific Computing, 28 (2006), pp. 337-359.