Common Fixed Point Iterations of a Finite Family of Quasi-nonexpansive Maps

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Abstract—**It is proved that Kuhfittig iteration process converges to a common fixed point of a finite family of quasi-nonexpansive maps on a Banach space. This result is extended to the random case. Our work improves upon several well-known results in the current literature.**

Keywords: ***Quasi-nonexpansive map, common fixed point, iteration process, Banach space, measurable space***

1 Introduction and Preliminaries

Approximation of fixed points of a quasi-nonexpansive map by iteration has been investigated in [8, 10, 16, 18]. Ghosh and Debnath [9] have approximated common fixed points of a finite family of quasi-nonexpansive maps in a uniformly convex Banach space. Rhoades [20] established weak convergence of Kuhfittig iteration scheme to a common fixed point of a finite family F of nonexpansive maps while Khan and Hussain [13] have obtained strong convergence of this scheme to a common fixed point of the family F on a nonconvex domain.

In this paper, we introduce an iteration process for any finite family of quasi-nonexpansive maps and study its convergence to a common fixed point of the family in a Banach space. We also provide a random version of this scheme and study its convergence.

Let C be a subset of a Banach space. A selfmap T of C is nonexpansive if $\|Tx-Ty\| \le \|x-y\|$, for all $x, y \in C$. We denote the set of all fixed points of T by $F(T)$. A generalization of a nonexpansive map with at least one fixed point is that of a quasi-nonexpansive map; T is quasi-nonexpansive if $\|Tx - p\| \le \|x - p\|$, for all $x \in C$ and all $p \in F(T)$. In general, a quasi-nonexpansive map may not be nonexpansive (see Dotson [6]). For various classes of quasi-nonexpansive maps and their strong connection with iterative methods, we refer to Berinde [4].

Let C be a convex set and $x_0 \in C$. Mann [17], in 1953, defined an iterative procedure as:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \tag{1.1}$$

where $\alpha_n \in [0, 1]$, $n = 0, 1, 2, \ldots$

Ishikawa [11], in 1974, devised an iteration scheme:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n,\ y_n = (1 - \beta_n)x_n + \beta_n T x_n, \tag{1.2}$$

where $\alpha_n, \beta_n \in [0, 1]$, $n = 0, 1, 2, \ldots$. If $\beta_n = 0$ for all n, then (1.2) becomes (1.1).

We introduce the Kuhfittig iteration scheme [14] as follows: Let $x_0 \in C$, $U_0 = I$ (identity map), $\alpha_n, \beta_{jn} \in (0, 1]$, $n = 0, 1, 2, \ldots$, $j = 1, 2, \ldots, k$,

$$\begin{aligned} U_1 &= (1 - \beta_{1n})I + \beta_{1n} T_1 U_0, \\ U_2 &= (1 - \beta_{2n})I + \beta_{2n} T_2 U_1, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ U_k &= (1 - \beta_{kn})I + \beta_{kn} T_k U_{k-1}, \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T_k U_{k-1}\, x_n . \end{aligned} \tag{1.3}$$

Indeed, if $k = 2$ and $T_1 = T_2 = T$ in (1.3), then we get the Ishikawa iteration (1.2).

We now state two useful conditions: A real sequence $\{\alpha_n\}$ is said to satisfy Condition A if $0 \le \alpha_n \le b < 1$, $n = 0, 1, 2, \ldots$, and $\sum_{n=0}^{\infty} \alpha_n = \infty$ (see [12]). The map $T : C \to C$ with $F(T) \ne \phi$ is said to satisfy Condition B if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that $\|x - Ty\| \ge f(d(x, F(T)))$ for $x \in C$ and all corresponding $y = (1 - t)x + tTx$, where $0 \le t \le \beta < 1$ and $d(x, F(T)) = \inf_{z \in F(T)} \|x - z\|$ (see [16]). Note that if $t = 0$, Condition B reduces to Condition I of Senter and Dotson, Jr. [21].

We need the following known results.

Lemma 1.1 [19, Lemma 2]. *If a sequence of numbers $\{a_n\}$ satisfies that $a_{n+1} \le a_n$ for all $n = 1, 2, \ldots$ and $\liminf_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} a_n = 0$.*

Theorem 1.2 [12, Theorem 1]. *Let C be a closed subset of a Banach space X, and T a nonexpansive map from*

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C into a compact subset of X. Suppose that there exists $\{\alpha_n\}$ satisfying the Condition A. If $\{x_n\}$ is defined by (1.1) with $x_n \in C$ for all n, then T has a fixed point in C and $\{x_n\}$ converges strongly to a fixed point of T.

Theorem 1.3 [16, Theorem 1]. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X, and T a quasi-nonexpansive selfmap of C satisfying the Condition B. Then, the Ishikawa iteration scheme (1.2), with $0 < a \leq \alpha_n \leq b < 1$ and $0 \leq \beta_n \leq \beta < 1$, converges strongly to a fixed point of T.*

Lemma 1.4 [5, Theorem 8.4]. *Let C be a bounded closed convex subset of a uniformly convex Banach space X, and $T : C \to X$ a nonexpansive map. Then:*

(i) If $\{x_n\}$ is a weakly convergent sequence in C with weak limit x_0 and if $(I - T)x_n$ converges strongly to y in X, then $(I - T)x_0 = y$.

(ii) $(I - T)(C)$ is a closed subset of X.

2 Convergence Theorems

Throughout this section, $F = \bigcap_{i=1}^{k} F(T_i)$ is assumed to be nonempty.

Theorem 2.1 *Let C be a nonempty closed convex subset of a Banach space X, and $\{T_i : i = 1, 2, \ldots, k\}$ a family of quasi-nonexpansive selfmaps of C. Then the sequence $\{x_n\}$ in (1.3) converges strongly to a common fixed point of the family if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.*

Proof. We will only prove sufficiency of the condition; the necessity is obvious. It can be shown by induction that $T_k U_{k-1}$ is quasi-nonexpansive. Let $z \in F$. Then

$$\begin{aligned} \|x_{n+1} - z\| &= \|(1 - \alpha_n)(x_n - z) + \alpha_n(T_k U_{k-1} x_n - z)\| \\ &\leq (1 - \alpha_n)\|x_n - z\| + \alpha_n\|T_k U_{k-1} x_n - z\| \\ &\leq (1 - \alpha_n)\|x_n - z\| + \alpha_n\|x_n - z\| \\ &= \|x_n - z\|. \end{aligned}$$

This implies that $d(x_{n+1}, F) \leq d(x_n, F)$ for all $n = 0, 1, 2, \ldots$ So, by Lemma 1.1 and $\liminf_{n\to\infty} d(x_n, F) = 0$, we get $\lim_{n\to\infty} d(x_n, F) = 0$.

Next we prove that $\{x_n\}$ is a Cauchy sequence. We have that

$$\|x_{n+m} - z\| \leq \|x_n - z\|, \tag{2.1}$$

for all $z \in F$ and $m, n = 0, 1, 2, \ldots$.

Since $\lim_{n\to\infty} d(x_n, F) = 0$, for each $\epsilon > 0$, there exists a natural number N_1 such that $d(x_n, F) \leq \frac{\epsilon}{3}$, for all $n \geq N_1$. Thus, there exists a $z_1 \in F$ such that

$$\|x_{N_1} - z_1\| = d(x_{N_1}, z_1) \leq \frac{\epsilon}{2} \tag{2.2}$$

From (2.1) and (2.2), for all $n \geq N_1$,

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - z_1\| + \|x_n - z_1\| \\ &\leq \|x_{N_1} - z_1\| + \|x_{N_1} - z_1\| \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Thus $\{x_n\}$ is a Cauchy sequence and so converges to $p \in X$.

Now we show that $p \in F$. For any $\overline{\epsilon} > 0$, there exists a natural number N_2 such that

$$\|x_n - p\| \leq \frac{\overline{\epsilon}}{4}, \quad \text{for all } n \geq N_2. \tag{2.3}$$

Again $\lim_{n\to\infty} d(x_n, F) = 0$ implies that there exists a natural number $N_3 \geq N_2$ such that $d(x_n, F) \leq \frac{\overline{\epsilon}}{12}$ for all $n \geq N_3$. Therefore there exists a $\overline{p} \in F$ such that

$$\|x_{N_3} - \overline{p}\| = d(x_{N_3}, \overline{p}) \leq \frac{\overline{\epsilon}}{8}. \tag{2.4}$$

From (2.3) and (2.4), we obtain, for any $T_i, i = 1, 2, \ldots, k$,

$$\|T_i p - p\| = \|T_i p - \overline{p} + \overline{p} - T_i x_{N_3} + T_i x_{N_3} - \overline{p} + \overline{p} - x_{N_3} + x_{N_3} - p\| \leq \overline{\epsilon}.$$

Since $\overline{\epsilon}$ is arbitrary, it follows that $T_i p = p$, $i = 1, 2, \ldots, k$. Thus $p \in F$. ■

Remark 2.2 (i) Theorem 2.1 is an extension of Corollary 1 of Qihou [19] for a family of quasi-nonexpansive maps.

(ii) If the family $\{T_i : i = 1, 2, \ldots, k\}$ is commutative, then the assumption $F \neq \phi$ may be omitted (see Theorem 4 in [7]).

In the sequel, we obtain some results for a family of maps $\{T_i : i = 1, 2, \ldots, k\}$ without the condition $\liminf_{n\to\infty} d(x_n, F) = 0$.

Theorem 2.3 *Let C be a nonempty compact convex subset of a strictly convex Banach space X, and $\{T_i : i = 1, 2, \ldots, k\}$ a family of nonexpansive selfmaps of C. Then the sequence $\{x_n\}$, in (1.3) with $\{\alpha_n\}$ satisfying Condition A and $\beta_{jn} = \beta_j$ for all n and $j = 1, 2, \ldots, k$, converges strongly to a common fixed point of the family.*

Proof. It is easy to show that U_j and $T_j U_{j-1}$, $j = 1, 2, \ldots, k$ are nonexpansive selfmaps of C, and the families $\{T_1, \ldots, T_k\}$ and $\{U_1, \ldots, U_k\}$ have the same set of common fixed points.

By Theorem 1.2, the sequence $\{x_n\}$ in (1.3) converges strongly to a fixed point y of T_kU_{k-1}. We show that y is a common fixed point of T_k and U_{k-1} $(k \geq 2)$. For this, we first show that $T_{k-1}U_{k-2}\, y = y$. Suppose not. Then the closed line segment $[y, T_{k-1}U_{k-2}\, y]$ has positive length. Let

$$z = U_{k-1}\, y = (1 - \beta_{(k-1)n})\, y + \beta_{(k-1)n} T_{k-1}U_{k-2}\, y.$$

Since $F \neq \phi$ and $\{T_1, \ldots, T_k\}$ and $\{U_1, \ldots, U_k\}$ have the same common fixed point set, $T_{k-1}U_{k-2}\, p = p$ for $p \in F$.

From the quasi-nonexpansiveness of T_kU_{k-2} and T_k,

$$\|T_{k-1}U_{k-2}\, y - p\| \leq \|y - p\| \tag{2.5}$$

and

$$\|T_k z - p\| \leq \|z - p\|.$$

In view of $T_k z = T_kU_{k-1}\, y = y$, it follows that $\|y - p\| \leq \|z - p\|$. As X is strictly convex, for noncollinear vectors a and b in X, we have $\|a + b\| < \|a\| + \|b\|$, which implies that

$$\begin{aligned} & \|y - p\| \leq \|z - p\| \\ = \;& \|(1 - \beta_{(k-1)n})\, y + \beta_{(k-1)n} T_{k-1}U_{k-2}\, y \\ & -(1 - \beta_{(k-1)n})\, p - \beta_{(k-1)n}\, p\| \\ < \;& (1 - \beta_{(k-1)n})\|y - p\| + \beta_{(k-1)n}\|T_{k-1}U_{k-2}\, y - p\|. \end{aligned}$$

So, we get

$$\|y - p\| < \|T_{k-1}U_{k-2}\, y - p\|$$

which contradicts (2.5). Hence, $T_{k-1}U_{k-2}\, y = y$. Subsequently,

$$U_{k-1}\, y = (1 - \beta_{(k-1)n})y + \beta_{(k-1)n} T_{k-1}U_{k-2}\, y = y$$

and

$$y = T_kU_{k-1}\, y = T_k\, y.$$

Thus, y is a common fixed point of T_k and U_{k-1}.

Since $T_{k-1}U_{k-2}\, y = y$, we may repeat the above procedure to show that $T_{k-2}U_{k-3}\, y = y$ and hence y must be a common fixed point of T_{k-1} and U_{k-2}. Continuing in this manner, we conclude that $T_1U_0\, y = y$ and y is a common fixed point of T_2 and U_1. Consequently, y is a common fixed point of $\{T_i : i = 1, 2, \ldots, k\}$. ■

Theorem 2.4 *Let C be a nonempty closed convex subset of a uniformly convex Banach space X, and $\{T_i : i = 1, 2, \ldots, k\}$ a family of quasi-nonexpansive selfmaps of C. Let $\{x_n\}$ be defined by (1.3) with $0 < a \leq \alpha_n \leq b < 1$ and $0 < \beta_{jn} \leq \beta < 1$. If the map T_kU_{k-1} satisfies the Condition B, then $\{x_n\}$ converges strongly to a common fixed point of the family.*

Proof. A uniformly convex space is strictly convex, so one can use the arguments of the proof of Theorem 2.3 with the exception that one employs Theorem 1.3 in lieu of Theorem 1.2. ■

Remark 2.5 Theorem 2.4 extends ([16], Theorem 1) and ([21], Theorems 1-2).

3 Random Iterative Procedures

Let $(\Omega, \sum)$ be a measurable space and C be a nonempty subset of a Banach space X. Let $\xi : \Omega \to C$ and $S, T : \Omega \times C \to X$. Then: (i) ξ is measurable if $\xi^{-1}(U) \in \sum$, for each open subset U of X; (ii) T is a random operator if, for each fixed $x \in C$, the map $T(., x) : \Omega \to X$ is measurable; (iii) ξ is a random fixed point of the random operator T if ξ is measurable and $T(\omega, \xi(\omega)) = \xi(\omega)$, for each $\omega \in \Omega$; (iv) ξ is a random common fixed point of S and T if ξ is measurable and for each $\omega \in \Omega$, $\xi(\omega) = S(\omega, \xi(\omega)) = T(\omega, \xi(\omega))$; (v) T is continuous (resp., nonexpansive) if the map $T(\omega, \cdot) : C \to X$ is continuous (resp., nonexpansive). A mapping $\xi : \Omega \to X$ is said to be a measurable selector of a mapping $T : \Omega \to CB(X)$, nonempty family of bounded and closed subsets of X, if ξ is measurable and for any $\omega \in \Omega$, $\xi(\omega) \in T(\omega)$.

The set of random fixed points of T will be denoted by $RF(T)$.

Proposition 3.1 [2, Proposition 3.4]. *Let C be a nonempty bounded closed convex subset of a separable Banach space X, and $T : \Omega \times C \to C$ a nonexpansive random operator. Suppose that $\{\xi_n\}$ is a sequence of maps from Ω to C defined by*

$$\xi_{n+1}(\omega) = (1 - \alpha)\xi_n(\omega) + \alpha T(\omega, \xi_n(\omega)), \textit{ for each } \omega \in \Omega, \tag{3.1}$$

where $0 < \alpha < 1, n = 1, 2, 3, \ldots,$ and $\xi_1 : \Omega \to C$ is an arbitrary measurable map. Then for each $\omega \in \Omega$, $\lim\limits_{n\to\infty} \|\xi_n(\omega) - T(\omega, \xi_n(\omega))\| = 0$.

If $\{\xi_n\}$, in (3.1), is pointwise convergent; that is, $\xi_n(\omega) \to \xi(\omega)$, for each $\omega \in \Omega$, then closedness of C implies that ξ is a map from Ω to C. For a continuous random operator T on C, it follows from ([1], Lemma 8.2.3) that $\omega \to T(\omega, f(\omega))$ is measurable for any measurable map f from Ω to C. Thus $\{\xi_n\}$ is a sequence of measurable maps and ξ, being the limit of a sequence of measurable maps, is itself measurable.

Let $\{T_i : i = 1, 2, \ldots, k\}$ be a family of random operators from $\Omega \times C$ to C. Let $\xi_n : \Omega \to C$ be a sequence of maps where ξ_1 is assumed to be measurable. We introduce random version of the iterative scheme (1.3) as follows: Let $0 < \alpha < 1$. For each $\omega \in \Omega$, define

$$\xi_{n+1}(\omega) = (1 - \alpha)\xi_n(\omega) + \alpha T_k(\omega, U_{k-1}(\omega, \xi_n(\omega))), \tag{3.2}$$

where $U_i : \Omega \times C \to C, i = 1, 2, \ldots, k$, are random operators given by

$$\begin{array}{rcl} U_0(\omega, \xi_n(\omega)) & = & \xi_n(\omega), \\ U_1(\omega, \xi_n(\omega)) & = & (1-\alpha)\xi_n(\omega) + \alpha\, T_1(\omega, U_0(\omega, \xi_n(\omega))), \\ U_2(\omega, \xi_n(\omega)) & = & (1-\alpha)\xi_n(\omega) + \alpha\, T_2(\omega, U_1(\omega, \xi_n(\omega))), \\ \ldots\ldots\ldots\ldots & & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ U_k(\omega, \xi_n(\omega)) & = & (1-\alpha)\xi_n(\omega) + \alpha\, T_k(\omega, U_{k-1}(\omega, \xi_n(\omega))), \end{array}$$

for each $\omega \in \Omega$.

Lemma 3.2 *Let C be a nonempty compact convex subset of a separable Banach space X, and $T : \Omega \times C \to C$ a nonexpansive random operator. Then T has a random fixed point ζ and $\{\xi_n\}$, in (3.1), converges strongly to ζ.*

Proof. For each n, define $G_n : \Omega \to K(C)$ by $G_n(\omega) = cl\{\xi_i(\omega) : i \geq n\}$ where $K(C)$ is the family of all nonempty compact subsets of C and cl denotes closure. Define $G : \Omega \to K(C)$ by $G(\omega) = \bigcap_{n=1}^{\infty} G_n(\omega)$. By the selection theorem ([15], p. 398), G has a measurable selector $\xi : \Omega \to C$. Fix $\omega \in \Omega$ arbitrarily. Now we can obtain a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ such that

$$\xi_{n_k}(\omega) \to \zeta(\omega), \text{ for each}\, \omega \in \Omega. \tag{3.3}$$

Thus, by Proposition 3.1, we have $\lim_{k\to\infty} \|\xi_{n_k}(\omega) - T(\omega, \xi_{n_k}(\omega))\| = 0$, for each $\omega \in \Omega$. We utilize nonexpansiveness of T to obtain $T(\omega, \zeta(\omega)) = \zeta(\omega)$, for each $\omega \in \Omega$. Moreover,

$$\begin{array}{rcl} \|\xi_{n+1}(\omega) - \zeta(\omega)\| & = & \|(1-\alpha_n)\xi_n(\omega) \\ & & +\alpha_n T(\omega, \xi_n(\omega)) - \zeta(\omega)\| \\ & \leq & (1-\alpha_n)\|\xi_n(\omega) - \zeta(\omega)\| \\ & & +\alpha_n\|T(\omega, \xi_n(\omega)) - T(\omega, \zeta(\omega))\| \\ & \leq & (1-\alpha_n)\|\xi_n(\omega) - \zeta(\omega)\| \\ & & +\alpha_n\|\xi_n(\omega) - \zeta(\omega)\| \\ & = & \|\xi_n(\omega) - \zeta(\omega)\|, \end{array} \tag{3.4}$$

for each $\omega \in \Omega$ and any positive integer n.

From (3.3), it follows that for any $\epsilon > 0$, there exists an integer n_0 such that $\|\xi_{n_0}(\omega) - \zeta(\omega)\| < \epsilon$, for each $\omega \in \Omega$. Thus, by (3.4), we get $\|\xi_n(\omega) - \zeta(\omega)\| < \epsilon$, for any integer $n \geq n_0$ and each $\omega \in \Omega$. Since ϵ is arbitrary, therefore $\xi_n(\omega) \to \zeta(\omega)$, for each $\omega \in \Omega$. The map $\zeta : \Omega \to C$, being the limit of a sequence of measurable maps, is also measurable. Thus ζ is a random fixed point of T. ∎

The following result generalizes Theorem 1 of Khufittig [14] for random operators.

Theorem 3.3 *Let C be a nonempty compact convex subset of a separable strictly convex Banach space X, and $\{T_i : i = 1, 2, \ldots, k\}$ a family of nonexpansive random operators from $\Omega \times C$ to C with $D = \bigcap_{i=1}^{k} RF(T_i) \neq \phi$. Then $\{\xi_n\}$, in (3.2), converges strongly to a random common fixed point of the family.*

Proof. It is easy to see that $\xi : \Omega \to C$ is a random common fixed point of $\{T_i : i = 1, 2, \ldots, k\}$ if and only if ξ is a random common fixed point of $\{U_i : i = 1, 2, \ldots, k\}$, for each $\omega \in \Omega$. Define $S_i : \Omega \times C \to C$ by

$$S_i(\omega, x) = T_i(\omega, U_{i-1}(\omega, x)), i = 1, 2, 3, \ldots, k.$$

Obviously, U_i and $S_i, i = 1, 2, \ldots, k$, are nonexpansive. By Lemma 3.2, $\{\xi_n\}$ in (3.2), converges strongly to a random fixed point $\zeta : \Omega \to C$ of S_k. By using the arguments of the proof of Theorem 2.3, we can show that ζ is a random common fixed point of $\{T_i : i = 1, 2, \ldots, k\}$. ∎

The compact subset C in Theorem 3.3 is replaced by a bounded closed subset of a uniformly convex space to obtain:

Theorem 3.4 *Let C be a nonempty bounded closed convex subset of a separable uniformly convex Banach space X, and $\{T_i : i = 1, 2, \ldots, k\}$ a family of nonexprnsive random operators from $\Omega \times C$ to C with $D = \bigcap_{i=1}^{k} RF(T_i) \neq \phi$. Then $\{\xi_n\}$, in (3.2), converges weakly to a random common fixed point of the family.*

Proof. Suppose that the maps $S_i, i = 1, 2, \ldots, k$, are defined as in the proof of Theorem 3.3. We note that C is weakly compact in a reflexive space X. Thus as in the proof of ([3], Theorem 3.2), $\{\xi_n\}$ has a subsequence $\{\xi_{n_j}\}$ converging weakly to $\zeta : \Omega \to C$. Now by Proposition 3.1, $\lim_{j\to\infty} \|\xi_{n_j}(\omega) - S_k(\omega, \xi_{n_j}(\omega))\| = 0$, for each $\omega \in \Omega$. Hence by Lemma 1.4, we get $S_k(\omega, \zeta(\omega)) = \zeta(\omega)$, for each $\omega \in \Omega$. That is, ζ is a random fixed point of S_k. A uniformly convex space is strictly convex, so one can use arguments similar to the proof of Theorem 2.3 to show that ζ is a random common fixed point of $\{T_i : i = 1, 2, \ldots, k\}$. ∎

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