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False-alarm and Non-detection Probabilities for On-line Quality Control via HMM

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Abstract—**On-line quality control during production calls for monitoring produced items according to some prescribed strategy. It is reasonable to assume the existence of system internal non-observable variables so that the carried out monitoring is only partially reliable. In this note, under the setting of a Hidden Markov Model (HMM) and assuming that the evolution of the internal state changes are governed by a two-state Markov chain, we derive false-alarm and non-detection malfunctioning probabilities.**

Keywords: false-alarm, Hidden Markov Model, on-line quality control

1 Introduction

Limited by the cost function a sampling interval m is selected and classical on-line quality control during production adopts the procedure of monitoring the sequence of items, independently produced, by examining a single item at every m produced items. Based on the quality requirements and on the distribution of the examined variable a control region C is pre-specified. If the examined item satisfies the control limits, the process is said to be in control and the production continues; otherwise, the process is declared out of control and the production is stopped for adjustments. After adjustments the production is resumed and it is in control again. At each stoppage a new production cycle is defined (see, for example, [4]). Suppose now the internal working status of the system is non-observable and may change from a good working condition (on control) to a deteriorating status (out of control). In [3] it is proposed a model where these changes are governed by a two-state Markov chain $\{\theta_n\}_{n\geq 0}$. When $\theta_n = 0$ the process is said to be in control at time n; and, if $\theta_n = 1$ the process is out of control. In their proposal, an observable random variable X_n, related to characteristics of interest, is examined at every m produced items. It is assumed that X_n has a Gaussian distribution, $\mathcal{N}(\mu, \sigma^2)$, with $\mu = \mu_{\theta_n}$. And values of m as well as the parameter d of the control region $C = (\mu_0 - d\sigma, \mu_0 + d\sigma)$ were determined by considering a given cost function to be minimized.

Here, in the framework of a Hidden Markov Model (HMM), we compute and propose estimation techniques for the false-alarm and non-detection probabilities. False-alarm occurs if the observed variable falls outside the control region but the non-observable internal system state is 0, a good working state. When the opposite occurs we have a non-detection situation. It is assumed that all the working (good) states are lumped together as state 0 and the deteriorating states are gathered as state 1. The Markov chain $\{\theta_n\}_{n\geq 0}$ describes the evolution of the state of the production system. Associated with this chain we observe a sequence of conditionally independent random variables $\{X_n\}_{n\geq 1}$, with the distribution of each X_n depending on the corresponding state θ_n. This process $\{\theta_n, X_n\}$ is generally referred to as a HMM. More specifically, we have

$$P(X_{n+1} \in A | X_1, \ldots, X_n, \theta_0, \ldots, \theta_n)$$

$$= P(X_{n+1} \in A | \theta_n) \tag{1}$$

and

$$P(X_1 \in A_1, \ldots, X_n \in A_n | \theta_1, \ldots, \theta_n)$$

$$= \prod_{j=1}^{n} P(X_j \in A_j | \theta_j). \tag{2}$$

In section 2, for given transition matrix of $\{\theta_n\}$ and conditional densities of X_n given θ_n the false-alarm and the non-detection probabilities are computed and, in section 3, using results from [1] estimates are presented.

2 False-alarm and Non-detection

For the HMM process $\{\theta_n, X_n\}$ assume that for some $0 < p < 1$ the chain $\{\theta_n\}$ has transition probabilities given by

$$P = \begin{pmatrix} 1-p & p \\ 0 & 1 \end{pmatrix}$$

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The conditional distribution of X_n given θ_n is known and based on this distribution a control region C is preselected. It is assumed that

$$P(X_n \in A|\theta_n = i) = \int_A f(x|i)dx,$$

$$0 < q_0 = \int_{C^c} f(x|0)dx < 1$$

and

$$0 < q_1 = \int_{C^c} f(x|1)dx < 1. \tag{3}$$

For the sampling interval m the on-line quality monitoring adopts the following strategy : items $X_m, X_{2m}, X_{3m}, \ldots$ are inspected and verified whether $X_m \in C$, $X_{2m} \in C, \ldots$; maintenance is required at time km if $X_m \in C, \ldots, X_{(k-1)m} \in C$ and $X_{km} \notin C$. Thus we can define the alert times by

$$\tau_X = \inf\{km : k \geq 1, X_{km} \notin C\}.$$

False-alarm occurs at time k if $\tau_X = k$ but the non-observable internal system state θ_k is 0, a good working state. Let τ_θ be defined as the first time, after time 0, the system reaches state 1,

$$\tau_\theta = \inf\{k : k \geq 1, \theta_k = 1\}.$$

Then false-alarm and non-detection correspond, respectively, to the events $(\tau_X < \tau_\theta)$ and $(\tau_X > \tau_\theta)$. Note that, starting from a good working status, $\theta_0 = 0$, we have from (2)

$$\begin{aligned}
P(\tau_X &= m, \tau_\theta > m) \\
&= P_0(X_m \notin C, \theta_1 = \cdots = \theta_m = 0) \\
&= P(\theta_1 = \cdots = \theta_m = 0)P(X_m \notin C|\theta_m = 0) \\
&= (1-p)^m q_0
\end{aligned}$$

and

$$\begin{aligned}
P(\tau_X &= km, \tau_\theta > km) \\
&= P(X_m \in C, \cdots, X_{(k-1)m} \in C, X_{km} \notin C, \\
&\quad \theta_1 = \cdots = \theta_{km} = 0) \\
&= (1-p)^{km}[\prod_{j=1}^{k-1} P(X_{jm} \in C|\theta_{jm} = 0)]. \\
&\quad P(X_{km} \notin C|\theta_{km} = 0) \\
&= (1-p)^{km}(1-q_0)^{(k-1)m} q_0.
\end{aligned}$$

It follows that

$$P(\tau_X < \tau_\theta) = \frac{(1-p)^m q_0}{1-(1-p)^m(1-q_0)}. \tag{4}$$

Similarly, we have

$$P(\tau_X = \tau_\theta = m) = (1-p)^{m-1} p q_1$$

and, in general,

$$P(\tau_X = \tau_\theta = km) = (1-p)^{km-1}(1-q_0)^{k-1} p q_1$$

Adding up we have

$$\begin{aligned}
P(\tau_X = \tau_\theta) &= \textstyle\sum_{k\geq 1}(1-p)^{km-1}(1-q_0)^{k-1} p q_1 \\
&= \frac{(1-p)^{m-1} p q_1}{1-(1-p)^m(1-q_0)}.
\end{aligned}$$

This along with (4) gives

$$P(\tau_X > \tau_\theta) = \frac{1-(1-p)^{m-1}(1-p+pq_1)}{1-(1-p)^m(1-q_0)}. \tag{5}$$

Proposition 1. If the chain $\{\theta_n, X_n\}$ has transition matrix P and $\theta_0 = 0$ then the false-alarm probability is given by (4) and the non-detection probability is given by (5).

Next, assume that some of the deteriorating states lumped together as state 1 can recuperate and this allows transitions from state 1 to 0 with a small probability $\epsilon > 0$. The corresponding transition matrix becomes

$$P_\epsilon = \begin{pmatrix} 1-p & p \\ \epsilon & 1-\epsilon \end{pmatrix}$$

$$0 < p < 1\ ,\ \epsilon > 0 \text{ and } p + \epsilon < 1. \tag{6}$$

Proposition 2. If the chain $\{\theta_n, X_n\}$ has transition matrix P_ϵ then, pending on the initial state, we have

$$P(\tau_X < \tau_\theta|\theta_0 = 0) = \frac{(1-p)^m q_0}{1-(1-p)^m(1-q_0)}$$

and

$$P(\tau_X < \tau_\theta|\theta_0 = 1) = \frac{\epsilon(1-p)^{m-1} q_0}{1-(1-p)^m(1-q_0)}.$$

For $m \geq 2$, the non-detection probabilities are

$$P(\tau_X > \tau_\theta|\theta_0 = 0) = \frac{1-(1-p)^{m-1}(1-p+pq_1}{1-(1-p)^m(1-q_0)}$$

and

$$P(\tau_X > \tau_\theta|\theta_0 = 1) = 1 - \frac{\epsilon(1-p)^{m-2}[(1-p)q_0 + pq_1}{1-(1-p)^m(1-q_0)}.$$

For further details and related results, see [1].

3 Estimation Results

In this section we assume that the hidden Markov chain $\{\theta_n\}$ has transition matrix P_ϵ given by (6), but unknown. As for the process $\{X_n\}$, we assume that the conditional densities are known and satisfy condition (3).

Note that, since all entries of P are strictly positive, $\{\theta_n\}$ is an ergodic chain and the stationary (limiting) distribution exists, $\pi P = \pi$,

$$\pi(0) = \frac{\epsilon}{p+\epsilon} \text{ and } \pi(1) = \frac{p}{p+\epsilon}. \tag{7}$$

Define the mixture density function

$$f(x) = \pi(0)f(x|0) + \pi(1)f(x|1). \tag{8}$$

Then, in some sense, $f(\cdot)$ represents the density of $\{X_n\}$ when the process reaches some "stable" regime. Though we are assuming known conditional densities, the stable regime density (8) may indicate which type of distribution one should assume for the variables X_n as well as gives insight concerning the control region C to be selected. Theorem 1 below gathers results [1] and [2] allowing us to estimate $f(x)$ as well as the equilibrium probabilities (7). For a probability density K on R define

$$\hat{f}_n(x) = \frac{1}{nh}\sum_{k=1}^{n} K(\frac{X_{km}-x}{h})$$

with

$$h = h_n \downarrow 0\,,\; nh_n \to \infty \text{ as } n \to \infty.$$

Typically one takes $K(\cdot)$ either a Gaussian density or an uniform density centered at x.

Theorem 1. Assume that the process $\{\theta_n, X_n\}$ satisfies (3) and (6). Then for any initial distribution of θ_0 and for any given $\delta > 0$, there exist constants $c_1 = c_1(\delta) > 0$ and $c_2 = c_2(\delta) > 0$ such that

$$P(\int |\hat{f}_n(y) - f(y)|dy \geq \delta) \leq c_1 \exp\{-c_2 n\}.$$

And, with probability 1

$$\hat{\pi}_n(0) \to \pi(0) \text{ and } \hat{\pi}_n(1) \to \pi(1) \text{ as } n \to \infty.$$

Where for some x_* chosen so that $f(x_*|0) \neq f(x_*|1)$ we define

$$\hat{\pi}_n(0) = \left|\frac{\hat{f}_n(x_*) - f(x_*|1)}{f(x_*|0) - f(x_*|1)}\right|$$

and

$$\hat{\pi}_n(1) = 1 - \hat{\pi}_n(0). \tag{9}$$

Observe that, in long-run, the false-alarm can be computed as

$$\begin{aligned} P(\theta_{km} &= 0|X_{km} \notin C) \\ &= \frac{P(X_{km} \notin C|\theta_{km} = 0)P(\theta_{km} = 0)}{P(X_{km} \notin C)}. \end{aligned}$$

But $P(X_{km} \notin C|\theta_{km} = 0) = q_0$ and by Theorem 1

$$\lim_{n\to\infty} P(\theta_{km} = 0) = \pi(0)$$

and

$$\lim_{n\to\infty} P(X_{km} \notin C) = \int_{C^c} f(x)dx$$

with $f(\cdot)$ given by (8). Since both $\pi(0)$ and $f(\cdot)$ can be estimated by Theorem 1, we can obtain an estimate for the false-alarm probability. Similarly, we have for non-detection

$$\begin{aligned} P(\theta_{km} &= 1|X_{km} \in C) \\ &= \frac{P(X_{km} \in C|\theta_{km} = 1)P(\theta_{km} = 1)}{P(X_{km} \in C)}. \end{aligned}$$

These results can be summarized by:

Corollary 1. Assume that the process $\{\theta_n, X_n\}$ satisfies (3) and (6). Then, regardless whether θ_0 is 0 or 1, we have

$$\frac{q_0\hat{\pi}_n(0)}{\int_{C^c}\hat{f}_n(x)dx} \to P(\text{false-alarm})$$

and

$$\frac{(1-q_1)\hat{\pi}_n(1)}{\int_{C}\hat{f}_n(x)dx} \to P(\text{non-detection}).$$

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