Condition Based Maintenance of Periodically Inspected Systems

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Abstract— **Condition based maintenance (CBM) is a powerful tool for improvement of system reliability and reduction of system downtime. This research considers CBM under which the system state is periodically observed with or without observational error, and maintenance is imperfect. System availability is maximized by determining the optimal maintenance threshold and the time interval between consecutive inspections of the state of the system. The optimal solution can be obtained numerically using a sequential uniform design algorithm.**

Keywords: condition based maintenance, imperfect maintenance, sensor error, uniform design

1 Introduction

This paper deals with condition based maintenance (CBM) of systems that experience soft failures that occur in a continuously degrading system when degradation reaches a failure threshold.

In CBM, the system is monitored from time to time. If the state of the system is detected to have deteriorated to a certain pre-determined level, maintenance is carried out. Since maintenance is carried out only when alarms occur, CBM is more effective than preventive maintenance (PM) under which maintenance is carried out at fixed time intervals irrespective of the condition of the system.

Most results in the literature on maintenance of degrading systems consider the case when the system is continuously monitored. Grall *et al.* [4] present a PM structure for a gradually degrading single-unit system, Liao *et al.* [7] consider a CBM model for continuously degrading systems under continuous monitoring, Marseguerra *et al.* [8] study a continuously monitored multi-component system and use generic algorithm to determine the optimal degradation level. In some cases, due to various constraints, it is impractical to monitor the system continuously. In such cases, the system may be monitored by inspecting the system at regular time intervals. For such a system, Sarker and Sarker [9] obtain recursive expressions for availability of two types of models under perfect repair. Biswas *et al.* [1] derive the expressions of availability of the system which is maintained through a fixed number of imperfect repairs before replacement or perfect repair, without considering CBM. Jamali *et al.* [6] considers joint optimal periodic and conditional maintenance policy, while their study involve only perfect maintenance.

Furthermore, in most research it is assumed that observations of the state of system are error-free, that is, the observations indicate the true state of the system without any error. In practice, however, not all observations are error-free. Common types of error includes systematic error [10, 11] which can be eliminated by precise calibration of the sensors, and random measurement error which can be regarded as Gaussian white noise [2, 5] and treated statistically.

In this paper, we consider a continuously degrading system which is being monitored at regular time intervals, and propose a CBM policy assuming maintenance is imperfect. It is assumed that the system deteriorates according to a Gamma process, and the system fails when its state of deterioration reaches a failure threshold. An optimal threshold to carry out maintenance and an optimal time interval for monitoring the system are determined to maximize the *Achieved Availability* (AA) of the system [7]. To maximize the AA, various heuristic methods can be employed.

2 Description of the system

Consider a system which experiences continuous degradation during operation. Let the state of the system at time t be represented by a continuous nondecreasing function $X(t)$, where $X(0) = 0$. As t increases, $X(t)$ increases as the system deteriorates. Suppose that there is a failure threshold D_F such that when $X(t)$ reaches D_F, failure of the system will occur immediately.

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Degrading of the system can be modeled as a gamma process, such that for any $s, t > 0$, the increment $X(s+t) - X(s)$ is a random variable depending on t only and having a gamma distribution with mean $\alpha\beta t$, variance $\alpha\beta^2 t$ and probability density function

$$Ga(x|\alpha, \beta, t) = \frac{\exp(-\frac{x}{\beta})x^{\alpha t-1}}{\Gamma(\alpha t)\beta^{\alpha t}},$$

where $\Gamma(z) = \int_0^\infty u^{z-1}e^{-u}\, du$ is the gamma function and $\alpha, \beta > 0$ are constants. Similar models of a continuously degrading system can also be found in [4].

Suppose that after the system has started to operate at time $t = 0$, the system is inspected periodically, under which the state $X(t)$ of the system is observed at discrete times and the time separation between two successive observations of $X(t)$ is $\tau > 0$, where τ is a constant. Under CBM, preventive maintenance for the system will be carried out whenever the observation of $X(t)$ is above D_L, where $D_L < D_F$ is a pre-determined positive maintenance threshold. Assume that the time taken to observe $X(t)$ and the waiting time for the maintenance service are small and negligible. After completion of maintenance, the system resumes operation.

Each time after $X(t)$ is observed, it is possible that $X(t)$ increases to D_F and the system fails before $X(t)$ is observed next time. When the system fails, it is replaced immediately. Suppose that the time required to replace the system is $\xi > 0$, and after replacement the system is as-good-as-new. A cycle starts when the system starts to operate, and ends when replacement of the system is completed.

As shown in Figure 1, R_0^+ denotes the starting time of a cycle, R_i $(1 \le i \le n \le N)$ denotes the starting time of the ith maintenance action, R_i^+ $(1 \le i \le N)$ denotes the completion time of the ith maintenance action, R_{N+1}, R_{n+1} and R_1 denote the starting time of replacement of the system, T_1 denotes the operating time (uptime) of the system before the 1st maintenance action, T_i $(2 \le i \le n \le N)$ denotes the operating time (uptime) of the system between the $(i-1)$th and the ith maintenance actions, M_i $(1 \le i \le n \le N)$ denotes the time required to perform the ith maintenance action (downtime).

Figure 1 shows two possible cases, Case A and Case B, for the system to complete a cycle. In Case A, the system has been maintained for N times; between the ith and $(i{+}1)$th maintenance actions $(i = 0, 1, ..., N{-}1)$, $X(t)$ has been observed for k_i times, where the first $k_i - 1$ of these observations give $X(t) < D_L$, but the last observation gives $D_L \le X(t) < D_F$ so that the system is maintained at $t = R_i$ $(i = 1, ...N)$; after the Nth maintenance action is completed at $t = R_N^+$, $X(t)$ has been observed for k_N times, where the first $k_N - 1$ of these observations give $X(t) < D_L$, but the last observation gives $D_L \le X(t) < D_F$ so that the system is replaced at $t = R_{N+1}$. In Case B, $X(t)$ has been maintained for n times, where $0 \le n \le N$; after the nth maintenance action is completed at $t = R_n^+$, $X(t)$ has been observed for k_n times, where all these observations give $X(t) < D_L$; after $X(t)$ has been observed for the last time, $X(t)$ increases to D_F within the next time period τ (which causes system failure) so that the system is replaced at the end of this time period. In Case B, as shown in Figure 1, the downtime of the system after failure until the start of replacement is T_n^W. Figure 1 shows the following three situations where either maintenance or replacement is to be carried out.

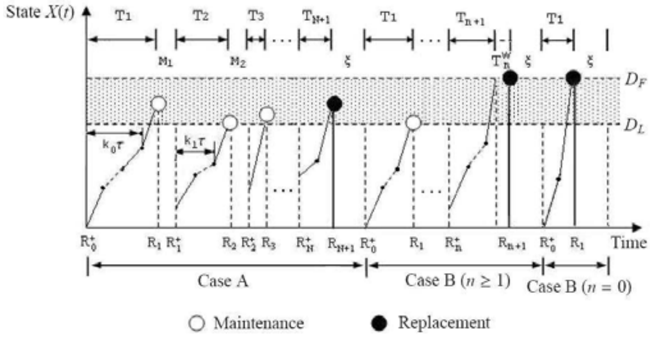

Figure 1: State Transition with Maintenance and Replacement

1) If $D_L \le X(R_n) < D_F$ and $1 \le n \le N$, carry out maintenance at R_n.

2) If $D_L \le X(R_{N+1}) < D_F$, replace the system at R_{N+1}.

3) If $X(t) = D_F$ at some time $t'' \in (R_n^+, R_{n+1}]$ so that the system fails at $t = t''$, replace the system at $t = R_{n+1}$ $(0 \le n \le N)$.

2.1 Maintenance result and maintenance time

Suppose that the ith maintenance action restores the system to state $g(i)$ $(i = 1, 2, ...)$, and that $g(0) = 0 = X(0)$. Hence $X(R_i^+) = g(i)$. Assuming that maintenance is imperfect, so that $g(i)$ is an increasing function of i. Perfect maintenance corresponds to $g(i) = X(0)$ (for all $i = 0, 1, 2, ...$). A possible form of $g(i)$ is

$$g(i) = c + di, \; i = 1, \cdots, N, \tag{1}$$

where c and d are constants (which may be known or estimated from historical data).

Assuming that a more severely degraded system needs longer time to maintain than one which is not so severely degraded, we let the expected value $E(M_i)$ of the maintenance time M_i be an increasing function of i, $X(R_{i-1}^+)$ and D_L. A model for $E(M_i)$ may be the following

$$E(M_i) = \gamma_0 D_L \exp(i\gamma_1 X(R_{i-1}^+)), \tag{2}$$

where $\gamma_0 > 0$ and $\gamma_1 \geq 0$ are constants, which are independent of D_L and can be estimated from historical data. It follows from (2) that $E(M_i) > E(M_j)$ for any $i > j \geq 1$ (since $X(R^+_{i-1}) > X(R^+_{j-1})$), and that $E(M_i)$ reduces to the constant $\gamma_0 D_L$ if $\gamma_1 = 0$.

2.2 Formulation of the optimization problem

An objective of optimal design for maintenance policy is to maximize the system availability. We consider the system's achieved availability [7] defined by

$$AA = \frac{E[\text{total uptime in a cycle}]}{E[\text{total time length in a cycle}]}. \qquad (3)$$

The value of AA depends on the threshold D_L and the time interval τ between two consecutive times when X_t is measured. Therefore, the optimal maintenance policy can be formulated as:

$$\begin{array}{rl} \text{Max} & AA(D_L, \tau) \\ \text{Subject to} & 0 < D_L \leq D_F,\ \tau > 0. \end{array} \qquad (4)$$

The optimal values of D_L and τ that correspond to the global maximum of AA can be obtained numerically.

3 Analytic results

A cycle may be completed under Case A or Case B described in Section 2. As shown in Figure 1, in a cycle with n maintenance actions, the expected uptime is

$$\begin{array}{rl} & E[\text{uptime}|n \text{ maintenance actions in a cycle}] \\ = & E(T_1 + \cdots + T_n + T_{n+1}) \quad (0 \leq n \leq N). \end{array} \qquad (5)$$

The maintenance time in a cycle is $M_1 + \cdots + M_n$ ($n \leq N$). For Case A, the change of $X(t)$ after time $t = R^+_N$ is depicted by the curve marked "Q_2" or "Q_3" in Figure 2 (with n replaced by N), and the expected downtime in a cycle is $E(M_1 + \cdots + M_N) + \xi$ which can be seen from Figure 1. For Case B, the change of $X(t)$ after time $t = R^+_N$ is depicted by the curve marked "Q_1" in Figure 2, the waiting time for replacement is $T^w_n > 0$, and the expected downtime in a cycle is

$$\begin{array}{rl} & E[\text{downtime}|n \text{ maintenance actions in a cycle}] \\ = & E(M_1 + \cdots + M_n) + E(T^w_n) + \xi, \end{array} \qquad (6)$$

which can be seen from Figure 1. The expression in (6) also covers Case A, noting that $E(T^w_n) = 0$ in Case A.

3.1 Expected uptime

Suppose that after the start of the system, n ($\leq N$) maintenance actions have been performed before the system is replaced. Between the ith and the $(i+1)$th ($1 \leq i+1 \leq n$) maintenance actions, $X(t)$ is observed for k_i times. The uptime T_{i+1} between the ith and the $(i+1)$th maintenance actions can be divided into two parts, the part

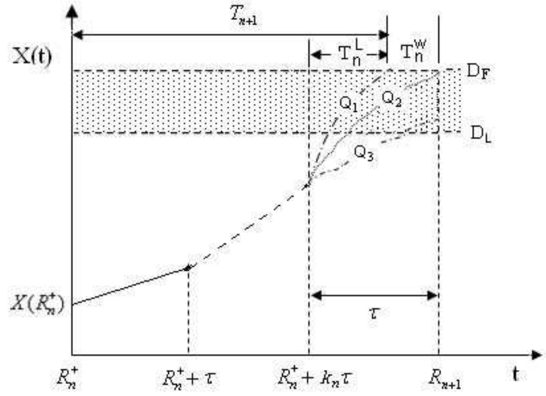

Figure 2: State after the nth maintenance actions

just before $X(t)$ is observed for the last k_ith time, and the part after. The first part obviously equals $k_i\tau$, and we denote the second part by T^L_i. The expected uptime $E(T_{i+1})$ is given by

$$E(T_{i+1}) = E(k_i) \cdot \tau + E(T^L_i). \qquad (7)$$

The value of k_i depends on the value of $X(R^+_i) = g(i)$ which is the state of the system immediately after the ith maintenance action. The expected value of k_i is given by

$$\begin{array}{rl} & E(k_i) = E[E(k_i|X(R^+_i))] \\ = & \sum_{j=0}^{\infty} j \times \Pr(X(R^+_i) + Y_1 + \cdots + Y_{j+1} \\ & > D_L \,|\, X(R^+_i) + Y_1 + \cdots + Y_j < D_L), \end{array} \qquad (8)$$

where $Y_1, Y_2, \ldots$ are independent Gamma random variables $Ga(\alpha\tau, \beta)$ under the Gamma model as described in Section 2. To obtain an approximate value for $E(k_i)$, we observe that $X(t)$ has to increase from $g(i)$ to D_L after it is maintained for i times, and that the mean degradation speed of $X(t)$ is $\alpha\beta$. Hence an approximated value for $E(k_i)$ is given by

$$E(k_i) = E[E(k_i|X(R^+_i))] \approx \frac{D_L - g(i)}{\alpha\beta\tau}. \qquad (9)$$

After $X(t)$ is observed for the k_ith time, it increases to D_L by the time $R_{i+1} = (k_i + 1)\tau$. If $i < n$, referring to Figure 1 we see that $T_{i+1} = (k_i+1)\tau$, and thus $E(T_{i+1}) = (k_i+1)\tau$. If $i = n$, referring to Figure 1 we see that either Case A or Case B may occur. In Case A, in which the change in $X(t)$ is described by the curve marked "Q_2" or "Q_3" in Figure 2, we have $X(R_{n+1}) \leq D_F$, $T^L_n = \tau$, $E[T_{n+1}] = (E(k_n) + 1)\tau$. In Case B, in which the change in $X(t)$ is described by the curve marked "Q_1" in Figure 2, we have $X(t) = D_F$ at $t = T^L_n < \tau$, the system fails at this point, and $E[T_{n+1}] = E(k_n)\tau + E(T^L_n)$. Case A occurs only when $n = N$, but Case B may occur when either $0 \leq n < N$ or $n = N$.

(1) $0 \leq n < N$. This happens only under Case B when the system has been maintained for n times. After the

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nth maintenance action is completed at R_n^+, the system continuous to operate within a time period of length $k_n\tau$ and continues to operate for a further time period $T_n^L \le \tau$. The movement of $X(t)$ is depicted by the curve marked "Q_1" in Figure 2. Replacement of the system takes place at $t = R_n^+ + k_n\tau + \tau$. The uptime T_n^L depends on the state $X(t)$ at $t = R_n^+ + k_n\tau < D_L$. As we assumed that $X(R_n^+) = g(n)$ and $X(R_n^+ + k_n\tau) - X(R_n^+) \sim Ga(k_n\tau\alpha, \beta)$, denoting the pdf of $X(R_n^+ + k_n\tau)$ by $f_{k_n\tau}(x)$, we have

$$\begin{aligned} & f_{k_n\tau}(x) \\ = \ & \frac{Ga(x-g(n)|k_n\tau\alpha,\beta)}{\int_{g(n)}^{D_L} Ga(u-g(n)|k_n\tau\alpha,\beta)du}, g(n) < x < D_L. \end{aligned} \quad (10)$$

Let $y = X(R_n^+ + k_n\tau)$. Then

$$\begin{aligned} E(T_n^L) &= E[E(T_n^L|X(R_n^+ + k_n\tau))] \\ &= \int_{g(n)}^{D_L} E(T_n^L|y) f_{k_n\tau}(y)dy. \end{aligned} \quad (11)$$

The CDF of the uptime T_n^L (which is the time to failure starting from $y = X(R_n^+ + k_n\tau)$ until $X(t)$ reaches the failure threshold D_F, as shown in Figure 2) is given by

$$F_y(t) = \frac{\Gamma\left(\frac{D_F - y}{\beta}, \alpha t\right)}{\Gamma(\alpha t)}. \quad (12)$$

The corresponding pdf $f_y(t)$ is given by $f_y(t) = \frac{\partial F_y(t)}{\partial t}$. Therefore, under the condition of $0 < t \le \tau$, the corresponding conditional density of $t = T_n^L$ is

$$f_{y|t\le\tau}(t) = \frac{f_y(t)}{\int_0^\tau f_y(v)dv}, \ 0 < t \le \tau. \quad (13)$$

The conditional expectation $E(T_n^L|y)$ based on the constraint $0 < T_n^L \le \tau$ is given by

$$\begin{aligned} E(T_n^L|y) &= E[E(t|y, t\le\tau)] = \int_0^\tau t f_{y|t\le\tau}(t)dt \\ &= \frac{\int_0^\tau t f_y(t)dt}{F_y(\tau)}, \end{aligned} \quad (14)$$

where $F_y(\cdot)$ is defined in (12). Moreover,

$$\begin{aligned} \int_0^\tau t f_y(t)dt &= \int_0^\tau \int_0^t f_y(t)dsdt = \int_0^\tau \int_s^\tau f_y(t)dtds \\ &= \int_0^\tau [F_y(\tau) - F_y(s)]ds \\ &= \tau F_y(\tau) - \int_0^\tau F_y(s)ds. \end{aligned} \quad (15)$$

Substituting (15) into (14) gives

$$E(T_n^L|y) = \tau - \frac{\int_0^\tau F_y(s)ds}{F_y(\tau)}. \quad (16)$$

Then, from (11), the expectation $E(T_n^L)$ is given by

$$E(T_n^L) = \tau - \int_{g(n)}^{D_L} \frac{1}{F_y(\tau)} \left(\int_0^\tau F_y(s)ds\right) f_{k_n\tau}(y)dy, \quad (17)$$

where $f_{k_n\tau}(y)$ is as in (10).

(2) $n = N$. This can happen under either Case A or Case B. The system has been maintained for N times. After the Nth maintenance action is completed at $t = R_N^+$, the system operates within a time period of length $k_N\tau$ and continues to operate for a further time period $T_n^L \le \tau$. The system is replaced at $t = R_N^+ + k_N\tau + \tau$. If the system fails at time $t < \tau$ (in which case the movement of $X(t)$ is depicted by the curve marked "Q_1" in Figure 2), the uptime $T_N^L = t$. If the system fails at time $t = \tau$ (in which case the movement of $X(t)$ is depicted by the curve marked "Q_2" or "Q_3" in Figure 2), the uptime is $T_N^L = R_{N+1} - (R_N^+ + k_N\tau) = \tau$. Then the conditional expectation of T_N^L given that $X(R_N^+ + k_N\tau) = y$ is

$$\begin{aligned} E(T_N^L|y) &= \int_0^\tau t f_y(t)dt + \int_\tau^\infty \tau f_y(t)dt \\ &= \int_0^\tau t f_y(t)dt + \tau(1 - F_y(\tau)). \end{aligned} \quad (18)$$

From (15), we have

$$E(T_N^L|y) = \tau - \int_0^\tau F_y(s)ds. \quad (19)$$

Substituting (19) into (11), we have

$$\begin{aligned} E(T_N^L) &= \int_{g(N)}^{D_L} \left(\tau - \int_0^\tau F_y(s)ds\right) f_{k_n\tau}(y)dy \\ &= \tau - \int_{g(N)}^{D_L} \left(\int_0^\tau F_y(s)ds\right) f_{k_n\tau}(y)dy, \end{aligned} \quad (20)$$

where $f_{k_n\tau}(y)$ is as in (10).

3.2 Expression for AA

Let p_i be the probability for the state of the system not reaching D_F between the ith maintenance action and the next action which may either be a maintenance action or a replacement. Let P_n be the probability for a cycle that consists of exactly n maintenance actions. It is easy to see that P_n is given by

$$P_n = \begin{cases} 1 - p_n, & n = 0 \\ \prod_{i=0}^{n-1} p_i(1-p_n), & n = 1, \cdots, N-1. \\ \prod_{i=0}^{N-1} p_i, & n = N \end{cases} \quad (21)$$

It follows from the definition of AA in (3) that

$$AA = \frac{\sum_{n=0}^N P_n\left(\sum_{i=1}^{n+1} E(T_i)\right)}{\sum_{n=0}^N P_n\left(\sum_{i=1}^{n+1} E(T_i) + \sum_{i=1}^n E(M_i) + E(T_n^w) + \xi\right)}, \quad (22)$$

Denote by $F_\tau(x)$ the CDF of $Ga(\alpha\tau, \beta)$. If $p_{i|x}$ represents the probability for $X(R_{i+1}) < D_F$ given the value of $x = X(R_i^+ + k_i\tau) < D_L$, we have $p_{i|x} = F_\tau(D_F - x)$. Furthermore, similar to (10), the pdf $f_{k_i\tau}$ of $X(R_i^+ + k_i\tau)$ is given by

$$f_{k_i\tau}(x) = \frac{Ga(x-g(i)|k_i\tau\alpha,\beta)}{\int_{g(i)}^{D_L} Ga(u-g(i)|k_i\tau\alpha,\beta)du}, \ g(i) < x < D_L. \quad (23)$$

Therefore,

$$p_i = \int_{g(i)}^{D_L} p_{i|x} f_{k_i\tau}(x)dx = \int_{g(i)}^{D_L} F_\tau(D_F - x) f_{k_i\tau}(x)dx. \quad (24)$$

Moreover, from the expression of $E(T_{i+1})$ in (7) and $T_n^L + T_n^W = \tau$, we have,

$$AA = \frac{\sum_{n=0}^N P_n\left(\sum_{i=0}^n E(k_i)\tau + n\tau + E(T_{n+1}^L)\right)}{\sum_{n=0}^N P_n\left(\sum_{i=0}^n E(k_i)\tau + \sum_{i=1}^n E(M_i) + (n+1)\tau + \xi\right)}, \quad (25)$$

where $E(M_i)$ is given by (2), $E(k_i)$ is given by (8) exactly and given by (9) approximately, and $E(T_{n+1}^L)$ is given by (17) if $n = 0, 1, \cdots, N-1$ and given by (20) if $n = N$.

4 System with sensor error

In the presence of observation error in inspection of the system, we let $Y(t)$ be the observed value of $X(t)$ and relate to $X(t)$ by

$$Y(t) = X(t) + \epsilon,$$

where ϵ is the Gaussian white noise that follows normal distribution with mean 0 and variance σ^2. In what follows, we shall denote by $\Phi_{(x,\sigma^2)}(u)$ the CDF of a normal random variable with mean x and variance σ^2. The system model is similar to the model in the previous sections expect the system uptime is need to redefine.

4.1 System uptime

The following symbols are needed.

M_i	i maintenance actions carried out
Re	system replacement
Re_j	the system is replaced after M_j $(= Re \cap (\cap_{i=0}^{j} M_i)))$
(M_i, O_j)	i maintenance actions carried out and $X(t)$ has been observed for j times
P_{ij}^C	the probability for the system to continue to operate (that is, $Y(t) \le D_L$), given (M_i, O_j)
P_{ij}^M	the probability that the $(i+1)$th maintenance action will be performed, given (M_i, O_j)
P_{ij}^R	the probability for the system not to fail but an observed value of $X(t)$ indicates that replacement is needed, given (M_i, O_j)
P_{ij}^F	the probability for the system to fail, given (M_i, O_j)
P_i^M	the probability that after M_i, the $(i+1)$th maintenance action will be performed $(= \Pr(M_{i+1}\|M_i))$
P_i^R	the probability that after M_i, the system will be replaced $(= \Pr(Re\|M_i))$
P_n	the probability that after M_n, the system will be replaced $(= \Pr(Re_n))$

We now obtain the system uptime. Since $0 \le n \le N$, the total expected uptime per cycle is given by

$$\begin{aligned} & E[\text{System Uptime per Cycle}] \\ = & \sum_{n=0}^{N} P_n \; E[\text{Uptime}|R_n] \\ = & \sum_{n=0}^{N} P_n \Big(\sum_{i=0}^{n-1} E\Big[T_i\Big|(M_{i+1}|M_i)\Big] \\ & + E\Big[T_n\Big|(R|M_n)\Big]\Big). \end{aligned} \tag{26}$$

It follows from the definitions of P_i^M and P_i^R that P_n is given by

$$P_n = \begin{cases} P_0^R, & n = 0; \\ \prod_{i=0}^{n-1} P_i^M P_n^R, & n = 1, \ldots, N-1; \\ \prod_{i=0}^{N-1} P_i^M, & n = N. \end{cases} \tag{27}$$

From the definition of P_{ij}^C and conditioning P_{ij}^C on $X(t)$, we have

$$P_{ij}^C = \int_{g(i)}^{D_F} (\Phi_{(x,\sigma^2)}(D_L)) \; Ga(x - g(i)|j\tau\alpha, \beta) \; dx. \tag{28}$$

$$\begin{aligned} P_{ij}^M = & \int_{g(i)}^{D_F} (\Phi_{(x,\sigma^2)}(D_F) - \Phi_{(x,\sigma^2)}(D_L)) \\ & Ga(x - g(i)|j\tau\alpha, \beta) \; dx, \end{aligned} \tag{29}$$

$$P_{ij}^R = \int_{g(i)}^{D_F} (1 - \Phi_{(x,\sigma^2)}(D_F)) \; Ga(x - g(i)|j\tau\alpha, \beta) \; dx, \tag{30}$$

$$P_{ij}^F = \int_{D_F}^{\infty} Ga(x - g(i)|j\tau\alpha, \beta) \; dx = 1 - P_{ij}^C - P_{ij}^M - P_{ij}^R. \tag{31}$$

Then according to the definition of P_i^M, for $i = 0, \ldots, N-1$ we have

$$\begin{aligned} P_i^M & = P_{i,1}^M + P_{i,1}^C P_{i,2}^M + \cdots + P_{i,1}^C \cdots P_{i,m_i-1}^C P_{i,m_i}^M \\ & = \sum_{j=1}^{m_i} \left(\prod_{s=1}^{j-1} P_{is}^C \right) P_{ij}^M. \end{aligned} \tag{32}$$

Note that when $i = N$, $P_i^M = 0$ as the system is replaced since it reaches the maximum number of maintenance. We also have

$$P_i^R = \begin{cases} 1 - P_i^M, & i = 0, \ldots, N-1; \\ 1, & i = N. \end{cases} \tag{33}$$

Thus P_n can be obtained from (27), (28), (29), (32) and (33). The expected uptime of the system depends on the number of time $Y(t)$ is observed before the next maintenance actions, and is given by

$$\begin{aligned} E\Big[T_i\Big|(M_{i+1}|M_i)\Big] & = \tfrac{1}{P_i^M} \big[P_{i,1}^M \tau + P_{i,1}^C P_{i,2}^M (2\tau) + \cdots \\ & \quad + P_{i,1}^C \cdots P_{i,k_i-1}^C P_{i,k_i}^M (k_i \tau) \big] \\ & = \tfrac{1}{P_i^M} \sum_{j=1}^{k_i} \left(\prod_{s=1}^{j-1} P_{is}^C \right) P_{ij}^M (j\tau). \end{aligned} \tag{34}$$

To find the expected uptime of the system after the nth maintenance action, we let T_{nj}^F $(n = 1, 2, \ldots, N,\ j = 1, 2, \ldots)$ be the uptime of the system between the last observation $Y(t)$ (i.e. the $(j-1)$th observation) after the nth maintenance action and system replacement. We consider the following two cases.

(1) $n < N$. In this case, after the nth maintenance action, an observed value $Y(t)$ indicates that system replacement is needed. After n maintenance actions, the expected

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uptime of the system before replacement is

$$\begin{aligned} E\Big[T_n\Big|(R|M_n)\Big] &= \tfrac{1}{P_n^R}\big[P_{n,1}^R\tau + P_{n,1}^F E[T_{n,1}^F] \\ &\quad + P_{n,1}^C\big(P_{n,2}^R(2\tau) + P_{n,2}^F(\tau + E[T_{n,2}^F])\big) \\ &\quad + \cdots \\ &\quad + P_{n,1}^C\cdots P_{n,k_n-1}^C\big(P_{n,k_n}^R(k_n\tau) \\ &\qquad + P_{n,k_n}^F((k_n-1)\tau + E[T_{n,k_n}^F])\big)\big] \\ &= \tfrac{1}{P_n^R}\sum_{j=1}^{k_n}\Big(\prod_{s=1}^{j-1}P_{ns}^C\Big)\big(P_{nj}^R(j\tau) \\ &\qquad + P_{nj}^F((j-1)\tau + E[T_{nj}^F])\big), \end{aligned} \tag{35}$$

To evaluate (35), we need to find $E[T_{nj}^F]$. Denote by $f_j(x)$ the pdf of $X(R_n^+ + (j-1)\tau)$. Since $X(R_{n+1}^+) = g(n)$, for $j = 2, 3, \ldots$ and $g(n) \le x \le D_F$ we have

$$f_j(x) = \frac{Ga(x-g(n)|(j-1)\tau\alpha,\beta)}{\int_{g(n)}^{D_F} Ga(u-g(n)|(j-1)\tau\alpha,\beta)du}. \tag{36}$$

By using the similar arguments in obtaining (17) the expectation $E[T_{nj}^F](j = 2, 3, \ldots)$ can be obtained as

$$E[T_{nj}^F] = \tau - \int_{g(n)}^{D_F}\frac{1}{F_y(\tau)}\Big(\int_0^\tau F_y(s)ds\Big)f_j(y)dy. \tag{37}$$

For the special case when $j = 1$, as the system fails before the first observation, the expectation $E[T_{n,1}^F]$ is simply given by

$$E[T_{n,1}^F] = \tau - \frac{\int_0^\tau F_{g(n)}(s)ds}{F_{g(n)}(\tau)}, \tag{38}$$

where $F_{g(n)}(t) = \frac{\Gamma\left(\frac{D_F-g(n)}{\beta},\alpha t\right)}{\Gamma(\alpha t)}$.

(2) $n = N$. In this case, the system is replaced after the N-th maintenance action, when either (i) an observed $Y(t)$ indicates that system replacement is needed, or (ii) an observed $Y(t)$ indicates that maintenance is needed. After N maintenance actions, the expected uptime of the system before replacement is

$$\begin{aligned} & E\Big[T_N\Big|(R|M_N)\Big] \\ =\ & (P_{N,1}^R + P_{N,1}^M)\tau + P_{N,1}^F E[T_{N,1}^F] \\ & + P_{N,1}^C\big((P_{N,2}^R + P_{N,2}^M)(2\tau) + P_{N,2}^F(\tau + E[T_{N,2}^F])\big) \\ & + \cdots \\ & + P_{N,1}^C\cdots P_{N,k_N-1}^C\big((P_{N,k_N}^R + P_{N,k_N}^M)(k_N\tau) \\ & \quad + P_{N,k_N}^F((k_N-1)\tau + E[T_{N,k_N}^F])\big) \\ =\ & \sum_{j=1}^{k_N}\Big(\prod_{s=1}^{j-1}P_{Ns}^C\Big)\big((P_{Nj}^R + P_{Nj}^M)(j\tau) \\ & \quad + P_{Nj}^F((j-1)\tau + E[T_{N,j}^F])\big), \end{aligned} \tag{39}$$

5 Conclusion

Assuming that degradation of the system is a Gamma process and maintenance is imperfect, the authors developed a maintenance model for maximizing the availability of the system. Heuristic method (such as an efficient discrete algorithm defined based on uniform design [3]) can be used to obtain the optimal solution.

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