High Order Conservative ENO/WENO Lagrangian Schemes for Euler Equations

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Abstract—**We develop a class of Lagrangian schemes for solving the Euler equations of compressible gas dynamics both on the quadrilateral meshes and curved quadrilateral meshes. The schemes are based on high order essentially non-oscillatory (ENO) and weighted ENO (WENO) reconstruction. They are conservative for the mass, momentum and total energy, can achieve at least uniformly second order accuracy both in space and time with the quadrilateral meshes and uniformly third order accuracy with the curved quadrilateral meshes, are essentially non-oscillatory, and have no parameters to be tuned for individual test cases. One and two dimensional numerical examples are presented to demonstrate the performance of the schemes in terms of accuracy, resolution for discontinuities, and non-oscillatory properties.**

Keywords: essentially non-oscillatory (ENO), weighted essentially non-oscillatory (WENO) , high order accuracy, Lagrangian scheme, Euler equations

1 Introduction

In numerical simulations of fluid flow, there are two typical choices: a Lagrangian framework, in which the mesh moves with the local fluid velocity, and an Eulerian framework, in which the fluid flows through a grid fixed in space. More generally, the motion of the grid can also be chosen arbitrarily, then the method is called the Arbitrary Lagrangian-Eulerian method (ALE; cf. [5]). Comparing with Eulerian methods, Lagrangian methods and certain ALE methods can capture contact discontinuities sharply, thus they are widely used in many fields for multi-material flow simulations in CFD and other applications (e.g. in astrophysics).

In the past near six decades, many Lagrangian schemes (cf. [7, 1, 2, 6]) were developed which have made successes in simulating the multi-material flows. These schemes usually have first or second order accuracy in space and first order in time. In this paper, we investigate high order Lagrangian type schemes. A class of high order Lagrangian type schemes are proposed for solving the Euler equations both on the quadrilateral meshes and the curved quadrilateral meshes respectively [3, 4]. The schemes are based on the high order ENO or simple WENO reconstruction, are conservative for the mass, momentum and total energy, and are essentially non-oscillatory. It can achieve at least uniformly second order accuracy on moving and distorted Lagrangian quadrilateral meshes and third order accuracy both in space and time on the curved quadrilateral meshes.

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In the Lagrangian framework, the Euler equations in the following integral form is approximated,

$$\frac{d}{dt}\int_{\Omega(t)} \mathbf{U} d\Omega + \int_{\Gamma(t)} \mathbf{F} d\Gamma = 0 \tag{1}$$

where $\Omega(t)$ is the moving control volume enclosed by its boundary $\Gamma(t)$. The vector of the conserved variables $\mathbf{U}$ and the flux vector $\mathbf{F}$ are given by

$$\mathbf{U} = \begin{pmatrix} \rho \\ \mathbf{M} \\ E \end{pmatrix}, \qquad \mathbf{F} = \begin{pmatrix} 0 \\ p \cdot \mathbf{n} \\ p\mathbf{u} \cdot \mathbf{n} \end{pmatrix}. \tag{2}$$

where ρ is the density, $\mathbf{u}$ is the velocity, $\mathbf{M} = \rho\mathbf{u}$ is the momentum, E is the total energy and p is the pressure, $\mathbf{n}$ denotes the unit outward normal to $\Gamma(t)$.

We use three steps to construct a high order ENO Lagrangian scheme, namely, spatial discretization, the determination of the vertex velocity and the time discretization[3]. In the step of spatial discretization, to get a high order scheme in space, the high order ENO or WENO reconstruction procedure is used to obtain high order and non-oscillatory approximations to the solution at the Gaussian points along the cell boundary from the neighboring cell averages. The determination of velocity at the vertex is a special step for a Lagrangian scheme which decides how the mesh moves at the next time. The reconstruction information of density and momentum from its four neighboring cell is used in this step which guarantees the approximation of velocity also has the same order accuracy as the primitive conserved variables. In the step of time discretization, TVD Runge-Kutta type methods are applied. One and two dimensional numerical examples both in the Cartesian and

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cylindrical coordinates are presented to demonstrate the performance of the schemes in terms of accuracy, resolution for discontinuities, and non-oscillatory properties.

In the accuracy test of the above ENO Lagrangian scheme with the quadrilateral meshes, a phenomenon for the high order Lagrangian type scheme is observed, that is, the third order Lagrangian type scheme can only achieve second order accuracy on two dimensional distorted Lagrangian meshes. This is analyzed to be due to the error from the mesh approximation. Since in a Lagrangian simulation, a cell with an initially quadrilateral shape may not keep its shape as a quadrilateral at a later time. It usually becomes a curved quadrilateral. Thus if during the Lagrangian simulation the mesh is always kept as quadrilateral with straight-line edges, this approximation of the mesh will bring second order error into the scheme. So for a Lagrangian type scheme in multi-dimensions, it can be at most second order accurate if curved meshes are not used. We further demonstrate the previous claim by developing a third order scheme on curved quadrilateral meshes in two space dimensions[4]. The reconstruction is based on the high order WENO procedure. Each curvilinear cell consists of four quadratically-curved edges by the information of the coordinates of its four vertices and the four middle points of its four edges. The accuracy test and some non-oscillatory tests are presented to verify the good performance of the scheme. The Lagrangian scheme can also be extended to higher than third order accuracy if a higher order approximation is used on both the meshes and the discretization of the governing equations.

3 Conclusions and Future Work

We have described a class of Lagrangian schemes for solving Euler equations which are based on high order essentially non-oscillatory (ENO) and weighted ENO (WENO) reconstruction both on the quadrilateral meshes and curved quadrilateral meshes. Although we have only performed tests on quadrilateral grids, the strategy can be used on any polygon grid such as triangles. The investigation and improvement of these high order schemes in multi dimensions in terms of robustness, constitute future work.

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