A Hybrid Finite Difference Method for Valuing American Puts

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Abstract—**This paper presents a numerical scheme that avoids iterations to solve the nonlinear partial differential equation system for pricing American puts with constant dividend yields. Upon applying a front-fixing technique to the Black-Scholes partial differential equation, a predictor-corrector finite difference scheme is proposed to numerically solve the discrete nonlinear scheme. In the comparison with the solutions from articles that cover zero dividend and constant dividend yields cases, our results are found accurate. The current method is conditionally stable since the Euler scheme is used, the convergency property of the scheme is shown by numerical experiments.**

Keywords:American Options, Predictor-Corrector, Finite Difference Method, Black-Scholes Equation

1 Introduction

Options are the most common securities that are frequently bought and sold in today's financial markets. Under the Black-Scholes partial differential equation (PDE) framework, Merton [1] casts the valuation problem of American options as a free-boundary problem in 1973. Ever since then, there have been two kinds of approximation methods in the literature, to solve the free-boundary problem associated with the valuation of American options. One approach is the analytical approximation method, e.g. the Quasi-analytical formula [2]. The other one is the numerical method, such as the Binomial Method [3], which are quite preferred by market practitioners, as they are usually much faster with acceptable accuracy.

In the last decade, various numerical methods have been presented by using the finite difference method (FDM), to solve the pricing problems of American options. For instance, Wu and Kwok [5] use a multilevel FDM to solve the nonlinear Black-Scholes PDE after applying a front-fixing technique [6], they adopt a so-called front-fixing technique or Landau transform [6] to fix the optimal exercise boundary on a vertical axis. To tackle the nonlinear nature of American option pricing problems, which is explicitly exposed after applying the front-fixing technique [6] to the original Black-Scholes PDE, they employ a two-level discretization scheme in time. However, since the scheme is a multilevel discretization scheme, the information at more than one time step is needed at the beginning to start the computation, which is referred to as the initialization for multilevel schemes in literature. The multilevel scheme of Wu and Kwok [5] motivates us a simpler version, while maintains the same level of computational accuracy.

To avoid the initialization and iteration, we propose a one-step scheme based on a prediction-correction framework. The approach adopts a predictor-corrector finite difference scheme at each time step to convert the nonlinear PDE to two linearized difference equations associated with the prediction and correction phase respectively. The predictor is constructed by an explicit Euler scheme, whereas the corrector is designed with the Crank-Nicolson scheme. The predictor is used only to calculate the optimal exercise price, as the literature shows that it is far more difficult to calculate the optimal exercise price with a high accuracy. The predicted optimal exercise price is then corrected in the correction phase together with the calculation of the option prices. The scheme maximizes the use of computational resources, as a high accuracy of the computed option price is easy to achieve as long as a high accuracy can be achieved in the computation of the optimal exercise price. The efficiency in the scheme results from the fact that only one set of linear algebraic equations needs to be solved at each time step.

The paper is organized as follows. Section 2 introduces the PDE system concerning the valuation of American put options. Section 3 presents a predictor-corrector scheme for computing the optimal exercise prices and the option values. In Section 4, some numerical examples are given to demonstrate the convergence and accuracy of the new scheme. Section 5 draws conclusions.

2 Partial Differential Equation System

This paper considers a general case in which a constant dividend yield is associated with the underlying asset and adopt the PDE given in Merton [1]. Let V denote the value of an American put option, which is a function of

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the value of underlying asset S and the time t. The value of an American put option also depends on the following parameters:
σ, the volatility of the underlying asset;
T, the life time of the contract;
X, the strike price;
r, the risk-free interest rate;
D_0, the dividend yield.
Without loss of generality, we assume that both the risk-free interest rate and the dividend yield be constants. The functions can be easily modified for the cases when they are some known functions of time and asset values.

Since American options can be decomposed into its European counterparts plus an early exercise premium, this early exercise premium is associated with the extra right embedded in American options in comparison with its European counterparts. Wilmott *et al.* [9] show that there are two boundary conditions of the optimal exercise price $S = S_f(t)$ for American options:

$$\begin{cases} V(S_f(t), t) = X - S_f(t), \\ \frac{\partial V}{\partial S}(S_f(t), t) = -1. \end{cases} \tag{1}$$

To close the system, another boundary condition at the end of large asset value, i.e. the payoff of the contract at the expiry is necessary,

$$\lim_{S\to\infty} V(S, t) = 0, \tag{2}$$

and the terminal condition for a put option is

$$V(S, T) = \max\{X - S, 0\}. \tag{3}$$

In summary, the differential system for pricing American put options can be written as:

$$\begin{cases} \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0)S\frac{\partial V}{\partial S} - rV = 0, \\ V(S_f(t), t) = X - S_f(t), \\ \frac{\partial V}{\partial S}(S_f(t), t) = -1, \\ \lim_{S\to\infty} V(S, t) = 0, \\ V(S, T) = \max\{X - S, 0\}. \end{cases} \tag{4}$$

To solve the differential system Eq. (4) effectively, we normalize all variables in the system by introducing the following scale of variables,
$V' = \frac{V}{X}$, $S' = \frac{S}{X}$, $\tau = (T - t)\frac{\sigma^2}{2}$, $\gamma = \frac{2r}{\sigma^2}$,
$D = \frac{2D_0}{\sigma^2}$, $S'_f(\tau) = \frac{S_f(T - 2\tau/\sigma^2)}{X}$.
After normalizing Eq. (4), dropping the primes, and imposing the Landau transform [6],

$$x = \ln \frac{S}{S_f(\tau)}, \tag{5}$$

the original system becomes:

$$\begin{cases} \frac{\partial P}{\partial \tau} - \frac{\partial^2 P}{\partial x^2} + (\gamma - D - 1)\frac{\partial P}{\partial x} + \gamma P = \\ \frac{\partial P}{\partial x}\frac{1}{S_f(\tau)}\frac{dS_f(\tau)}{d\tau}, \\ P(0, \tau) = 1 - S_f(\tau), \\ \frac{\partial P}{\partial x}(0, \tau) = -S_f(\tau), \\ \lim_{x\to\infty} P(x, \tau) = 0, \\ P(x, 0) = 0. \end{cases} \tag{6}$$

After this rather simple manipulation, the nonlinear nature of the problem is explicitly exposed in the inhomogeneous term on the right hand side of Eq. (6), which consists the product of the Delta of the unknown option price under the Landau transform, the time derivative of the unknown optimal exercise boundary $S_f(\tau)$ and its reciprocal.

One should note that we have replaced the unknown function $V(S, t)$ in Eq. (4), with a new unknown function P, which is defined as $P(x, \tau) = V(S(x, (\tau)), \tau)$ through the transform defined in Eq. (5). This is to facilitate the introduction of a relation between $P(0, \tau)$ and the $S_f(\tau)$ on the boundary $x = 0$, which is used to design the predictor of the numerical scheme. Moreover, one should also note that the transform in Eq. (5) only holds if $S_f(\tau) > 0$. This condition poses no problem since it is easy to show that the $S_f(\tau)$ for an American put option is a monotonically decreasing function of τ; the minimum value $S_f(\tau)$ is the optimal exercise price of the corresponding perpetual contract. For a perpetual American put on a constant dividend yield paying asset, this value was shown as follows:

$$\lim_{\tau\to\infty} S_f(\tau) = \frac{\eta + \sqrt{\eta^2 + 4\gamma}}{2 + \eta + \sqrt{\eta^2 + 4\gamma}}, \tag{7}$$

with $\eta = \gamma - D - 1$. It is then very trivial to show that $S_f(\tau) > 0$ for any η values. Therefore, the differential system Eq. (5) defines a well-posed problem, other than a well-known singular point at $\tau = 0$ (see Barles *et al.* [10]). We now propose an efficient and accurate numerical scheme to solve this system.

3 The Predictor-Corrector FDM Scheme

This section presents the predictor-corrector scheme. We propose to solve the nonlinear PDE in differential system Eq. (6) in two phases within a time step, a prediction phase in which an initial guess of the $S_f(\tau)$ is worked out before its final value is calculated together with the option value $P(x, \tau)$ in the correction phase of the scheme.

Beginning with truncating the bounded x domain, as well as the time domain τ, the computational domain is discretized with uniformly spread $M + 1$ grids placed in the x direction and $N + 1$ grids in the τ direction (i.e., M and N are the number of steps in these two directions, respectively). For the easiness of presentation, we denote the step length in the x direction by $\Delta x = \frac{x_{max}}{M}$ and that in the τ direction by $\Delta\tau = \frac{\tau_{exp}}{N}$, in which τ_{exp} is the normalized tenor of the contract with respect to half of the variance of the underlying asset, i.e., $\tau_{exp} = T\sigma^2/2$. Consequently, the value of unknown function P at a grid point is denoted by P^n_m with the superscript n denoting the nth time step and the subscript m denoting the mth log-transformed asset grid point.

To facilitate the numerical computation, we derive an additional boundary condition to construct our predictor-corrector scheme. This condition is not independent from all those boundary conditions prescribed in Eq. (6). Rather, it is derived by making use of the PDE in Eq. (6) as well as the boundary conditions that have already made the system closed. Firstly, we take a partial derivative with respect to τ on both sides of the first boundary condition in Eq. (6), which yields

$$\frac{\partial P}{\partial \tau}(0,\tau) = -\frac{dS_f(\tau)}{d\tau}. \tag{8}$$

In fact, one easily shows that Eq. (8) is consistent with the condition $\frac{\partial V}{\partial \tau}(S_f(\tau),\tau) = 0$ in Bunch and Johnson's paper [7]. Then, if we evaluate the PDE in Eq. (6) at $x = 0$, utilizing Eq. (8) and the second boundary condition in Eq. (6), we obtain

$$\frac{\partial^2 P}{\partial x^2}_{|x=0} - (D+1)S_f(\tau) + \gamma = 0, \quad if \quad \tau > 0. \tag{9}$$

Eq. (9) reveals a relationship of the put option price and the optimal exercise price at any time, except on the expiry day. This relation is important to our scheme in eliminating the value of the unknown function defined on the fictitious grid point near the boundary $x = 0$, when the second-order central difference scheme is applied. The reason that it is only valid for $\tau > 0$ is the inherent singular behavior of the Black-Scholes PDE at $\tau = 0$ (see Barles *et al.* [10]).

Applying a second-order central difference scheme to the equation, one has the asset price discretization in the x direction. Eq. (9) and the boundary conditions in Eq. (6) is written as

$$-\frac{P_1^{n+1} - 2P_0^{n+1} + P_{-1}^{n+1}}{\Delta x^2} - (D+1){S_f}^{n+1} + \gamma = 0, \tag{10}$$

and

$$\begin{cases} P_0^{n+1} = 1 - {S_f}^{n+1}, \\ \frac{P_1^{n+1} - P_{-1}^{n+1}}{2\Delta x} = -{S_f}^{n+1}, \\ P_M^{n+1} = 0, \\ P_m^0 = 0, \end{cases} \tag{11}$$

respectively. Upon eliminating the fictitious nodal value P_{-1}^{n+1} from Eq. (10) and the second equation in Eq. (11), we obtain a relation between S_f and P_1 at the $(n+1)th$ time step as

$$P_1^{n+1} = \alpha - \beta {S_f}^{n+1}, \tag{12}$$

in which $\alpha = 1 + \frac{\gamma}{2}\Delta x^2$ and $\beta = 1 + \Delta x + \frac{D+1}{2}\Delta x^2$. Eq. (12) is used in the predictor and corrector construction.

Predictor: The predictor is constructed by using the explicit Euler scheme to calculate a guessed value of ${S_f}^{n+1}$, which is denoted as $\hat{S_f}^{n+1}$. Applying the explicit Euler scheme to the PDE in Eq. (6) results in

$$\frac{\hat{P}_1^{n+1} - P_1^n}{\Delta\tau} - \frac{P_2^n - 2P_1^n + P_0^n}{\Delta x^2} - (\gamma - D - 1)\frac{P_2^n - P_0^n}{2\Delta x} + \gamma P_1^n = \frac{P_2^n - P_0^n}{2\Delta x}\frac{1}{{S_f}^n}\frac{\hat{S_f}^{n+1} - {S_f}^n}{\Delta\tau}, \tag{13}$$

which is coupled with Eq. (12) to generate the $\hat{S_f}^{n+1}$ value. The boundary condition of $\hat{P}_0^{n+1}$ used in the corrector is also predicted here; with the calculated $\hat{S_f}^{n+1}$ value, $\hat{P}_0^{n+1}$ is calculated from the first equation in Eq. (11), which is nothing but the payoff function. Like the predicted $\hat{S_f}^{n+1}$ value, this predicted boundary value of $\hat{P}_0^{n+1}$ will also be corrected once the $\hat{S_f}^{n+1}$ is corrected in the following corrector scheme.

Corrector: The corrector is based on the Crank-Nicolson scheme, applied to the linearized PDE in Eq. (6). The linearlization is designed with an alternating term being valued at the current time step in comparison with that in the predictor. In the predictor, we let the time derivative of the S_f in the nonlinear inhomogeneous term be valued at the current time step, whereas now we let the asset price derivative of P be valued at the current time step through the Crank-Nicolson scheme. This alternating approach, inspired by the idea of the ADI approach in solving two dimensional time-dependent PDEs [11], has an advantage of reducing the numerical errors induced in the prediction-correction process. The finite difference scheme used for the corrector is

$$\begin{aligned} &\frac{P_m^{n+1} - P_m^n}{\Delta\tau} + \gamma\frac{P_m^{n+1} + P_m^n}{2} - \\ &\quad \frac{P_{m+1}^{n+1} - 2P_m^{n+1} + P_{m-1}^{n+1} + P_{m+1}^n - 2P_m^n + P_{m-1}^n}{2\Delta x^2} \\ &\quad - (\gamma - D - 1)\frac{P_{m+1}^{n+1} - P_{m-1}^{n+1} + P_{m+1}^n - P_{m-1}^n}{4\Delta x} + \\ &= \frac{P_{m+1}^{n+1} - P_{m-1}^{n+1} + P_{m+1}^n - P_{m-1}^n}{4\Delta x} \times \frac{2}{\hat{S_f}^{n+1} + {S_f}^n} \\ &\times \frac{\hat{S_f}^{n+1} - {S_f}^n}{\Delta\tau}. \end{aligned} \tag{14}$$

In Eq. (14), m value starts from 1 to $M-1$, which indicates that $M-1$ equations are solved simultaneously to obtain the corrected option values at the $(n+1)th$ time step. P_1^{n+1} is obtained upon solving Eq. (14). Then, by means of Eq. (12), the newly-obtained P_1^{n+1} is used to correct the ${S_f}^{n+1}$, which is then used to correct the P_0^{n+1} value before it is used in the calculation of the next time step. And Eq. (14) can be written in matrix form which is a more condensed way for Matlab computation. This predictor-corrector process is repeated until the expiry time is reached. We solve these matrix equations in Matlab, Version 7 on a Intel P4 machine.

4 Numerical Examples

Although the Crank-Nicolson scheme for the corrector is unconditionally stable [11], our predictor-corrector finite difference scheme is only conditionally stable since the explicit Euler scheme for the predictor is conditionally stable. In this section, the conditional stability of our approach, as well as the accuracy shall be verified empirically.

4.1 Discussion on Convergence

For the linearized system, the proof of the consistency is trivial through the application of Taylor's expansion and thus is omitted here. A theoretical proof the stability for the linearized system, on the other hand, is not so trivial because of the presence of the singularity at $\tau = 0$ (see Barles *et al.* [10]). Therefore, we establish a stability criterion empirically. Based on preliminary numerical experiments, we were convinced that the stability criterion $\frac{\Delta\tau}{\Delta x^2} \leq 1$ should be imposed in the selection of time step length for a given grid size in the x direction. Between the option price and the optimal exercise price, the latter is far more difficult to calculate accurately; once the $S_f(\tau)$ is determined accurately, the calculation of the option price itself is straight forward. Therefore, in this subsection we focus on the calculation of the $S_f(\tau)$ first.

The example we chose for our numerical tests has been used by researchers for the discussion of American puts on an asset without any dividend payment [4, 5]. The relevant parameters are: the strike price $X = \$100$, the interest rate $r = 10\%$, the volatility of the underlying asset $\sigma = 30\%$ and the tenor of the option being one year. In this subsection, we focus only on the zero-dividend case, i.e., we set the constant dividend yield to zero. For the convenience of those readers who prefer to see the results in dimensional form, all results presented in this section are those associated with the original dimensional quantities before the normalization process was introduced.

Firstly, we examined a point-wise convergence by focusing on a specific point of the S_f value first. As an indicator, the differences of S_f values at a specific time to expiry, say 1 year, are calculated with time step size being consecutively halved. Table 1 shows the differences of the computed S_f values with the total number of grid points in the x direction being fixed to 51, while the number of time step intervals is consecutively doubled from 200 to 3200 (the time step size is consecutively halved). One should note that in Table 1, the "difference" refers to the absolute change in S_f values when the time step size is halved, while the "ratio" refers to the ratio of successive differences. Theoretically, the order of convergence is related to calculated ratio by $ratio = 2^k$, in which k is the order of convergence. Clearly, when the grid size in the x direction is fixed, the ratios of the differences of two S_f values at $\tau = 1$ year with two consecutive calculations of time step length being halved indeed approach 2, which indicates that our scheme is indeed of the first order in time.

Table 1: Ratios for the order of convergence in time

Time steps	S_f (\$)	difference	ratio
200	76.126		
400	76.121	0.00000406	
800	76.120	0.00000125	3.261
1600	76.120	0.00000060	2.072
3200	76.119	0.00000030	2.036

Table 2: Ratios for the order of convergence in asset price

Grid intervals	S_f (\$)	difference	ratio
50	76.120		
100	76.150	0.00030017	
200	76.160	0.00010254	2.927
400	76.164	0.00003342	3.068
800	76.164	0.00001114	3.000

Then we fixed the time step size to $\Delta\tau = \frac{\tau_{max}}{1600}$ instead and examine the ratios of the of the differences of two S_f values at $\tau = 1$ year with the two consecutive calculations of x grid length being halved, we find that these ratios are close to 3, as shown in Table 2. This indicates that the order of convergence in the x direction is certainly higher than one but lower than theoretically predicted 2nd order convergence of the Crank-Nicolson scheme. One plausible reason for this is that the errors introduced in the predictor somehow reduced the order of convergence in the x direction a bit, so that now it is of an order of one and half rather than two.

Having discussed the point-wise convergence, we tested the convergence of the new scheme on the entire solution of S_f. We first ran our code with an extremely fine grid, e.g., the N and M are set up as 102,400 and 1,000, respectively. Naturally, this takes a long time to compute. But, once the S_f values are computed on this fine grid, we used these values as the reference values to verify the convergence of computed S_f values based of some coarse grid. To measure the overall difference between the results of the coarse grid and those of the finest grid, we use two error measures. The root mean square absolute errors (RMSAE), which is usually referred as root mean square errors. In order to tell relative errors, a modification of root mean square errors is used here, we refer it as the root mean square relative errors (RMSRE). The two measures are defined respectively as

$$RMSAE = \sqrt{\frac{1}{I}\sum_{i=1}^{I}(\tilde{a}_i - a_i)^2}, \qquad (15)$$

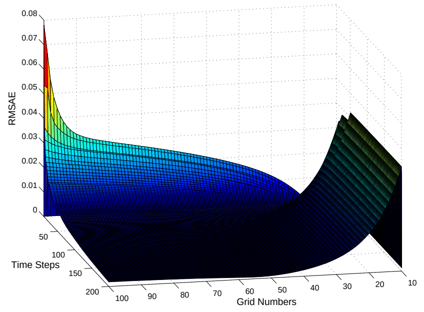

Figure 1: The RMSAE with Increased Grid numbers

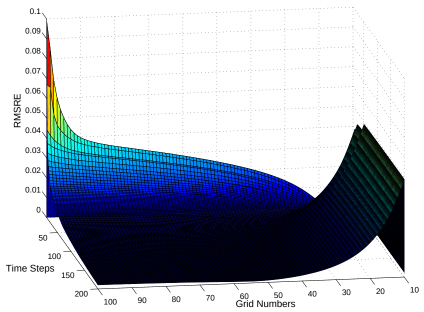

Figure 2: The RMSRE with Increased Grid Numbers

$$RMSRE = \sqrt{\frac{1}{I}\sum_{i=1}^{I}(\frac{\tilde{a}_i - a_i}{a_i})^2}, \qquad (16)$$

where $\tilde{a}_i$s are the nodal S_f values associated with coarse grid; a_is are the S_f values associated with finest grid and I is the number of sample points used in the RMSAE and RMSRE. In the following experiments, I was set to be 50 for all the results shown in the following diagrams.

By demonstrating the RMSAE and RMSRE, we obtain an overall measure of the convergence to make sure what we observed from analyzing the order of convergence previously based on one point only is also true for other grid points. Figure 1 and 2 show the RMSAE and RMSRE respectively, for the S_f values when the number steps in the x direction and the τ direction are gradually increased. As can be clearly seen from these figures, the RMSAE reduces by nearly 10 folds when the grid size changes from $N = 10$ and $M = 10$ (with $RMSAE = 0.0254$) to $N = 200$ and $M = 100$ (with $RMSAE = 0.0028$). In fact, the difference between the results obtained with a coarse grid and those obtained with the finest grid is better reflected by the RMSRE, which shows very similar trend as that of the RMSAE; when the number of grid has increased to $N = 200$ and $M = 100$, the RMSRE has reached 0.3%, which is quite an acceptable accuracy in comparison with the solution based on an extremely refined grid. This confirms that analysis of convergence order presented earlier can be extended to other grid points as well. Therefore, we are confident that the scheme can lead to a uniform order of convergence for the calculation of the optimal exercise prices.

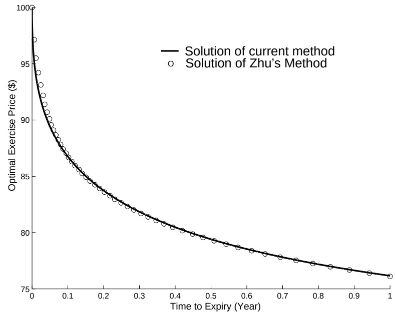

Figure 3: Comparison: Two Optimal Exercise Boundaries

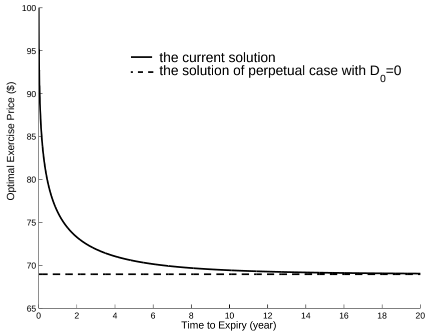

Figure 4: Optimal Exercise Prices with A Long Tenor

4.2 Discussion on Accuracy

This subsection proves that the numerical solution of the linearized PDE does converge to that of the original nonlinear PDE system. We firstly compare our results with Zhu's semi-closed solution in the non-dividend case [4]. If one can demonstrate that the converged solution approaches Zhu's solution, it is confident to say that the linearization process we took before the predictor-corrector scheme was applied. Figure 3 shows such a comparison with the optimal exercise boundaries being computed by using the current scheme with $N = 102,400$, $M = 1,000$ and Zhu's solution [4]. As it can be seen from this figure, the two results agree with each other almost perfectly, especially when the time to expiry increases. A close examination reveals that the current approach slightly underestimates the S_f values when the time is close to expiry. When the time to expiry increases from 0 to 1 year, the under-estimation gradually improves from a roughly 2.16% at the time to expiry $(T - t)$ being 0.0067 year, to 0.05% at the time to expiry being 1 year. Given the presence of the well-known singularity at the expiry [10], which is not possible for any numerical scheme to deal with, the performance of the proposed numerical scheme is certainly very satisfactory.

Another test that a good numerical scheme must pass is

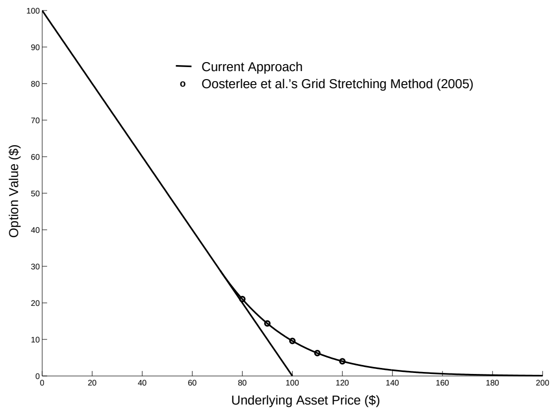

Figure 5: The Option Value with $D_0 = 5\%$, $T = 1$ year

that the optimal exercise price asymptotically approaches to that of its corresponding perpetual counterpart when the lifetime of the put option becomes infinite. In this extreme case, it was reported in the literature that some approaches lead to an oscillatory and non-monotonic optimal exercise price when the lifetime of an option is very long. We have prolonged the lifetime to 20 years to artificially make the option in this example a long-lifetime option. Again, using the finest grid $N = 102,400$ and $M = 1,000$, We calculated the $S_f(\tau)$ and plotted its value against the theoretical perpetual optimal exercise price given in Eq. (7), as shown in Figure 4. Clearly, the numerical solution exhibits a nice asymptotical approach to the optimal exercise price of the corresponding perpetual put option; no oscillation was observed at all. This shows that our scheme is very stable and can be used for for long-lifetime options as all.

4.3 Option Prices in Constant Dividend Yield Cases

This subsection presents option prices from the current method for constant dividend yields case discussion. The relevant option parameters used in the following example are the same as those used in the non-dividend case, except the constant dividend yield D_0 is now set at 5%. The results presented in this section were obtained using a grid resolution of $N = 200$ and $M = 100$.

Figure 5 shows a comparison of the option values calculated by using the current approach and the ones from Oosterlee *et al.* [12], who employ the so-called Grid Stretching Method. The option values in Figure 5 are plotted against the underlying asset prices at time to expiry being 1 year. The agreement between the two appears to be excellent, reinforcing the fact that once the optimal exercise price can be accurately calculated, the accurate calculation of the option price itself naturally follows.

5 Conclusion

This paper presents a new predictor-corrector scheme to numerically tackle American put option pricing with constant dividend yields. The key feature of the current scheme is its high efficiency since there is neither iteration nor initialization required. Through a couple of numerical examples, we have demonstrated the convergency and accuracy of the proposed scheme.

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