Valuation of Fixed Rate Mortgages by Moving Boundary Approach

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Abstract

In this communication a moving boundary approach has been proposed to valuate a fixed rate mortgage contract which allows the mortgage holder to prepay the outstanding balance of the mortgage. This new method transforms the free-boundary problem into a monotonic sequence of moving boundary problems each of which can be solved by standard techniques. Illustrative examples demonstrate that the moving boundary approach is able to provide an accurate approximation to both the early exercise boundary and the contract value very efficiently. Furthermore, the approximate results can be systematically improved by the multi-stage approximation to the early exercise boundary.

Keywords: Vasicek model; Fixed rate mortgages; Free boundary problems; Moving boundary approach

1. Introduction

The analysis of contingent claims (financial derivatives) of the American style remains to be a fundamental problem in mathematical finance. Despite more than three decades of efforts, no exact analytical solution to these free boundary problems is known yet, and one often resorts to time-consuming direct numerical valuation. In this communication we present a moving boundary approach to tackle one of these free boundary problems, namely the valuation of a fixed rate mortgage contract which provides the mortgage holder the right to prepay the outstanding balance of the mortgage. In analogy to the standard problem of bond valuation, the fixed rate mortgage contract can be formulated as a derivative product of the stochastic interest rate. Assuming that the stochastic interest rate r follows the Vasicek model:

$$dr = \kappa\left(\theta - r\right)dt + \sigma dZ \qquad , \tag{1}$$

where κ is the mean reverting speed, θ is the long term mean of r, σ is the volatility of r and dZ refers to a standard Wiener process, the standard risk-neutral pricing of an amortized mortgage contract $V\left(r,\tau\right)$ with a duration T and a fixed mortgage interest rate c is obtained by solving the partial differential equation

$$\begin{aligned}\frac{\partial V(r,\tau)}{\partial\tau} &= \frac{1}{2}\sigma^2\frac{\partial^2 V(r,\tau)}{\partial r^2} + \kappa\left(\theta - r\right)\frac{\partial V(r,\tau)}{\partial r} \\ &\quad -rV(r,\tau) + m\end{aligned} \tag{2}$$

for $r \geqslant h\left(\tau\right)$ and $\tau \geqslant 0$, subject to the boundary conditions

$$\begin{aligned} V\left(r,\tau\right) &= M\left(\tau\right) && \text{for} \quad r = h\left(\tau\right) \\ \frac{\partial V\left(r,\tau\right)}{\partial r} &= 0 && \text{for} \quad r = h\left(\tau\right) \\ V\left(r,0\right) &= 0 && \text{for} \quad r \geqslant h\left(0\right) = c \end{aligned} \tag{3}$$

Here $\tau = T - t$ is the time to maturity, m is the rate of payment of the mortgage, and $M\left(\tau\right)$ is the outstanding loan balance given by

$$M\left(\tau\right) = \frac{m}{c}\left\{1 - \exp\left(-c\tau\right)\right\} \qquad . \tag{4}$$

Provided that the mortgage holder always has sufficient funds for prepayment, the decision on exercising the right to prepay obviously depends upon the rate of return r that can be obtained by investing the amount $M\left(\tau\right)$ in other financial instruments. The optimal strategy for the mortgage holder is to exercise the option to pay off the mortgage the first time that the rate r falls below $h\left(\tau\right)$, *i.e.* the unknown early exercise boundary. By means of a variational

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analysis Jiang *et al.* (2005) has proven the mathematical well-posedness of this problem and the existence of a unique solution.

Standard numerical techniques such as the finite difference method and binomial tree method have been applied to this problem, but they all suffer from poor accuracy and stability in addition to slow convergence. Recently, an algorithm using integral equations was proposed by Xie *et al.* (2007 and 2008) to determine the early exercise boundary and valuate the mortgage contract. Owing to the iterative nature of the method, we find that the computing time required appears to increase rapidly with the time to maturity of the contract. Nevertheless, these shortcomings do not appear in the proposed moving boundary approach. Our method is based upon solving the pricing equation of the mortgage contract for a parametric class of moving boundaries, among which the optimal solution is selected to approximate the exact result. Thus, the moving boundary approach merely transforms the free-boundary problem into a monotonic sequence of moving boundary problems, each of which is linear and can be solved by standard techniques. As shown in the following, the moving boundary approach allows us to reduce the problem to one involving simple one-dimensional numerical integrals only and thus is capable of producing accurate estimation of the contract value efficiently. It should be pointed out that the approximate optimal exercise boundary is not only very close to the exact one but it is also an upper bound of the exact boundary. From the financial point of view, the upper bound provides an early signal for optimal exercising of the mortgage contract, which is believed to be useful information for any interested parties such as the borrower or potential buyers of the contract. Furthermore, the approximate results can be systematically improved by applying the multi-stage approximation proposed by Lo *et al.* (2003) to the early exercise boundary.

In next section we outline the proposed moving boundary approach for obtaining an accurate estimation of the exact result of the free boundary problem. Then, in section 3 some numerical results are presented and the performance of our method is discussed. Finally, a brief summary of our investigation is presented in the last section.

2. Moving boundary approach

The essence of the moving boundary approach is that instead of solving the free boundary problem defined by Eqs.(2)-(4), we tackle a less difficult problem, namely a boundary-value problem associated with Eq.(2) subject to the constraints given in Eq.(3) for a parametric class of moving boundaries:

$$h(\tau) = \{c - 2\beta\eta(\tau) - \alpha_3(\tau)\}\exp\{-\alpha_1(\tau)\} \tag{5}$$

where β is an adjustable real parameter and

$$\begin{aligned}
\alpha_1(\tau) &= -\kappa\tau \\
\alpha_3(\tau) &= \left(\theta - \frac{\sigma^2}{2\kappa^2}\right) - \left(\theta - \frac{\sigma^2}{\kappa^2}\right)\exp(-\kappa\tau) \\
&\quad -\frac{\sigma^2}{2\kappa^2}\exp(-2\kappa\tau) \\
\eta(\tau) &= \frac{\sigma^2}{4\kappa}\{1 - \exp(-2\kappa\tau)\} \quad .
\end{aligned} \tag{6}$$

Among this parametric class of solutions, the optimal solution is then selected to approximate the exact solution of the free boundary problem. To solve the boundary-value problem, we first assume that the solution $V(r,\tau)$ takes the form

$$\begin{aligned}
&V(r,\tau) \\
= \;& \exp\left\{\alpha_4(\tau) + \beta\alpha_3(\tau) + \beta^2\eta(\tau)\right\} \times \\
& \exp\{[\beta + \alpha_2(\tau)]\exp\{\alpha_1(\tau)\} r\} \times \\
& U\left(re^{\alpha_1(\tau)} + 2\beta\eta(\tau) + \alpha_3(\tau) - c, \eta(\tau)\right)
\end{aligned} \tag{7}$$

where

$$\begin{aligned}
\alpha_2(\tau) &= \frac{1}{\kappa}\{1 - \exp(\kappa\tau)\} \\
\alpha_4(\tau) &= \left(\frac{\theta}{\kappa} - \frac{3\sigma^2}{4\kappa^3}\right) - \left(\theta - \frac{\sigma^2}{2\kappa^2}\right)\tau - \\
&\quad \left(\frac{\theta}{\kappa} - \frac{\sigma^2}{\kappa^3}\right)\exp(-\kappa\tau) - \\
&\quad \frac{\sigma^2}{4\kappa^3}\exp(-2\kappa\tau) \quad .
\end{aligned} \tag{8}$$

By direct substitution, we are able to re-cast Eq.(2) into an inhomogeneous diffusion equation:

$$\frac{\partial U(r,\eta(\tau))}{\partial \eta} - \frac{\partial^2 U(r,\eta(\tau))}{\partial r^2} = p(r,\eta(\tau)) \tag{9}$$

where

$$\begin{aligned}
p(r,\eta(\tau)) &= \frac{2m}{\sigma^2}\exp\{-2\alpha_1(\tau)\} \times \\
&\quad \exp\{-[\beta + \alpha_2(\tau)](r + c)\} \times \\
&\quad \exp\{-\alpha_4(\tau) + \alpha_2(\tau)\alpha_3(\tau) + \\
&\quad \beta^2\eta(\tau) + 2\beta\eta(\tau)\alpha_2(\tau)\} \quad .
\end{aligned} \tag{10}$$

Accordingly, the boundary conditions defined by Eq.(3) becomes

$$\begin{aligned}
U(0,\eta(\tau)) &= F(\eta(\tau)) \\
\left.\frac{\partial U(r,\eta(\tau))}{\partial r}\right|_{r=0} &= -[\beta + \alpha_2(\tau)]F(\eta(\tau)) \\
U(r,\eta(\tau) = 0) &= 0
\end{aligned} \tag{11}$$

where

$$F(\eta(\tau)) = M(\tau)\exp\{-[\beta+\alpha_2(\tau)]c\}\times \exp\{-\alpha_4(\tau)+\alpha_2(\tau)\alpha_3(\tau)+ \beta^2\eta(\tau)+2\beta\eta(\tau)\alpha_2(\tau)\} \quad . \quad (12)$$

As a result, we are now left with a boundary-value problem associated with an inhomogeneous diffusion equation.

By introducing $W(r,\eta) = U(r,\eta) - F(\eta)$, the boundary-value problem can be reduced to

$$\frac{\partial W(r,\eta)}{\partial\eta} - \frac{\partial^2 W(r,\eta)}{\partial r^2} = p(r,\eta) - \frac{dF(\eta)}{d\eta} \quad (13)$$

with

$$\begin{aligned} W(0,\eta) &= W(r,0) = 0 \\ \left.\frac{\partial W(r,\eta)}{\partial r}\right|_{r=0} &= -[\beta+\alpha_2(\tau)]F(\eta) \quad . \quad (14) \end{aligned}$$

In terms of the classical Green's function of the diffusion equation (Kevorkian, 2000):

$$\begin{aligned} & G(r,\eta;r',\eta') \\ = & \frac{1}{\sqrt{4\pi(\eta-\eta')}}\exp\left\{-\frac{(r-r')^2}{4(\eta-\eta')}\right\} \quad , \quad (15) \end{aligned}$$

the solution of Eq.(13), subject to the homogeneous boundary conditions: $W(0,\eta) = W(r,0) = 0$, is given by

$$\begin{aligned} & W(r,\eta) \\ = & \int_0^\eta d\eta' \int_0^\infty dr' \left[G(r,\eta;r',\eta') - G(r,\eta;-r',\eta')\right] \\ & \times\left\{p(r',\eta') - \frac{dF(\eta')}{d\eta'}\right\} \\ = & F(\eta) - 2\int_0^\eta d\eta' \left[\frac{dF(\eta')}{d\eta'} N\left(\frac{r}{\sqrt{2(\eta-\eta')}}\right) - \right. \\ & \left\{N\left(\frac{r-2(\eta-\eta')[\beta+\alpha_2(\tau')]}{\sqrt{2(\eta-\eta')}}\right)\times \right. \\ & e^{-[\beta+\alpha_2(\tau')]r} - e^{[\beta+\alpha_2(\tau')]r}\times \\ & \left. N\left(-\frac{r+2(\eta-\eta')[\beta+\alpha_2(\tau')]}{\sqrt{2(\eta-\eta')}}\right)\right\}\times \\ & \left.\frac{mF(\eta')}{\sigma^2 M(\tau')} e^{-2\alpha_1(\tau')+(\eta-\eta')[\beta+\alpha_2(\tau')]^2}\right] \quad (16) \end{aligned}$$

where $N(\cdot)$ denotes the cumulative normal distribution function,

$$\eta'(\tau') = \frac{\sigma^2}{4\kappa}\{1-\exp(-2\kappa\tau')\} \quad (17)$$

and the one-dimensional integral over η' can be efficiently evaluated by Gaussian quadrature. Once $W(r,\eta)$ is known, the price $V(r,\tau)$ of the mortgage contract can be obtained readily.

The remaining task is to determine the parameter β that characterizes the optimal moving boundary to approximate the free boundary of the mortgage problem. It was rigorously proven by Jiang *et al.* (2005) that the price $V(r,\tau)$ of the mortgage contract is a monotonic decreasing function of the short term interest rate r.[1] That is, the admissible values of β must ensure that

$$\left.\frac{\partial V(r,\tau)}{\partial r}\right|_{r=h(\tau)} \leqslant 0 \quad . \quad (18)$$

Thus, to fix the optimal value of β, we can simply require that the solution $V(r,\tau)$ of Eq.(2) satisfies the boundary condition:

$$\begin{aligned} & \left.\frac{\partial V(r,\tau)}{\partial r}\right|_{r=h(\tau)} \\ = & M(\tau)\exp\{\alpha_1(\tau)\}\times \\ & \left\{[\beta+\alpha_2(\tau)] + \frac{1}{F(\eta)}\left.\frac{\partial W(r,\eta)}{\partial r}\right|_{r=0}\right\} \\ = & 0 \quad , \quad (19) \end{aligned}$$

where

$$\begin{aligned} & \left.\frac{\partial W(r,\eta)}{\partial r}\right|_{r=0} \\ = & -2\int_0^\eta d\eta' \left[\frac{2mF(\eta')[\beta+\alpha_2(\tau')]}{\sigma^2 M(\tau')}\times \right. \\ & e^{-2\alpha_1(\tau')+(\eta-\eta')[\beta+\alpha_2(\tau')]^2}\times \\ & N\left(-[\beta+\alpha_2(\tau')]\sqrt{2(\eta-\eta')}\right) + \\ & \frac{1}{\sqrt{4\pi(\eta-\eta')}}\left\{\frac{dF(\eta')}{d\eta'} - \right. \\ & \left.\left.\frac{2mF(\eta')\exp\{-2\alpha_1(\tau')\}}{\sigma^2 M(\tau')}\right\}\right] \quad , \quad (20) \end{aligned}$$

at $\tau = T$. It is not difficult to show that the requirement in Eq.(18) is automatically satisfied by this optimal β for $\tau < T$, too. The root-finding task which can be easily achieved by standard algorithms is significantly simplified by considering the behaviour of the free boundary $h(\tau)$ near $\tau = 0^+$. From Eq.(5), we can easily see that for $0 < \tau \ll 1$,

$$h(\tau) \approx c - \left[\beta + (\theta - c)\frac{\kappa}{\sigma^2}\right]\sigma^2\tau \quad . \quad (21)$$

[1] As the market return rate increases while the mortgage interest rate remains the same, the relative return rate of the mortgage contract would decrease and hence the fair price of the mortgage contract drops.

Since $h(\tau) < c$ for $\tau > 0$, we must require that

$$\beta > -(\theta - c)\frac{\kappa}{\sigma^2} \quad . \tag{22}$$

Moreover, in Jiang *et al.* (2005) it has been proven that the exact free boundary $h_{\text{exact}}(\tau)$ is a smooth curve with $dh_{\text{exact}}(\tau)/d\tau$ being always negative, and that near $\tau = 0^+$ it has the asymptotic behaviour:

$$h_{\text{exact}}(\tau) \approx c - \frac{\zeta}{\sqrt{2}}\sqrt{\sigma^2\tau} \tag{23}$$

where $\zeta\ (\geqslant 0)$ is the solution of the following equation:

$$\begin{aligned}&\left(\zeta^5 + 10\zeta^3\right)\exp\left\{\frac{1}{4}\zeta^2\right\}\int_{-\infty}^{\zeta}\exp\left\{-\frac{1}{4}\phi^2\right\}d\phi \\ =\ & 16 - 16\zeta^2 - 2\zeta^4 \quad .\end{aligned} \tag{24}$$

On the contrary, the free boundary $h(\tau)$ is not monotonically decreasing and has a minimum point at

$$\tau = \frac{1}{2\kappa}\ln\left\{\frac{1/\kappa + \beta}{1/\kappa - \beta + 2\kappa(c-\theta)/\sigma^2}\right\} \quad . \tag{25}$$

Hence, we can conclude that the approximate optimal exercise boundary is an upper bound of the exact boundary. Finally, once the optimal β is obtained, the price $V(r,\tau)$ of the mortgage contract can be evaluated readily.

3. Numerical results

For illustration, in Figure 1 we plot $V(r,\tau)$ computed by our moving boundary approach for a mortgage contract of a duration of 10 years versus the short term interest rate r. The exact prices generated by Xie *et al.*'s approach (2007 and 2008) are also included for comparison. Following Xie *et al.* (2008), other input model parameters are selected as follows: $c = 0.085$, $m = 1.0$, $\theta = 0.07$, $k = 0.25$ and $\sigma = 0.03$. It is found that not only our estimates are very close to the exact results, but our moving boundary approach is also much more efficient. The percentage errors of our estimates are found to be not more than 1.2% for $r \leqslant 0.5$. The corresponding moving boundary and the exact early exercise boundary are displayed in Figure 2. It is evident that our method provides an upper bound for the exact early exercise boudary at any time t.

Moreover, in order to systematically improve the approximate results, we hereby adopt the multi-stage approximation scheme proposed by Lo *et al.* (2003) for the moving boundary $h(\tau)$. The essence of the approximation scheme is to replace the above smooth boundary $h(\tau)$ by a continuous and piecewise smooth boundary in order that the deviation from the exact early exercise boundary is minimized in a systematic manner, as shown in Figures 3. We then need to perform some simple one-dimensional numerical integrations (*e.g.* using the Gauss quadrature method) at the connecting points of the piecewise smooth boundary in order to evaluate the price $V(r,\tau)$ of the mortgage contract. As expected, the multi-stage approximation becomes better and better as the number N of stages increases. In practice, even a rather low-order approximation can yield very accurate estimates of the exact results. As demonstrated by Figure 4, the estimates of the price $V(r,\tau)$ of the mortgage contract of a duration of 30 years obtained by the three-stage approximation with 10 years each stage are apparently very accurate. The percentage errors of the estimates are less than 0.6% for $r \leqslant 0.5$.

4. Conclusion

In this communication a moving boundary approach has been proposed to valuate a fixed rate mortgage contract which allows the mortgage holder to prepay the outstanding balance of the mortgage. Unlike previous approaches, this new method transforms the free-boundary problem into a monotonic sequence of moving boundary problems each of which can be solved by standard techniques. It has been found that the moving boundary approach is able to provide an accurate approximation to both the early exercise boundary and the contract value very efficiently. Furthermore, the approximate results can be systematically improved by the multi-stage approximation to the early exercise bounday.

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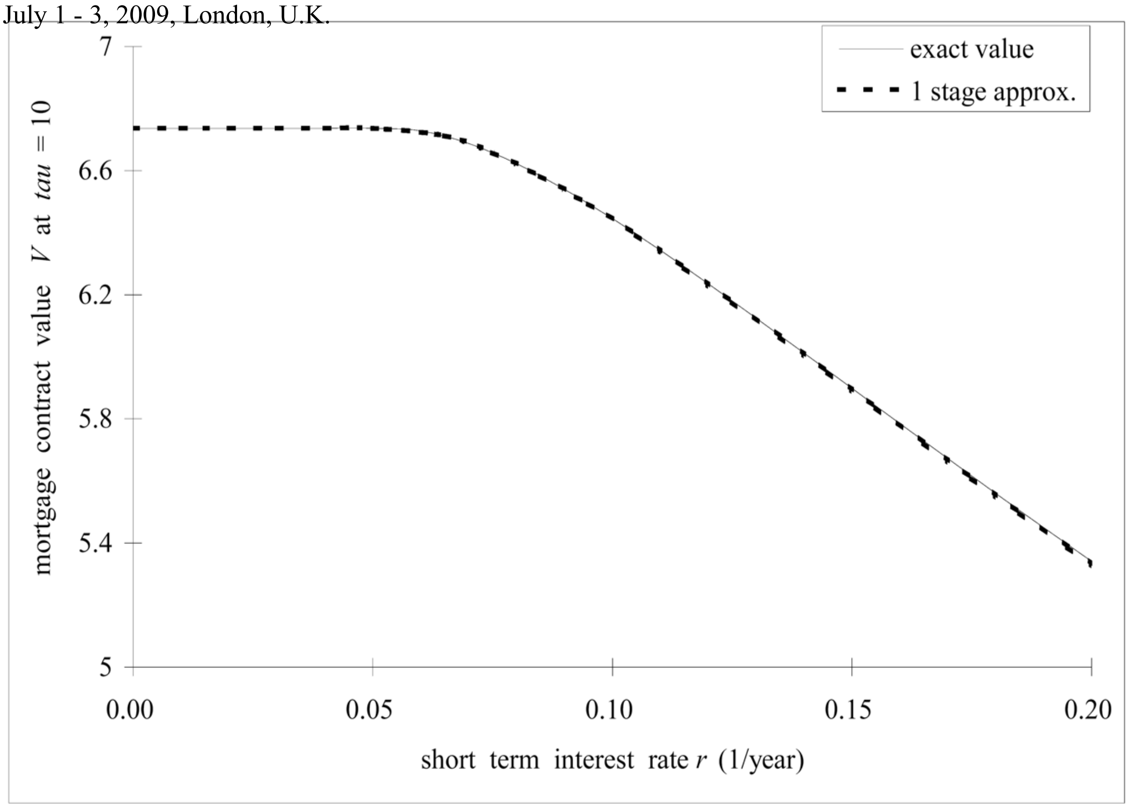

Figure 1 : Mortgage contract value $V(r,\tau=10)$ versus the short term interest rate r. Other input model parameters are selected as follows: $c = 0.085$, $m = 1.0$, $\theta = 0.07$, $k = 0.25$ and $\sigma = 0.03$.

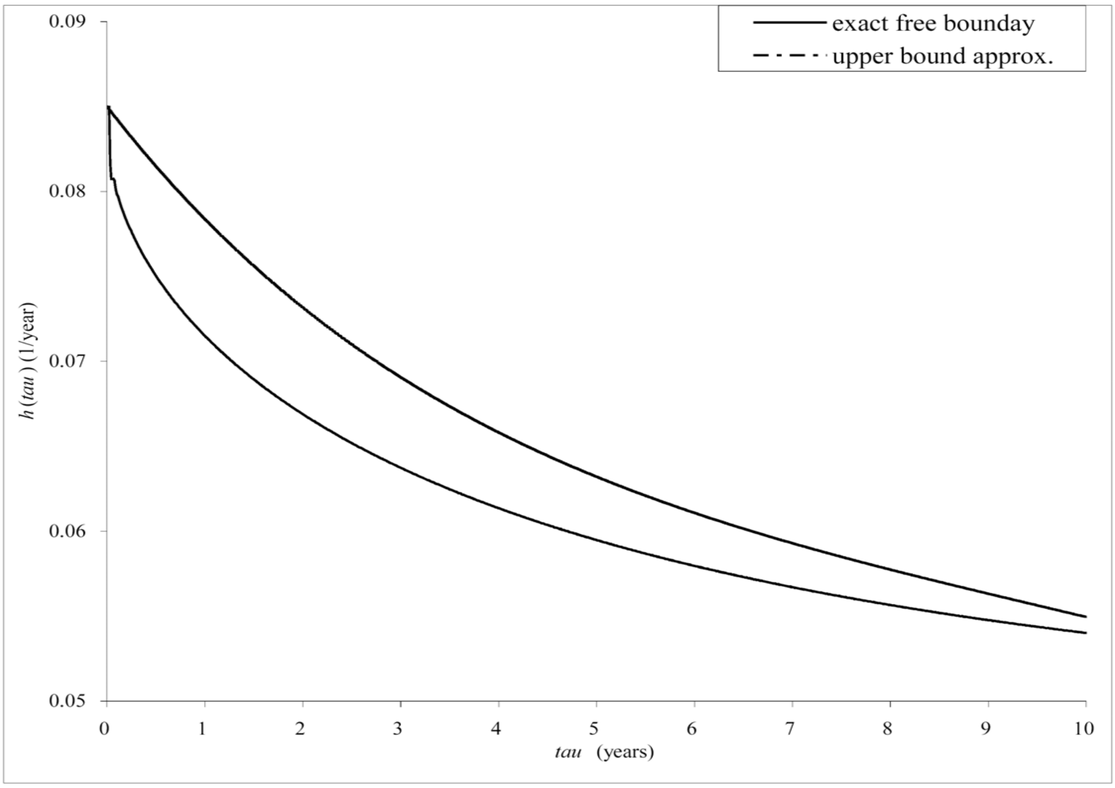

Figure 2 : The single-stage approximate moving boundary and the exact early exercise boundary.

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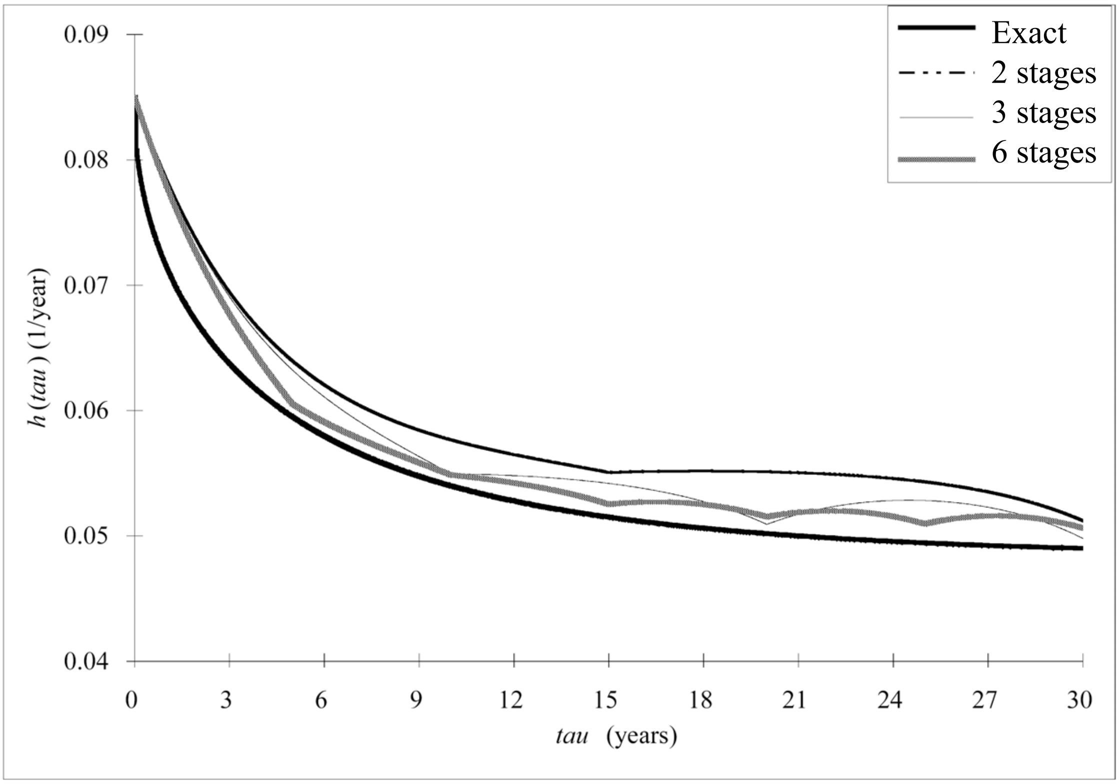

Figure 3 : The multi-stage approximate moving boundary and the exact early exercise boundary.

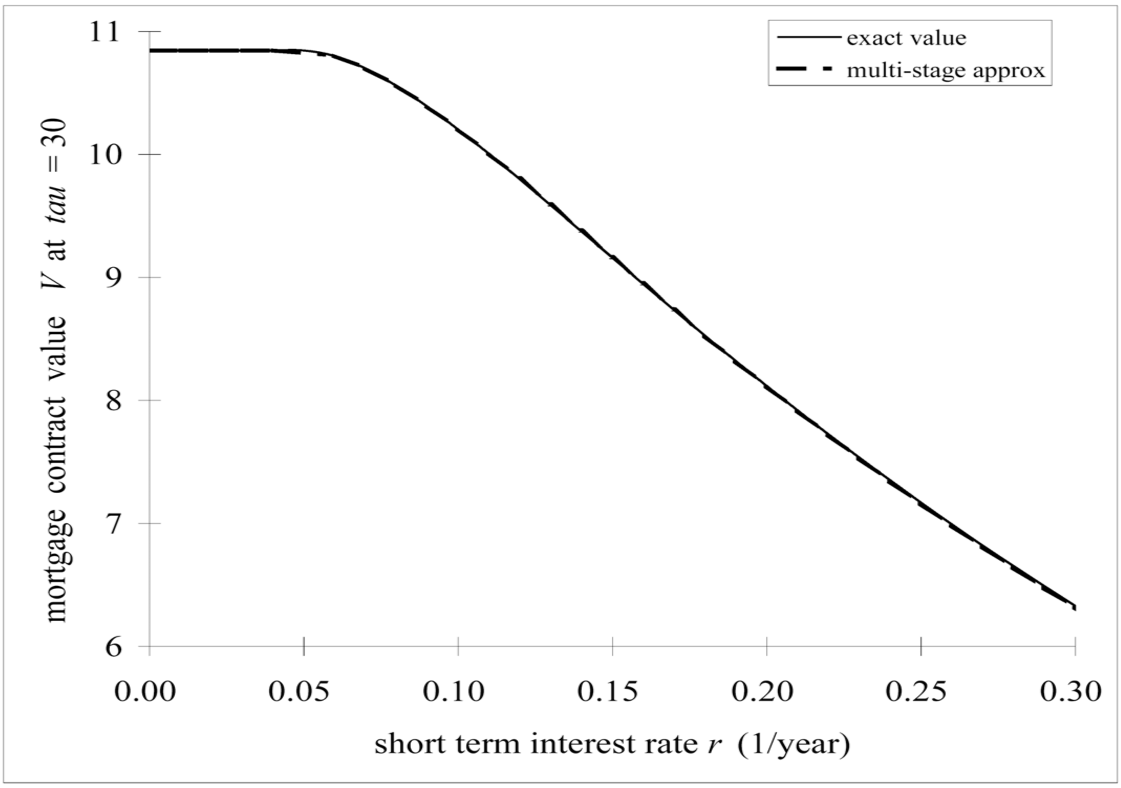

Figure 4 : Mortgage contract value $V(r,\tau=30)$ versus the short term interest rate r. Other input model parameters are selected as follows: $c = 0.085$, $m = 1.0$, $\theta = 0.07$, $k = 0.25$ and $\sigma = 0.03$.