Proceedings of the World Congress on Engineering 2009 Vol II
WCE 2009, July 1 - 3, 2009, London, U.K.

Geometrically Non-Linear Dynamic Analysis of Thin-Walled Beams

Kuo Mo Hsiao, Wen Yi Lin, and Ren Haw Chen

***Abstract*—A co-rotational finite element formulation for the geometrically nonlinear dynamic analysis of thin-walled beam with large rotations but small strain is presented. The element developed here has two nodes with seven degrees of freedom per node. The element nodes are chosen to be located at the centroid of the end cross sections of the beam element and the centroid axis is chosen to be the reference axis. The kinematics of the beam element is described in the current element coordinate system constructed at the current configuration of the beam element. The element nodal forces are conventional forces, moments and bimoments. Both the element deformation nodal forces and inertia nodal forces are systematically derived by consistent linearization of the fully geometrically non-linear beam theory, the d'Alembert principle and the virtual work principle in the current element coordinates. An incremental-iterative method based on the Newmark direct integration method and the Newton-Raphson method is employed here for the solution of the nonlinear equations of motion. Numerical examples are presented to demonstrate the accuracy and efficiency of the proposed method.**

***Index Terms*—Co-rotational formulation, Dynamics, Geometrical nonlinearity, Thin-walled beam.**

I. INTRODUCTION

The nonlinear dynamic behavior of beam structures, e.g., framed structures, flexible mechanisms, and robot arms, has been the subject of considerable research. However, the application of co-rotational formulation in the nonlinear dynamic analysis of three-dimensional beams has been rather limited (e.g. [1-3]). In [3] a consistent co-rotational finite element formulation for the nonlinear dynamic analysis of three-dimensional elastic Euler beam using consistent linearization of the fully geometrically non-linear beam theory was presented. The formulation was proven to be very effective by numerical examples studied in [3]. However, the effect of warping restraint was neglected in [3]. To the authors' knowledge, the application of co-rotational formulation in the geometric nonlinear dynamic analysis for thin-walled beams with the consideration of the warping rigidity has not been reported in the literature. The object of this paper is to present a co-rotational finite element formulation for the geometric nonlinear dynamic analysis of thin-walled beams with open section.

In order to capture correctly all coupling among bending, twisting, and stretching deformations of the beam elements, the formulation of beam elements might be derived by the fully geometrically non-linear beam theory. The exact expressions for the element nodal forces, which are required in a total Lagrangian formulation for large displacement/small strain problems, are highly nonlinear functions of element nodal parameters. However, the dominant factors in the geometrical nonlinearities of beam structures are attributable to finite rotations, the strains remaining small. For a beam structure discretized by finite elements, this implies that the motion of the individual elements to a large extent will consist of rigid body motion. If the rigid body motion part is eliminated from the total displacements and the element size is properly chosen, the deformational part of the motion is always small relative to the local element axes. Thus in conjunction with the co-rotational formulation, the higher order terms of nodal parameters in the element nodal forces may be neglected by consistent linearization. The element deformation and inertia nodal forces are systematically derived by using the d'Alembert principle and the virtual work principle. An incremental-iterative method based on the Newmark direct integration method and the Newton-Raphson method is employed here for the solution of the nonlinear equations of motion. Numerical examples are presented and compared with the results reported in the literature to demonstrate the accuracy and efficiency of the proposed method.

II. FINITE ELEMENT FORMULATION

The kinematics of the beam element presented in [4] and the co-rotational finite element formulation proposed in [3] are employed here. In the following only a brief description of the beam element is given.

A. *Basic Assumptions*

The following assumptions are made in derivation of the beam element behavior: (1) The beam is prismatic and slender, and the Euler-Bernoulli hypothesis is valid. (2) The cross section of the beam is doubly symmetric. (3) The unit extension of the centroid axis of the beam element is uniform. (4) The cross section of the beam element does not deform in its own plane and strains within this cross section can be neglected.

Manuscript received February 20, 2009. This work was supported in part by the National Science Council, Taiwan under Grant NSC96-2221-E009-176.

Kuo Mo Hsiao is with Department of Mechanical Engineering, National Chiao Tung University, 1001 Ta Hsueh Road, Hsinchu, 300, Taiwan (phone: 886-3-5712121-55107; fax: 886-3-5720634; e-mail: kmhsiao@mail.nctu.edu.tw).

Wen Yi Lin is with Department of Mechanical Engineering, De Lin Institute of Technology, 1 Alley 380, Ching Yun Road, Tucheng, Taiwan (e-mail: wylin@dlit.edu.tw).

Ren Haw Chen is with Department of Mechanical Engineering, National Chiao Tung University, 1001 Ta Hsueh Road, Hsinchu, 300, Taiwan (e-mail: chenrh@mail.nctu.edu.tw).

ISBN:978-988-18210-1-0

B. Coordinate Systems

In this paper, a co-rotational formulation is adopted. In order to describe the system, we define three sets of right handed rectangular Cartesian coordinate systems:

1. A fixed global set of coordinates, X_i^G (i = 1, 2, 3) (see Fig. 1); the nodal coordinates, displacements, rotations, velocities, and accelerations, and the equations of motions of the system are defined in this coordinates.

2. Element cross section coordinates, x_i^S (i = 1, 2, 3) (see Fig. 1); a set of element cross section coordinates is associated with each cross section of the beam element. The origin of this coordinate system is rigidly tied to the centroid of the cross section. The x_1^S axes are chosen to coincide with the normal of the unwrapped cross section and the x_2^S and x_3^S axes are chosen to be the principal directions of the cross section.

3. Element coordinates, x_i (i = 1, 2, 3) (see Fig. 1); a set of element coordinates is associated with each element, which is constructed at the current configuration of the beam element. The origin of this coordinate system is located at node 1, and the x_1 axis is chosen to pass through two end nodes of the element; the x_2 and x_3 axes are determined by the method proposed in [5]. Note that this coordinate system is just a local coordinate system not a moving coordinate system. The deformations, deformation nodal forces, inertia nodal forces, stiffness matrix, and mass matrix of the elements are defined in terms of these coordinates. In this paper the element deformations are determined by the rotation of element cross section coordinate systems relative to this coordinate system.

C. Kinematics of Beam Element

In this study only the doubly symmetric cross section is considered. Let Q (Fig. 1) be an arbitrary point in the beam element, and P be the point corresponding to Q on the centroid axis. The position vector of point Q in the undeformed and deformed configurations may be expressed as [4]:

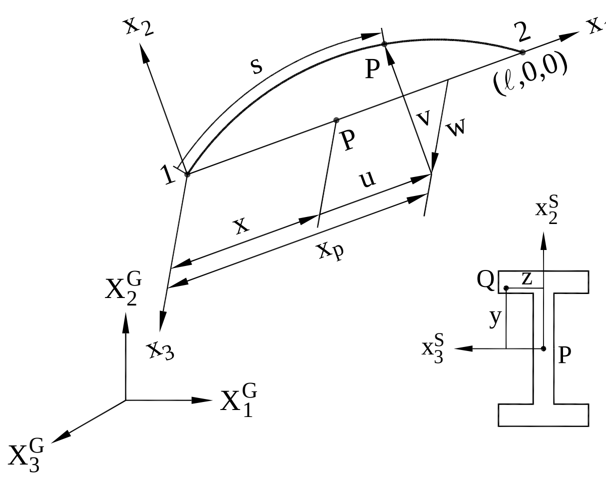

Fig. 1. Coordinate systems

$$\mathbf{r}_0 = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 \tag{1}$$

$$\mathbf{r} = x_p(x,t)\mathbf{e}_1 + v(x,t)\mathbf{e}_2 + w(x,t)\mathbf{e}_3 + \theta_{1,x}\omega\mathbf{e}_1^S + y\mathbf{e}_2^S + z\mathbf{e}_3^S \tag{2}$$

where $x_p(x,t)$, $v(x,t)$ and $w(x,t)$ are the x_1, x_2 and x_3 coordinates of point P, respectively, in the deformed configuration, $\theta_{1,x} = \theta_{1,x}(x,t)$ is the twist rate of the deformed centroid axis, $\omega(y,z)$ is the Saint Venant warping function for a prismatic beam of the same cross section, and $\mathbf{e}_i$ and $\mathbf{e}_i^S$ (i = 1, 2, 3) denote the unit vectors associated with the x_i and x_i^S axes, respectively. Note that $\mathbf{e}_i$ and $\mathbf{e}_i^S$ are coincident in the undeformed state. The relationship between $\mathbf{e}_i$ and $\mathbf{e}_i^S$ is given in [4] and not repeated here. Here, the lateral deflections of the centroid axis, $v(x,t)$ and $w(x,t)$, and the rotation about the centroid axis, $\theta_{1,x}$, are assumed to be the Hermitian polynomials of x.

The relationship among $x_p(x,t)$, $v(x,t)$, and $w(x,t)$, and x may be given as [4]

$$x_p(x,t) = u_1 + \int_0^x [(1+\varepsilon_c)^2 - v_{,x}^2 - w_{,x}^2]^{1/2} dx \tag{3}$$

where u_1 is the displacement of node 1 in the x_1 direction. Note that due to the definition of the element coordinate system, the value of u_1 is equal to zero. However, the variation and time derivatives of u_1 are not zero.

Making use of the assumption of uniform unit extension, ε_c and the axial displacements of the centroid axis may be calculated using (3) and the current chord length of the beam element.

D. Element Nodal Force Vector, Stiffness Matrix and Mass Matrix

The element proposed here has two nodes with seven degrees of freedom per node. The nodal parameters are chosen to be u_{ij} ($u_{1j} = u_j$, $u_{2j} = v_j$, $u_{3j} = w_j$), the x_i (i = 1, 2, 3) components of the translation vectors $\mathbf{u}_j$ at node j (j = 1, 2), ϕ_{ij}, the x_i (i = 1, 2, 3) components of the rotation vectors $\boldsymbol{\phi}_j$ at node j (j = 1, 2), and β_j, the twist rate of the centroid axis at node j. Here, the values of $\boldsymbol{\phi}_j$ are reset to zero at current configuration. Thus, $\delta\phi_{ij}$, the variation of ϕ_{ij}, represents infinitesimal rotations about the x_i axes [5], and the generalized nodal forces corresponding to $\delta\phi_{ij}$ are m_{ij}, the conventional moments about the x_i axes. The generalized nodal forces corresponding to δu_{ij}, the variations of u_{ij}, are f_{ij}, the forces in the x_i directions. The generalized nodal forces corresponding to $\delta\beta_j$, the variations of β_j, are bimoment B_j.

The element nodal force vector is obtained from the virtual work principle and the d'Alembert principle in the current element coordinates. The virtual work principle requires that

$$\begin{aligned}&\delta W_{ext} = \delta\mathbf{q}^t\mathbf{f} = \delta W_{int}\\ &= \int_V (\sigma_{11}\delta\varepsilon_{11} + 2\sigma_{12}\delta\varepsilon_{12} + 2\sigma_{13}\delta\varepsilon_{13} + \rho\delta\mathbf{r}^t\ddot{\mathbf{r}})dV = \delta\mathbf{q}_\theta^t\mathbf{f}_\theta\end{aligned} \tag{4}$$

$$\mathbf{f} = \mathbf{f}^D + \mathbf{f}^I = \{\mathbf{f}_1, \mathbf{m}_1, \mathbf{f}_2, \mathbf{m}_2, \mathbf{B}\} \tag{5}$$

$$\mathbf{f}_\theta = \mathbf{f}_\theta^D + \mathbf{f}_\theta^I = \{\mathbf{f}_1^\theta, \mathbf{m}_1^\theta, \mathbf{f}_2^\theta, \mathbf{m}_2^\theta, \mathbf{B}\}$$

$$\begin{aligned}&\delta\mathbf{q}_\theta = \{\delta\mathbf{u}_1, \delta\boldsymbol{\theta}_1^*, \delta\mathbf{u}_2, \delta\boldsymbol{\theta}_2^*, \delta\boldsymbol{\beta}\}\\ &\delta\mathbf{q} = \{\delta\mathbf{u}_1, \delta\boldsymbol{\phi}_1, \delta\mathbf{u}_2, \delta\boldsymbol{\phi}_2, \delta\boldsymbol{\beta}\}\end{aligned} \tag{6}$$

$$\varepsilon_{11} = \frac{1}{2}(\mathbf{r}_{,x}^t\mathbf{r}_{,x} - 1),\ \varepsilon_{12} = \frac{1}{2}\mathbf{r}_{,x}^t\mathbf{r}_{,y}.\ \varepsilon_{13} = \frac{1}{2}\mathbf{r}_{,x}^t\mathbf{r}_{,z}. \tag{7}$$

where $\delta\mathbf{u}_j = \{\delta u_j, \delta v_j, \delta w_j\}$, $\delta\boldsymbol{\phi}_j = \{\delta\phi_{1j}, \delta\phi_{2j}, \delta\phi_{3j}\}$, $\delta\boldsymbol{\theta}_j^* = \{\delta\theta_{1j}, -\delta w_j', \delta v_j'\}$, $\mathbf{f}_j = \{f_{1j}, f_{2j}, f_{3j}\}$, $\mathbf{m}_j = \{m_{1j}, m_{2j}, m_{3j}\}$, $\mathbf{f}_j^\theta = \{f_{1j}^\theta, f_{2j}^\theta, f_{3j}^\theta\}$, $\mathbf{m}_j^\theta = \{m_{1j}^\theta, m_{2j}^\theta, m_{3j}^\theta\}$ (j = 1, 2), $\delta\boldsymbol{\beta} = \{\delta\beta_1, \delta\beta_2\}$ and $\mathbf{B} = \{B_1, B_2\}$. $\mathbf{f}^D$ and $\mathbf{f}^I$ are element deformation nodal force vector and inertia nodal force vector, respectively. V is the volume of the undeformed beam element, $\delta\varepsilon_{1i}$ (i = 1, 2, 3) are the variation of ε_{1i} in (7) corresponding to $\delta\mathbf{q}_\theta$. Note that because $\delta\varepsilon_{1i}$ are function of $\delta\mathbf{q}_\theta$, δW_{int} may be expressed by $\delta\mathbf{q}_\theta^t\mathbf{f}_\theta$. $\mathbf{f}_\theta^D$ and $\mathbf{f}_\theta^I$ are generalized deformation nodal force vector and inertia nodal force vector corresponding to $\delta\mathbf{q}_\theta$. σ_{1i} (i = 1, 2, 3) are the second Piola-Kirchhoff stress. For linear elastic material, $\sigma_{11} = E\varepsilon_{11}$, $\sigma_{12} = 2G\varepsilon_{12}$, and $\sigma_{13} = 2G\varepsilon_{13}$, where E is Young's modulus and G is the shear modulus. ρ is the density, $\delta\mathbf{r}$ and $\ddot{\mathbf{r}}$ are the variation and the second time derivative of $\mathbf{r}$ in (2), respectively. Note that because the element coordinate system is just a local coordinate system not a moving coordinate system, $\ddot{\mathbf{r}}$ is the absolute acceleration. The higher order terms of nodal parameters in the element nodal forces are neglected by consistent second order linearization in this study.

The relation between $\delta\mathbf{q}$ and $\delta\mathbf{q}_\theta$, and the relation between $\mathbf{f}$ and $\mathbf{f}_\theta$ may be expressed as [4]

$$\delta\mathbf{q}_\theta = \mathbf{T}_{\theta\phi}\delta\mathbf{q}, \quad \mathbf{f} = \mathbf{T}_{\theta\phi}^t\mathbf{f}_\theta \tag{8}$$

where $\mathbf{f}_\theta$ may be calculated using (2-7).

The element stiffness matrix and mass matrix may be expressed as

$$\mathbf{k} = \frac{\partial\mathbf{f}}{\partial\mathbf{q}}, \quad \mathbf{m} = \frac{\partial\mathbf{f}}{\partial\ddot{\mathbf{q}}} \tag{9}$$

E. Equations of Motion

The nonlinear equations of motion may be expressed by

$$\mathbf{F}^R = \mathbf{F}^I + \mathbf{F}^D - \mathbf{P} = \mathbf{0} \tag{10}$$

where $\mathbf{F}^R$ is the unbalanced force among the inertia nodal force $\mathbf{F}^I$, deformation nodal force $\mathbf{F}^D$, and the external nodal force $\mathbf{P}$. $\mathbf{F}^I$ and $\mathbf{F}^D$ are assembled from the element nodal force vectors, which are calculated using (4) and (8) first in the current element coordinates and then transformed from element coordinate system to global coordinate system before assemblage using standard procedure.

III. Numerical Studies

An incremental iterative method based on the Newmark direct integration method and the Newton-Raphson method [3] is employed here.

The first example considered is a simply supported beam subjected to uniform load as shown in Fig. 2. This example was analyzed by [6]. Twenty elements are used for discretization. A time step size of $\Delta t = 0.001$sec is used. The time histories of displacements at point C are shown in Fig. 3 together with the solution given in [6]. As can be seen, the discrepancy between these two solutions is distinct. The discrepancy may be attributed at least in part to that the moment of inertia of the beam cross section is not considered in [6].

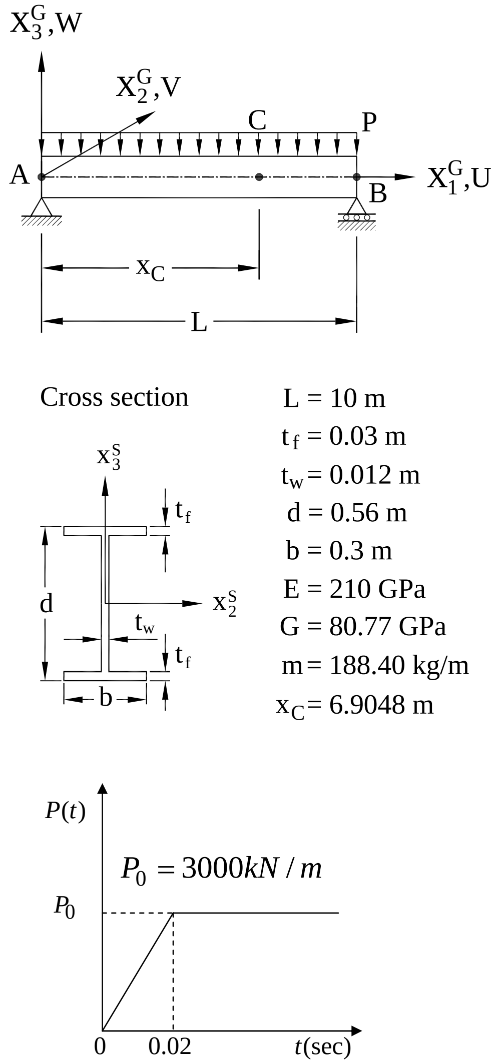

Fig. 2. Simply supported beam subjected to uniform load.

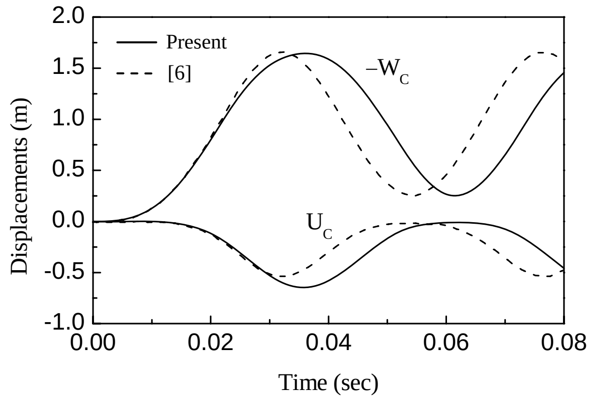

Fig. 3. Time histories of displacements at point C.

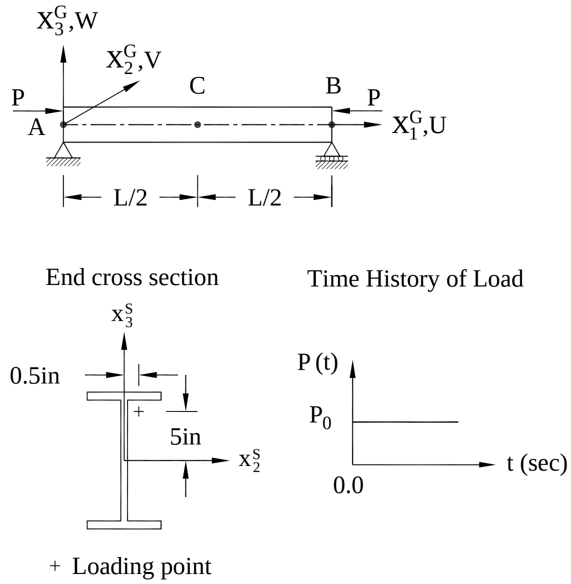

Fig. 4. Simply supported beam subjected to eccentric axial force.

The second example considered is a simply supported W14 × 43 beam subjected to an eccentric axial step loading with magnitude $P_0 = 50\ kip$ as shown in Fig. 4. The ends of the beam are free to warp and free to rotate about X_2^G and X_3^G axes, but restrained from rotation about X_1^G axis. The translation is restrained at end point *A*, and is free only in the direction of X_1^G axis at points *B*. The geometrical and material properties are $L = 264.6\ in$, $b = 7.995\,in$, $t_f = 0.53\ in$, $d = 13.66\ in$, $t_w = 0.305\ in$, Young's modulus $E = 29000\ ksi$, and the shear modulus $G = 11200\ ksi$, $\rho = 0.283\,lb \cdot s^2 / in^4$.

The first axial natural frequency corresponding to the undeformed state may be given by $\frac{\pi}{2L}\sqrt{E/\rho} = 60.095\,rad/\sec$ .The static buckling load for this example given in [4] is $P_{cr} = 139.1\,kip$. Twenty elements are used for discretization. The first five natural frequencies and vibration modes corresponding to the static equilibrium configuration at $P_0 = 50\,kip$ are calculated and given in Fig. 5. It can be seen that the first and fourth vibration modes are dominated by the lateral vibration in the X_2^G direction, the second and fifth vibration mode is a coupled lateral-torsional vibration, and the third vibration mode is dominated by the lateral vibration in the X_3^G direction. A time step size of $\Delta t = 0.001\sec$ is used. The time histories of lateral displacements at point C and axial displacement at point B are shown in Fig. 6. It can be seen from Figs. 5 and 6 that the

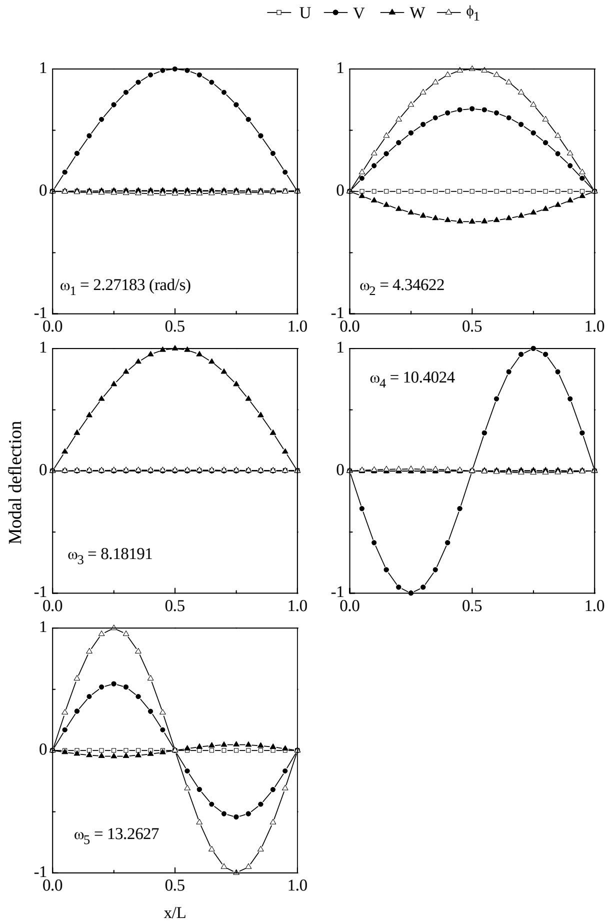

Figure 5. Vibration modes for simply supported beam subjected to eccentric axial force.

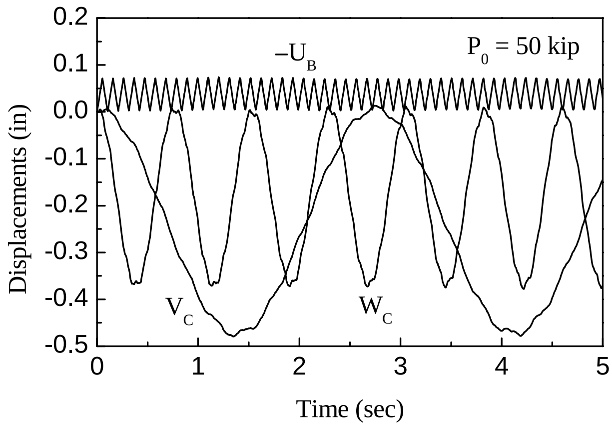

Fig. 6. Time history for simply supported beam subjected to eccentric axial force.

time histories given in Fig. 6 are dominated by the first axial vibration mode, and the first and the third vibration modes.

IV. CONCLUSIONS

A consistent co-rotational total Lagrangian finite element formulation for the geometrically nonlinear vibration analysis of doubly symmetric thin-walled beams with open section is presented.

The nodal coordinates, displacements, rotations, velocities, accelerations, and the equations of motion of the structure are defined in a fixed global set of coordinates. The beam element has two nodes with seven degrees of freedom per node. The element nodal forces are conventional forces and moments. The kinematics of beam element is defined in terms of element coordinates which are constructed at the current configuration of the beam element. Both the element inertia and deformation nodal forces are systematically derived by using consistent second order linearization of the fully geometrically nonlinear beam theory, the d'Alembert principle and the virtual work principle. In conjunction with the co-rotational formulation, the higher order terms of nodal parameters in element nodal forces are consistently neglected. However, in order to include the nonlinear coupling among the bending, twisting, and stretching deformations, terms up to the second order of nodal parameters are retained in element deformation nodal forces. It should be noted that the element coordinate system is just a local coordinate system, which is updated at each iteration, not a moving coordinate system. Thus, the velocity and acceleration described in the element coordinates are the absolute velocity and acceleration. The element equations are constructed first in the element coordinate system and then transformed to the global coordinate system by using standard procedure.

From the numerical examples studied, the accuracy and efficiency of the proposed method are well demonstrated.

It is believed that the consistent co-rotational formulation for beam element and numerical procedure presented here may represent a valuable engineering tool for the dynamic analysis of three dimensional thin-walled beam structures.

REFERENCES

[1] T. Belytschko and B. J. Hsieh, Nonlinear transient finite element analysis with convected coordinates, *Internat. J. Numer. Meths. Engrg* , Vol. 7, 1973, pp. 255-271.

[2] T. Belytschko and L. Schwer, Large displacement, transient analysis of space frames, *Internat. J. Numer. Meths. Engrg*., Vol. 11, 1977, pp. 65-84.

[3] K. M. Hsiao, J. Y. Lin and W. Y. Lin, A Consistent Co-Rotational Finite Element Formulation for Geometrically Nonlinear Dynamic Analysis of 3-D Beams, *Comput. Methods Appl. Mech. Engrg*., Vol. 169, 1999, pp. 1-18.

[4] W. Y. Lin and K. M. Hsiao, Co-Rotational Formulation for Geometric Nonlinear analysis of Doubly Symmetric Thin-Walled Beams, *Comput. Methods Appl. Mech. Engrg.*, Vol. 190, 2001, pp. 6023-6052.

[5] K. M. Hsiao, A Co-rotational Total Lagrangian Formulation for Three Dimensional Beam Element, *AIAA Journal*, Vol. 30, 1992, pp. 797-804.

[6] J. T. Katsikadelies and G. C. Tsiatas, Nonlinear Dynamic Analysis of Beams with Variable Stiffness, *J. Sound Vib*., Vol. 270, 2004, pp. 847-863.