Elasto-Plastic EFGM Analysis of an Edge Crack

B Aswani Kumar, I V Singh, and V H Saran

***Abstract*-In this paper, elasto-plastic analysis of an edge crack has been performed using element free Galerkin method. A model problem has been solved in plane stress condition under mode-I loading. A system of nonlinear equations has been obtained by using incremental theory of plasticity, and the equations are solved by assuming piecewise linear approximation. A code has been written in Matlab for elasto-plastic analysis of cracked components. The size of plastic zone is calculated around the tip of the crack by EFGM code.**

***Keywords*- Elasto-plastic, Edge crack, Element free Galerkin method, Plane stress.**

I. INTRODUCTION

Now a day, Finite Element Method (FEM) is a well established simulation approach, and is widely used in many branches of engineering and sciences. However, it still has some shortcomings. The reliance of the method on a mesh leads to complications for certain classes of problems. Consider the modeling of large deformation processes; considerable loss in accuracy arises due to element distortion. Examining the growth of cracks with arbitrary and complex paths, and the simulations of phase transformations is also difficult.

To overcome some of FEM shortcomings, a number of meshless methods were proposed such as EFGM, MLPG, and PIM as discussed by Liu [1]. In a meshless method, unlike FEM, a predefined mesh is not necessary, at least for field variables interpolation.

Two main characteristics of the EFGM seem to be a unique approximate function for the whole field and rather an easier kind of crack modeling. In this paper, the EFGM has been applied to solve a single edge crack problem having elasto-plastic material behavior. The nonlinear equations are solved by assuming each load step as a piecewise linear.

II. REVIEW OF EFGM

In EFGM, a field variable u is approximated by moving least square approximation (MLS) function $u^h(\mathbf{x})$ [2], which is given by

$$u^h(\mathbf{x}) = \sum_{j=1}^{m} p_j(\mathbf{x})\, a_j(\mathbf{x}) \equiv \mathbf{p}^T(\mathbf{x})\,\mathbf{a}(\mathbf{x}) \tag{1}$$

where, $\mathbf{p(x)}$ is a vector of basis functions, $\mathbf{a(x)}$ are unknown coefficients, and m is the number of terms in the basis.

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The unknown coefficients $\mathbf{a}(\mathbf{x})$ are obtained by minimizing a weighted least square sum of the difference between local approximation, $u^h(\mathbf{x})$ and field function nodal parameters u_I. The weighted least square sum $L(\mathbf{x})$ can be written in the following quadratic form:

$$L(\mathbf{x}) = \sum_{I=1}^{n} w(\mathbf{x}-\mathbf{x}_I)[\mathbf{p}^T(\mathbf{x})\mathbf{a}(\mathbf{x}) - u_I]^2 \tag{2}$$

where, u_I is the nodal parameter associated with node I at $\mathbf{x}_I$. u_I are not the nodal values of $u^h(\mathbf{x}-\mathbf{x}_I)$ because $u^h(\mathbf{x})$ is used as an approximant and not an interpolant. $w(\mathbf{x}-\mathbf{x}_I)$ is the weight function having compact support associated with node I, and n is the number of nodes with domain of influence containing the point $\mathbf{x}$, $w(\mathbf{x}-\mathbf{x}_I) \neq 0$. By setting $\partial L/\partial \mathbf{a}$ =0, following set of linear equation is obtained:

$$\mathbf{A}(\mathbf{x})\,\mathbf{a}(\mathbf{x}) = \mathbf{B}(\mathbf{x})\,\mathbf{u} \tag{3}$$

By substituting Eq. (3) in Eq. (1), the approximation function is obtained as:

$$u^h(\mathbf{x}) = \sum_{I=1}^{n} \Phi_I(\mathbf{x})\, u_I \tag{4}$$

III. DISCRETIZED VARIATIONAL FORMULATION

Consider two-dimensional (2D) problem with small displacements on the domain Ω bounded by Γ. The governing equilibrium equation is given as:

$$\nabla.\boldsymbol{\sigma} + \mathbf{b} = 0 \ \text{ in}\, \Omega \tag{5}$$

with the following essential and natural boundary conditions:

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{on} \quad \Gamma_u \tag{6}$$

$$\boldsymbol{\sigma}.\bar{\mathbf{n}} = \bar{\mathbf{t}} \ \text{ on} \quad \Gamma_t \tag{7}$$

where, $\boldsymbol{\sigma}$ is the stress tensor which is defined as $\boldsymbol{\sigma} = \mathbf{D}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_T)$, $\mathbf{D}$ is the linear elastic material property matrix, $\boldsymbol{\varepsilon}$ is the strain vector, $\mathbf{b}$ is the body force vector, $\mathbf{u}$ is the displacement vector, $\bar{\mathbf{t}}$ is the traction force and $\bar{\mathbf{n}}$ is the unit normal, Enforcing essential boundary conditions [3] using Lagrange multiplier approach [4], and applying variational principle, the following discrete equations are obtained from Eq. (4):

$$\begin{bmatrix} \mathbf{K} & \mathbf{G} \\ \mathbf{G}^T & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \lambda \end{Bmatrix} = \begin{Bmatrix} \mathbf{f} \\ \mathbf{q} \end{Bmatrix} \tag{8}$$

where,

$$K_{IJ} = \int_{\Omega} \mathbf{B}_I^T \mathbf{D} \mathbf{B}_I \, d\Omega \tag{8a}$$

$$f_I = \int_{\Gamma_t} \bar{\mathbf{t}} \, \Phi_I \, d\Gamma_t + \int_{\Omega} \Phi_I \mathbf{b} \, d\Omega \tag{8b}$$

$$G_{IK} = -\int_{\Gamma_u} \Phi_I \, \mathbf{N}_K \, d\Gamma_u \tag{8c}$$

$$q_K = -\int_{\Gamma_u} \mathbf{N}_K \, \bar{u} \, d\Gamma_u \tag{8d}$$

$$\mathbf{B}_I = \begin{bmatrix} \Phi_{I,x} & 0 \\ 0 & \Phi_{I,y} \\ \Phi_{I,y} & \Phi_{I,x} \end{bmatrix}, \qquad \mathbf{N}_K = \begin{bmatrix} N_K & 0 \\ 0 & N_K \end{bmatrix}$$

$$\mathbf{D} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} = \mathbf{D}_{\mathrm{e}} \quad \text{(for plane stress)}$$

where, E is the modulus of elasticity and ν is the Poisson's ratio.

IV. ELASTO-PLASTIC CONSTITUTIVE EQUATION

The elasto-plastic constitutive relation for a material can be modeled using incremental theory of plasticity. The material behavior [5] is modeled in the form of an incremental stress vector $\mathrm{d}\boldsymbol{\sigma}$ and incremental strain vector $\mathrm{d}\boldsymbol{\varepsilon}$ such that $\mathrm{d}\boldsymbol{\sigma} = \mathbf{D}_{\mathrm{ep}} \mathrm{d}\boldsymbol{\varepsilon}$. In this relation, $\mathbf{D}_{\mathrm{ep}}$ is called elasto-plastic stiffness matrix. To find $\mathbf{D}_{\mathrm{ep}}$ matrix, the following relations are assumed to be known

a) Total strain increment is the sum of elastic and plastic parts i.e.,
$$\mathrm{d}\varepsilon = \mathrm{d}\varepsilon_{\mathrm{e}} + \mathrm{d}\varepsilon_{\mathrm{p}} \tag{9}$$
b) Elastic stress–strain relation in the incremental form is similar to the relation in its total form, e.g.,
$$\mathrm{d}\boldsymbol{\sigma} = \mathbf{D}_{\mathrm{e}} \mathrm{d}\boldsymbol{\varepsilon}_{\mathrm{e}} \tag{10}$$
c) Failure criteria is given as
$$F(\boldsymbol{\sigma}) = f(\bar{\sigma}) \tag{11}$$
in which F and f are two different forms of failure functions, $\boldsymbol{\sigma}$ is the stress tensor and $\bar{\sigma}$ is the equivalent stress.
d) Flow rule that relates strain increment to other quantities, is the gradient of a function called plastic potential. If one assumes that the plastic potential function is the same as the failure function, then one can get the following relation known as normality rule as the flow rule,
$$\mathrm{d}\boldsymbol{\varepsilon}_{\mathrm{p}} = \nabla F . \mathrm{d}\lambda \tag{12}$$
e) Plastic modulus H' is given as
$$\mathrm{H}' = \frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\bar{\varepsilon}_{\mathrm{p}}} \tag{13}$$
f) For a given strain energy δw, and according to the definition of $\mathrm{d}\bar{\varepsilon}_{\mathrm{p}}$ we must have,
$$\delta w = \bar{\sigma} . \mathrm{d}\bar{\varepsilon}_{\mathrm{p}} \tag{14}$$
g) According to the von Mises criteria [6], $F = J_2$, where J_2 is the second invariant of deviatoric stress tensor. So, we must have $f(\bar{\sigma}) = \bar{\sigma}^2/3$. The Eqs. (9) and (10) results in
$$\mathrm{d}\boldsymbol{\sigma} = \mathbf{D}_{\mathrm{e}} \{\mathrm{d}\boldsymbol{\varepsilon} - \mathrm{d}\boldsymbol{\varepsilon}_{\mathrm{p}}\} \tag{15}$$

After taking the derivatives from both sides of Eq. (11), one obtains

$$\left(\frac{\partial F}{\partial \boldsymbol{\sigma}} . \mathrm{d}\sigma \right) = \left(\frac{\partial f}{\partial \bar{\sigma}} . \frac{\partial \bar{\sigma}}{\partial \bar{\varepsilon}_{\mathrm{p}}} . \frac{\partial \bar{\varepsilon}_p}{\partial w} . \frac{\partial w}{\partial \bar{\varepsilon}_{\mathrm{p}}} . \mathrm{d}\varepsilon_{\mathrm{p}} \right) \tag{16}$$

For simplicity we take $\partial F / \partial \boldsymbol{\sigma} = \mathbf{a}$, and of $\partial f / \partial \bar{\sigma} = \bar{a}$ also by means of Eqs. (13) and (14), Eq (16) can be written in the following form

$$\mathbf{a}.\mathbf{d}\boldsymbol{\sigma} = \bar{a}.\mathrm{H}' \left(\frac{1}{\bar{\sigma}} \right) \boldsymbol{\sigma} . d\varepsilon_p \tag{17}$$

$\mathrm{d}\lambda$ is calculated by omitting $\mathbf{d}\boldsymbol{\sigma}$ between Eq. (15) and (16) and substituting $\mathbf{d}\boldsymbol{\varepsilon}_{\mathrm{p}}$ from Eq. (12). By substituting $\mathrm{d}\lambda$ in Eq. (12), the final form of material matrix is obtained as,

$$\mathbf{D}_{\mathrm{ep}} = \mathbf{D}_{\mathrm{e}} - \mathbf{D}_{\mathrm{p}} \tag{18}$$

where

$$\mathbf{D}_{\mathrm{p}} = \frac{\mathbf{D}\mathbf{a}\mathbf{a}^{\mathbf{T}}\mathbf{D}}{\frac{\bar{\mathrm{a}}}{\bar{\sigma}} \mathrm{H}' \boldsymbol{\sigma}^{\mathbf{T}} \mathbf{a} + \mathbf{a}^{\mathbf{T}} \mathbf{D} \mathbf{a}} \tag{19}$$

V. ELASTO-PLASTIC EFGM ALGORITHM

As indicated earlier, the elasto-plastic constitutive relations are required to get the incremental solution. If in Eqs. (8) and (8a), $\mathbf{u}$ is replaced by incremental auxiliary nodal displacement $\Delta\mathbf{u}$ and $\mathbf{D}$ is replaced by $\mathbf{D}_{\mathrm{ep}}$, then a new set of equations will be obtained which describes incremental elastoplastic behavior. Hence, the incremental form of Eq. (8) is written as

$$\begin{bmatrix} \mathbf{K}_{\mathrm{ep}} & \mathbf{G} \\ \mathbf{G}^T & 0 \end{bmatrix} \begin{Bmatrix} \Delta\mathbf{u} \\ \Delta\lambda \end{Bmatrix} = \begin{Bmatrix} \Delta\mathbf{f} \\ \Delta\mathbf{q} \end{Bmatrix} \tag{20}$$

this can be shown in more compact form as,

$$\mathbf{S}_{\mathrm{ep}} u = \mathrm{f} \tag{21}$$

for the sake of brevity, $\mathbf{S}_{\mathrm{ep}}$ will be called as the stiffness matrix, $\mathbf{u}$ as the auxiliary nodal displacement vector and $\mathbf{f}$ as the force vector. In this manner, to obtain the total displacement, the boundary conditions should be changed gradually and related incremental equations need to be solved and finally field quantities are obtained by summation of incremental values.

Apart from incremental behavior, there is another difference between the forms of Eqs. (8) and (20) i.e. in this model (Eq.20), the behavior of elasto-plastic EFGM incremental stiffness matrix is nonlinear. The stiffness matrix, $\mathbf{S}_{\mathrm{ep}}$ (which depends on material properties) is used to obtain the displacement field. In other words, it can be easily verified that the elasto-plastic material property matrix $\mathbf{D}_{\mathrm{ep}}$ indirectly depends on displacement field. So, in order to obtain the unknown displacement vector $\Delta\mathbf{u}$ in Eq. (20), the nonlinear equations are solved by assuming the piecewise linear behavior with in the load step. In this method, the total stress is applied in small increments through various load steps in such a way that the behavior of the material in plastic region is assumed to be linear in each load step.

V. RESULTS AND DISCUSSIONS

A. Numerical example

A rectangular plate with an edge crack of dimensions L x D is considered, subjected to a traction at the free end as shown in Fig. 1. The problem has been solved for the plane stress case with the following material properties: modulus of elasticity $E = 20\text{x}10^5$ unit, Poisson's ratio ν =0.3, tangent modulus E_t =$2\text{x}10^4$, yield stress = 200 units, and the plate dimensions are L = 1 unit D = 2 units, length of the crack (a) = 0.4 unit. In each integration cell 6×6 Gauss quadrature is used over the domain and 8×8 is used near the crack tip region to evaluate EFGM stiffness matrix. The solutions were obtained using a linear basis function with the cubicspline weight function and a $d_{\max}$ value of 1.5 is used for domain nodes and 3.0 is used for the nodes along the crack. The total stress σ_0 = 2400 units has been applied to the plate in small increments in each load step.

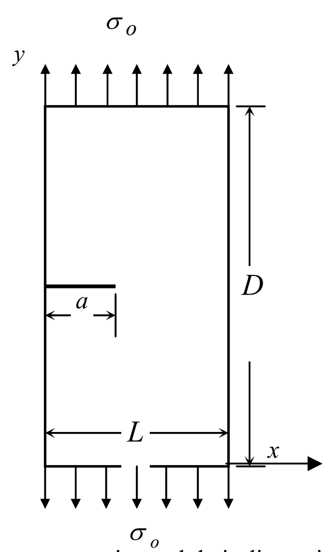

Fig. 1: Problem geometries and their dimensions along with boundary conditions

Fig. 2 shows the stress vs strain plot for an evaluation point near the crack tip as well as for a point which is near to the lower left corner. From Fig. 2, it is clearly seen that the point at the crack tip reached the plastic region where as the base point is still in the elastic region up to the final load step, and the increments in stress and strain are a very small in each load step of the plastic region leading to piecewise linear approximation.

Fig. 3a-3j show the Gaussian points and their stress distribution over the domain which are in plastic region from initial load step to the final load step.

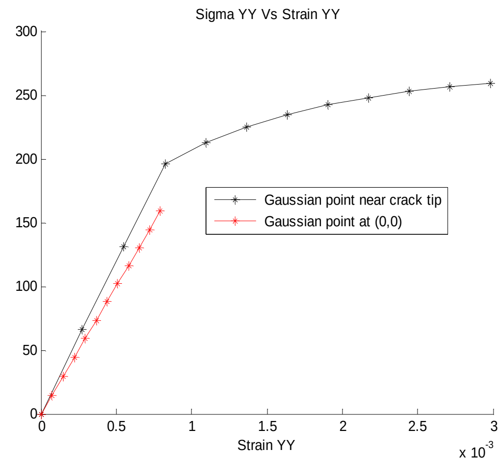

Fig. 2: Stress vs. Strain plot for the Gaussian point situated near the crack tip and the lower left corner Gaussian point

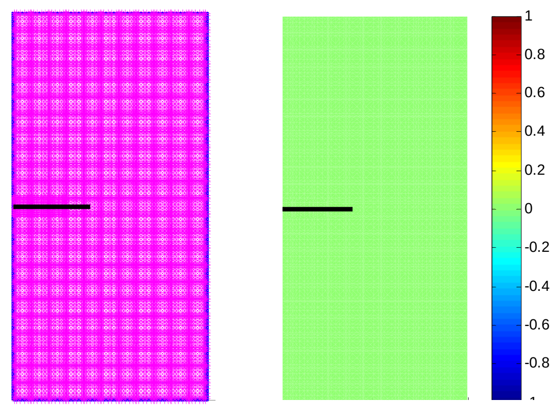

Fig. 3a: Distribution of Gaussian points and stress around the tip of the crack at the end of First, Second & Third load steps.

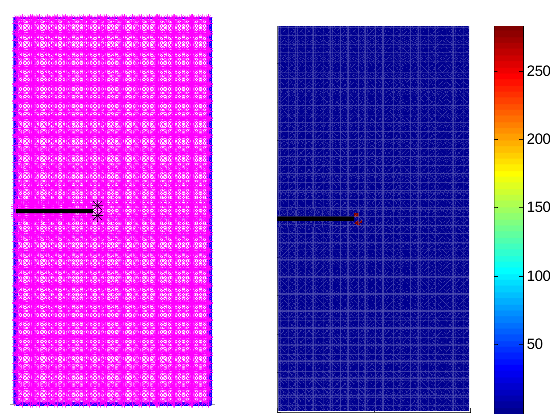

Fig. 3b: Distribution of Gaussian points and stress around the tip of the crack at the end of Fourth load step.

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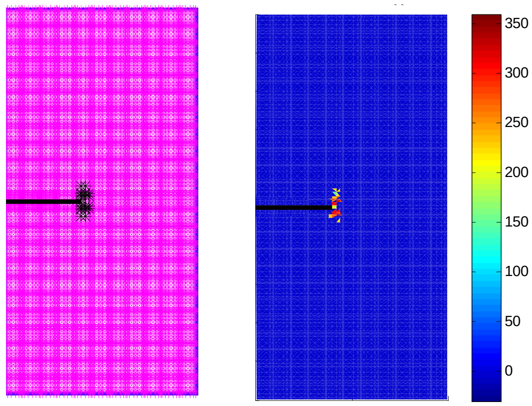

Fig. 3c: Distribution of Gaussian points and stress around the tip of the crack at the end of Fifth load step

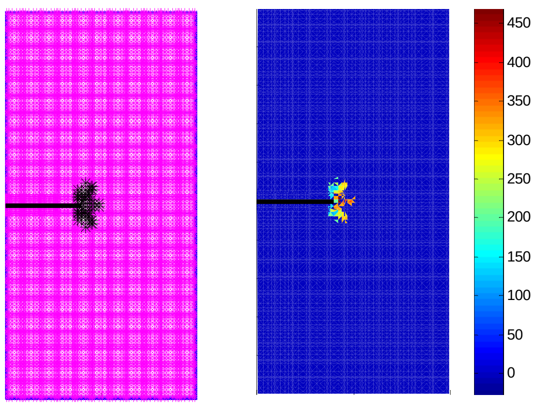

Fig. 3d: Distribution of Gaussian points and stress around the tip of the crack at the end of Sixth load step

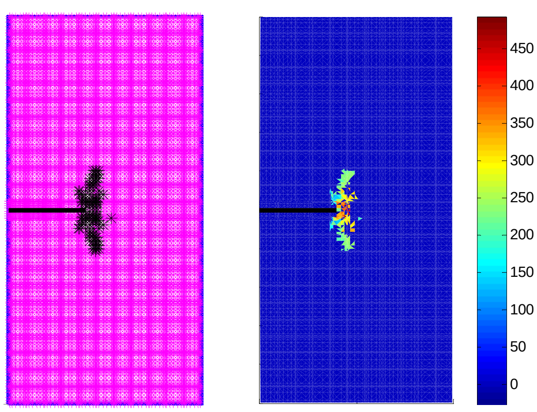

Fig. 3e: Distribution of Gaussian points and stress around the tip of the crack at the end of Seventh load step

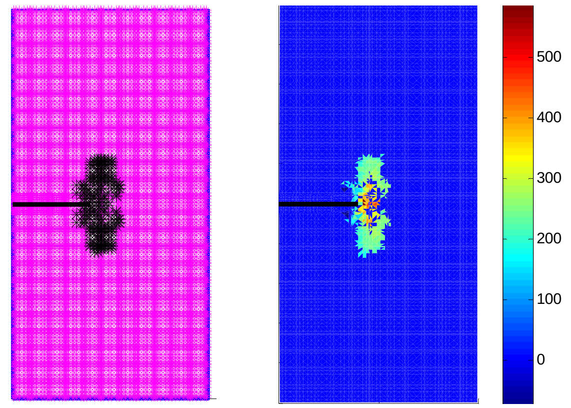

Fig. 3f: Distribution of Gaussian points and stress around the tip of the crack at the end of Eighth load step

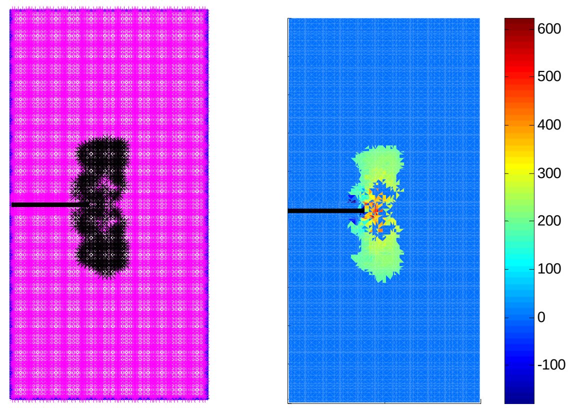

Fig. 3g: Distribution of Gaussian points and stress around the tip of the crack at the end of Ninth load step

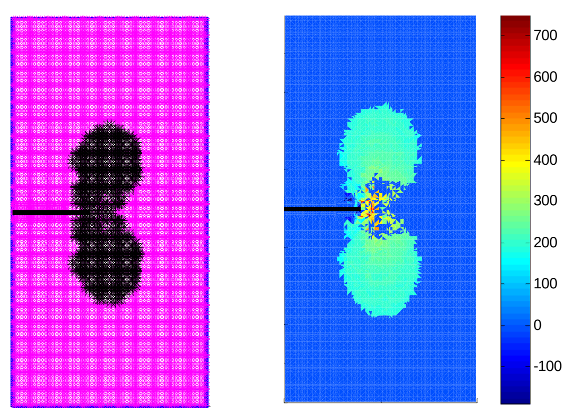

Fig. 3h: Distribution of Gaussian points and stress around the tip of the crack at the end of Tenth load step

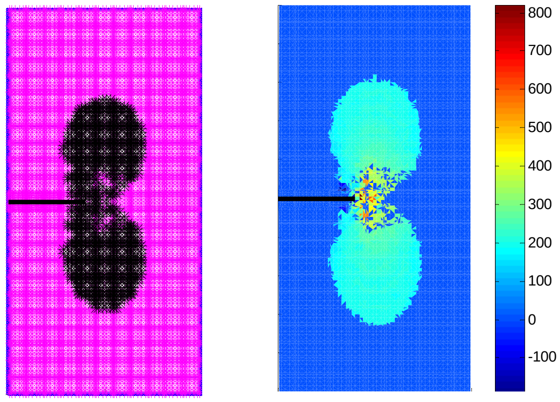

Fig. 3i: Distribution of Gaussian points and stress around the tip of the crack at the end of Eleventh load step

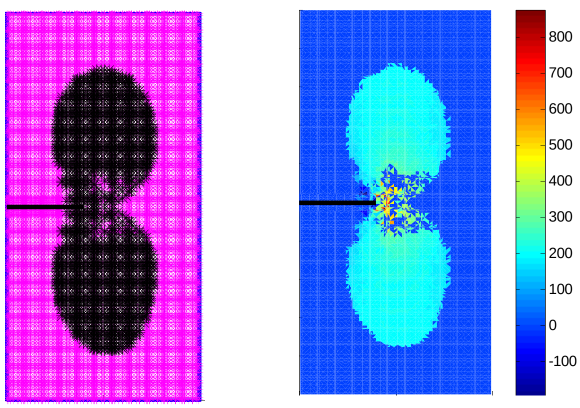

Fig. 3j: Distribution of Gaussian points and stress around the tip of the crack at the final load step

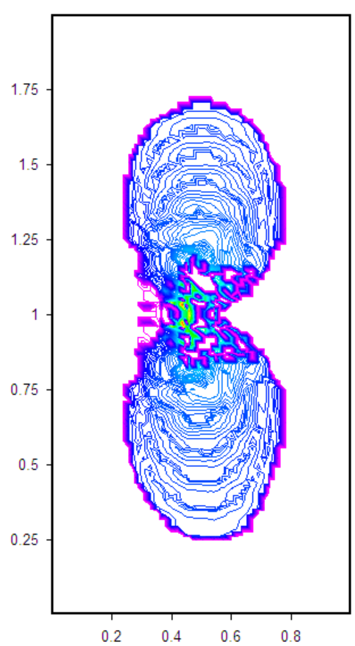

Fig. 4: Stress contours of the total plastic region around the crack tip at the end of the final load step

VI. CONCLUSIONS

In this paper, the EFGM has been used to simulate elastoplastic solid mechanics problems with geometrical discontinuity such as crack. The elasto-plastic formulation has been derived and implemented for an edge crack problem. The nonlinear equations are solved assuming piecewise linear approximation with in the each load step. Distribution of Gaussian points and stress around the tip of the crack are obtained for different load steps. From this analysis, it is observed that elasto-plastic analysis of cracked components can be done by taking small load steps in plastic region, and with in each load step, system of equations can be solved by linear solvers.

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