Proceedings of the World Congress on Engineering 2009 Vol II
WCE 2009, July 1 - 3, 2009, London, U.K.

A New Infinite Boundary Element Formulation Applied to Three-Dimensional Domains

[1]Dimas Betioli Ribeiro and [2]João Batista de Paiva *

Abstract—**This work presents a bi-dimensional mapped infinite boundary element (IBE), based on a triangular boundary element (BE) with linear shape functions. Kelvin fundamental solutions are employed, considering the static analysis of infinite, three-dimensional, linear-elastic and isotropic solids. One advantage of the proposed formulation is that no additional degrees of freedom are added to the original BE mesh by the presence of the IBEs. Thus, the IBEs allow reducing the mesh without compromising the result accuracy. An example is presented, in which the numerical results show good agreement with an analytical solution.**

Keywords: ***boundary elements, three-dimensional solids, infinite domains, static analysis***

1 Introduction

Practical engineering problems that involve truly infinite domains are unusual. In some cases, however, it is feasible to consider one or more domains infinite in order to simplify their numerical simulation. As an example, one may consider soil-structure interaction problems. In this case detailing the soil limits have little impact on the results, becoming more practical so simulate it as an infinite domain. In these situations, some numerical tools become more advantageous than others.

One option is to employ the finite element method (FEM), as performed in [1] and [2]. This numerical tool, however, requires the domain discretization into FEs, which may become impractible due to the data storage and time processing. One way to reduce this computational cost is to model the far field behavior using infinite elements (IEs) together with the original mesh of FEs, as performed in [3] and [4].

Another option is to employ the Boundary Element Method (BEM), which requires only boundary discretization and therefore reduces the problem dimension. This characteristic implies in a significant computational cost reduction, therefore the BEM becomes more advantageous than the FEM for infinite domain simulation. In such a way, many authors use the BEM to model infinite solids, as performed in [5] and [6]. It is possible to obtain an even more advantageous formulation if infinite boundary elements (IBE) are used together with BEs, applying the same concept of the IEs. In such a way, it is possible to analyze even three-dimensional infinite domain problems with relatively low computational cost, as performed in [7] and [8].

In this work, an IBE is proposed for the static analysis of infinite domains. The Kelvin fundamental solutions are employed, considering the static analysis of three-dimensional, homogeneous, isotropic and linear-elastic solids. The strategy is to use the same formulation of a triangular finite BE with linear shape functions, only changing the Jacobian that relates the local system of equations with the global one. This new Jacobian is obtained using special mapping functions, which are different from the ones usually found in the literature. Although only homogeneous domains are considered in this work, in the future the authors intent to use this same formulation in non-homogeneous problems.

2 Boundary Element Formulation

The equilibrium of a solid body can be represented by a boundary integral equation called Somigliana Identity, which for homogeneous, isotropic and linear-elastic domains is:

$$\begin{gathered} c_{ij}(y)\, u_j(y) + \int_S T_{ij}(x,y)\, u_j(x)\, dS(x) = \\ \int_S U_{ij}(x,y)\, t_j(x)\, dS(x) \end{gathered} \tag{1}$$

This equation is written for a source point y at the boundary, where the displacement is $u_j(y)$. The constant $c_{ij}(y)$ depends on the Poisson ratio and the boundary geometry at y. The field point x goes through all boundary S, where displacements are $u_j(x)$ and tractions are $t_j(x)$. The integrals kernels $U_{ij}(x,y)$ and $T_{ij}(x,y)$ are Kelvin three-dimensional fundamental solutions for displacements and tractions, respectively. Kernel $U_{ij}(x,y)$ has order ${}^{1}\!/_{r}$ and kernel $T_{ij}(x,y)$ order ${}^{1}\!/_{r^2}$ with $r = |x-y|$, therefore the integrals have singularity problems when x approaches y. Therefore the stronger singular integral, over the traction kernel, has to be defined in the sense of a Cauchy Principal Value (CPV).

*University of São Paulo, São Carlos Engineering School, Structural Engineering Department, Av. Trabalhador Sãocarlense, 400, 13566-590, São Carlos, SP, Brasil. Tel: 55-16-3373-9470 Fax: 55-16-3373-9482 Email: [1]dimas@sc.usp.br [2]paiva@sc.usp.br

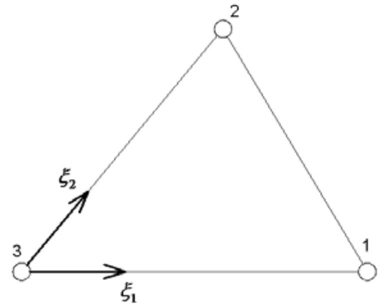

Figure 1: Triangular boundary element.

To solve equation 1 numerically, the boundary S is divided into sub-regions where displacements and tractions are approximated by known shape functions. In this work these sub-regions are of two types, finite boundary elements (BEs) and infinite boundary elements (IBEs). The BEs employed are triangular, as shown in figure 1 with the local system of coordinates, $\xi_1\xi_2$, and the local node numeration. The subsequent approximations are used for this BE:

$$u_j = \sum_{k=1}^{3} N^k u_j^k, \quad t_j = \sum_{k=1}^{3} N^k t_j^k \tag{2}$$

In expressions 2, the boundary values u_j and t_j are related to the nodal values of the BE. The BEs have 3 nodes and for each node there are three components of displacement u_j^k and traction t_j^k. The shape functions N^k used for these approximations are:

$$N^1 = \xi_1 \ , \quad N^2 = \xi_2 \ , \quad N^3 = 1 - \xi_1 - \xi_2 \tag{3}$$

The same shape functions are used to approximate the boundary geometry:

$$x_j = \sum_{k=1}^{3} N^k x_j^k \tag{4}$$

where x_j^k are the node coordinates. For IBEs, displacements and tractions are interpolated with the same functions:

$$u_j = \sum_{k=1}^{Np} N^k u_j^k, \quad t_j = \sum_{k=1}^{Np} N^k t_j^k \tag{5}$$

Each IBE has Np nodes and not Nod as the BEs. The IBEs geometry, on the other hand, is approximated using special mapping functions, as discussed with more details in Section 3.

By substituting expressions 2 and 5 at equation 1, the subsequent equation is obtained:

$$\begin{aligned} c_{ij}(y)\, u_j(y) + \sum_{e=1}^{N_{BE}} \left\{ \sum_{k=1}^{3} \left[\Delta T_{ij}^{ek} u_j^k \right] \right\} + \\ + \sum_{e=1}^{N_{IBE}} \left\{ \sum_{k}^{Np} \left[\Delta^\infty T_{ij}^{ek} u_j^k \right] \right\} = \sum_{e=1}^{N_{BE}} \left\{ \sum_{k=1}^{3} \left[\Delta U_{ij}^{ek} t_j^k \right] \right\} + \\ + \sum_{e=1}^{N_{IBE}} \left\{ \sum_{k=1}^{Np} \left[\Delta^\infty U_{ij}^{ek} t_j^k \right] \right\} \end{aligned} \tag{6}$$

N_{BE} is the number of BEs and N_{IBE} is the number of IBEs. For BEs:

$$\Delta T_{ij}^{ek} = \int_{\gamma_e} |J|\, N^k T_{ij} d\gamma_e, \quad \Delta U_{ij}^{ek} = \int_{\gamma_e} |J|\, N^k U_{ij} d\gamma_e \tag{7}$$

In equation 7 the global system of coordinates is transformed to the local one using the Jacobian $|J| = 2A$, where A is the element area in the global system. On the other hand, for IBEs:

$$\begin{aligned} \Delta^\infty T_{ij}^{ek} &= \int_{\gamma_e} |^\infty J|\, N^k T_{ij} d\gamma_e, \\ \Delta^\infty U_{ij}^{ek} &= \int_{\gamma_e} |^\infty J|\, N^k U_{ij} d\gamma_e \end{aligned} \tag{8}$$

Special mapping functions are used to calculate $|^\infty J|$, as detailed in Section 3.

The integrals of equations 7 and 8 are calculated employing standard BEM techniques. Non-singular integrals are numerically evaluated using integration points. The singular ones, on the other hand, are evaluated using the technique presented in reference [9]. In the end, the free term c_{ij} may be obtained by rigid body motions.

Writing equation 6 for all boundary nodes, as described in [10], one obtains the following system of equations:

$$\Delta T \cdot u = \Delta U \cdot t \tag{9}$$

The ΔT_{ij}^{ek} element contributions, including the free term c_{ij}, are assembled to matrix ΔT and ΔU_{ij}^{ek} contributions are assembled to ΔU. Vectors u and t contain all boundary displacements and tractions, respectively. Reorganizing this system separating the known boundary values from the unknown, one obtains a system of equations which solution is all the unknown boundary values.

3 Infinite Boundary Elements

In this work, the field variables are considered to vanish at infinity. Hence, nodes placed at infinity have no contribution in the integrals defined in expressions 8. It is also more practical not to write mapping functions for these nodes, replacing them by auxiliary ones that are placed at a finite distance. Three types of mapping are considered, as illustrated in figure 2.

In the first type of mapping, as represented in Figure 2a, only direction ξ_1 is mapped and node 1 is placed at infinity. The IBE is represented in the local coordinate system on the left side, and in the global coordinate system on the right side. The global coordinates x_i are related to the local ones using special mapping functions, M^k, and the nodal global coordinates, x_i^k. Node 4 is created only to replace node 1 for the mapping and do not contribute with the integrals.

Figure 2b is analogous to Figure 2a, but in this case only direction ξ_2 is mapped and node 2 is placed at infinity. Therefore, node 5 is created to auxiliate the mapping. Finally, in Figure 2c both local directions are mapped and nodes 1 and 2 are placed at infinity. As a result, the auxiliary nodes 4 and 5 must be created to replace them in the mapping.

In this work, an auxiliary coordinate $\bar{\xi}_i(\xi_i)$ is created in order to obtain the mapping functions. After analyzing some options and considering reference [11], the following function was chosen:

$$\bar{\xi}_i = \frac{\xi_i}{1-\xi_i} \tag{10}$$

Using expression 10 and considering direction ξ_1 mapped, the following relation is obtained:

$$\bar{\xi}_1(\xi_1) = \frac{\xi_1}{1-\xi_1} \tag{11}$$

and to map direction ξ_2:

$$\bar{\xi}_2(\xi_2) = \frac{\xi_2}{1-\xi_2} \tag{12}$$

In the end, the mapping functions are obtained by substituting the relations 11 and 12 in the shape functions 3 of the original BE. In such a way, to map only direction ξ_1 as illustrated in figure 2a, equation 11 must be substituted in the shape functions 3. As a result, one obtains:

$$M^4_{1\infty} = \bar{\xi}_1(\xi_1) = \frac{\xi_1}{1-\xi_1} \tag{13}$$

$$M^2_{1\infty} = \xi_2 \tag{14}$$

$$M^3_{1\infty} = 1 - \bar{\xi}_1(\xi_1) - \xi_2 = 1 - \frac{\xi_1}{1-\xi_1} - \xi_2 \tag{15}$$

The symbol "1∞" was used to indicate that these expressions are valid case only direction ξ_1 is mapped. It is important to notice that mapping function $M^2_{1\infty}$ is equal to the original shape function, meaning that direction ξ_2 is not influenced by this mapping.

In reference [12] it is recommended to verify the mapping functions as follows:

- Their sum must be equal to 1;
- The sum of their derivatives must be equal to zero;
- Any mapping function is equal to 1 on the corresponding node, and to zero on the other nodes;
- For nodes at infinity, mapping functions tent to $-\infty$ or $+\infty$;

The last item do not apply to function $M^2_{1\infty}$ because it refers to a direction that is not mapped. For all other cases, it is demonstrable that the items are verified by the defined mapping functions. These functions are then employed to relate the local system of coordinates to the global one. In other words:

$$x_i = M^4_{1\infty} x_i^4 + M^2_{1\infty} x_i^2 + M^3_{1\infty} x_i^3 \tag{16}$$

After obtaining expression 16, the Jacobian used when only direction ξ_1 is mapped may be calculated as follows:

$$|{}^{\infty}J_1| = \frac{\partial x_1}{\partial \xi_1}\frac{\partial x_2}{\partial \xi_2} - \frac{\partial x_2}{\partial \xi_1}\frac{\partial x_1}{\partial \xi_2} = \frac{2A_1}{(1-\xi_1)^2} \tag{17}$$

where A_1 is the area of triangle defined by nodes 2, 3 and 4 in the global system of coordinates.

The same steps are repeated to obtain the Jacobian for mapping only in the ξ_2 direction. That is, by substituting 12 in the shape functions, one obtains:

$$M^1_{2\infty} = \xi_1 \tag{18}$$

$$M^5_{2\infty} = \bar{\xi}_2(\xi_2) = \frac{\xi_2}{1-\xi_2} \tag{19}$$

$$M^3_{2\infty} = 1 - \xi_1 - \bar{\xi}_2(\xi_2) = 1 - \xi_1 - \frac{\xi_2}{1-\xi_2} \tag{20}$$

The symbol "2∞" is used to indicate that only direction ξ_2 is mapped. As a result, mapping function $M^1_{2\infty}$ is equal to the original mapping function. These functions also satisfy the verifications suggested in reference [12], as presented before. Therefore, the global system is related to the local one as follows:

$$x_i = M^1_{2\infty} x_i^1 + M^5_{2\infty} x_i^5 + M^3_{2\infty} x_i^3 \tag{21}$$

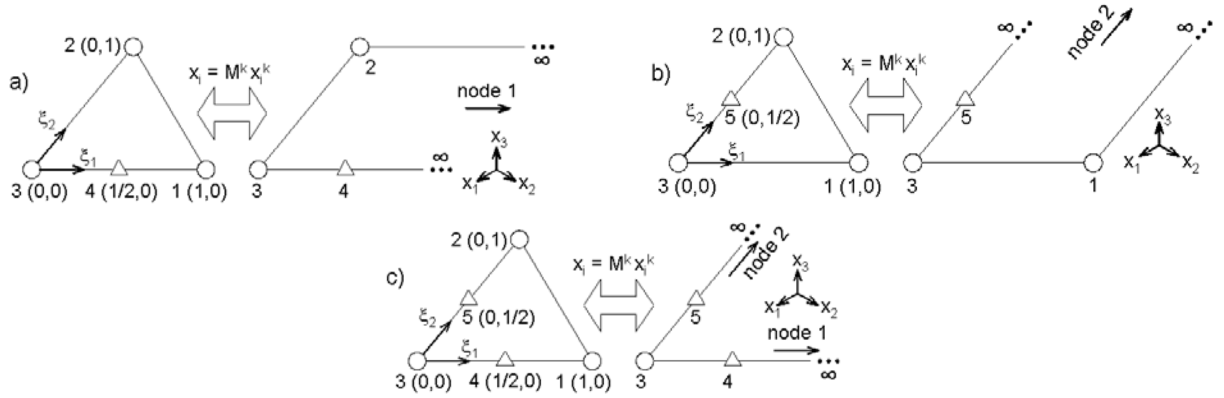

Figure 2: Types of mapping.

and the Jacobian is:

$$|^{\infty}J_2| = \frac{2A_2}{(1-\xi_2)^2} \tag{22}$$

where A_2 refers to the area of the triangle defined by nodes 1, 3 and 5 in the global system of coordinates.

Finally, to map directions ξ_1 and ξ_2 both expressions 11 and 12 must be substituted at the shape functions. The result is:

$$M_{\infty}^4 = \frac{\xi_1}{1-\xi_1} \tag{23}$$

$$M_{\infty}^5 = \frac{\xi_2}{1-\xi_2} \tag{24}$$

$$M_{\infty}^3 = 1 - \frac{\xi_1}{1-\xi_1} - \frac{\xi_2}{1-\xi_2} \tag{25}$$

The symbol "∞" is used to indicate that both directions are mapped. Once more the functions satisfy the verifications and may be used to relate the local and global systems of equations, as follows:

$$x_i = M_{\infty}^4 x_i^4 + M_{\infty}^5 x_i^5 + M_{\infty}^3 x_i^3 \tag{26}$$

and the Jacobian becomes:

$$|^{\infty}J_3| = \frac{2A_3}{(1-\xi_1)^2(1-\xi_2)^2} \tag{27}$$

where A_3 is the area of the triangle defined by nodes 3, 4 and 5 in the global system.

4 Results

An example is now analyzed, as illustrated in Figure 3a, in order to test the presented formulation. A circular uniform load of 2 kN/m^2 and with a radius of 2.5 m is applied at the surface of an homogeneous half-space. The domain has an elasticity module of 9000 kN/m^2 and a Poisson ratio of 0.0.

In reference [13] an analytical expression is presented for the vertical displacement at the central node of the loaded area, which is identified in figure 3a as C. This expression is:

$$d = 2rp\frac{(1-\nu^2)}{E} \tag{28}$$

where d is the vertical displacement at point C, r is the radius of the circular load, p is the load value, ν is the Poisson ratio of the half space and E is its elasticity module. Substituting the values of figure 3a in expression 28, a displacement of 1.1111×10^{-3} m is obtained.

To simulate this problem a mesh with 57 nodes was generated, totalizing 96 BEs and 32 IBEs, as illustrated in figure 3b. The circle detached in the center corresponds to the loaded area, the dashed lines represent the IBEs and the rest of the mesh is composed by BEs. The points marked at the limits of the BE mesh are the ones that receive the influence of the IBEs. It is important to notice that no additional degrees of freedom are included by the presence of the IBEs, which is an advantage of this formulation.

Simulating this problem with the mesh of Figure 3b, a vertical displacement of 1.0850×10^{-3} m was obtained. This value agrees with the analytical solution, with an error of 2.4 %. In order to evaluate the influence of the IBEs, the example was simulated with the same BE mesh but no IBEs. A displacement of 1.0107×10^{-3} m was than obtained, with the higher error of 9.0 % comparing to the analytical value.

The error increases when no IBEs are used because the domain is not well simulated as an infinite media. In order to improve this precision, more BEs and degrees of freedom need to be added at the mesh limits. In such a way, different meshes were tested and the results obtained are presented in Table 4.

With the increasing number of BEs the domain becomes closer to an infinite half-space and consequently better

ISBN:978-988-18210-1-0

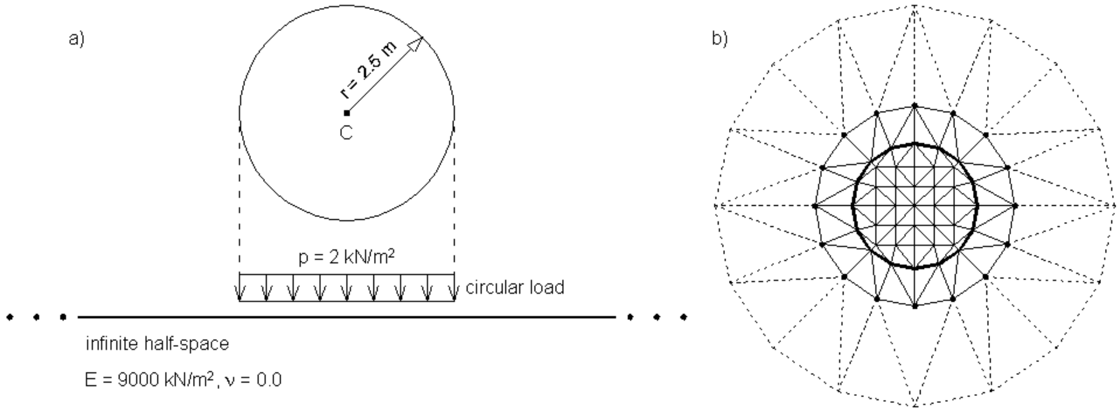

Figure 3: *a*) Infinite domain problem; *b*) Mesh generated.

Table 1: Displacement calculation with BEs only.

Nodes	Displacement (m)	Error (%)
57	1.0107×10^{-3}	9.0
73	1.0391×10^{-3}	6.5
89	1.0580×10^{-3}	4.8
105	1.0699×10^{-3}	3.7
121	1.0774×10^{-3}	3.0
137	1.0820×10^{-3}	2.6
153	1.0849×10^{-3}	2.4

results are achieved, however more degrees of freedom are needed. As may be observed, 153 nodes were needed for the BE mesh to equalize the precision of the first 57 node mesh, that was with IBEs. Comparing these two values it may be concluded that, in this example and maintaining the error of 2.4 %, the use of IBEs allows a mesh reduction of 63 %.

In tests with finer meshes at the central area, errors below 1 % were obtained. In such cases, comparing the number of nodes with and without IBEs, the mesh reduction is also very significant.

5 Conclusions and Future Work

This article presents a new mapped infinite boundary element (IBE) formulation, based on a triangular boundary element (BE) with linear shape functions. To obtain the mapping functions the local coordinate to be mapped is replaced by an auxiliary function, which was defined using reference [11]. The resulting functions are then used to relate the global system of equations with the local one, obtaining a Jacobian for each case.

The resulting IBE, when associated with a BE mesh, has the advantage of not increasing the original number of degrees of freedom, as demonstrated in the example. The results obtained with the IBEs showed good agreement with an analytical solution, and the use of this formulation promoted a mesh reduction of 63 %. The reduction calculated for other levels of precision was also very significant.

In future works, the authors intent to use IBEs together with BEs in problems involving infinite non-homogeneous half-spaces. The results here presented show that this formulation is adequate to this type of analysis.

References

[1] Karakus, M., Ozsan, A., Basarir, H., "Finite element analysis for the twin metro tunnel constructed in Ankara Clay, Turkey," *Bulletin of Engineering Geology and the Environment*, V66, pp. 71-79, 2007.

[2] Yin, L. Z., Yang, W., "Topology optimization for tunnel support in layered geological structures," *International Journal for Numerical Methods in Engineering*, V47, pp. 1983-1996, 2000.

[3] Sadecka, L., "A finite/infinite element analysis of thick plate on a layered foundation," *Computers and Structures*, V76, N5, pp. 603-610, 2000.

[4] Liu, D. S., Chiou, D. Y., Lin, C. H., "3D IEM formulation with an IEM/FEM coupling scheme for solving elastostatic problems," *Advances in Engineering Software*, V34, N6, pp. 309-320, 2003.

[5] Ribeiro, D. B., Almeida, V. S., Paiva, J. B., "Uma formulação alternativa para analisar a interação solo não-homogêneo/ fundação/ superestrutura via acoplamento MEC-MEF," *Revista Sul-Americana de Engenharia Estrutural*, V2, N2, pp. 27-46, 2005.

[6] Andersen, L., Jones, C. J. C., "Coupled boundary and finite element analysis of vibration from railway tunnels - a comparison of two- and three-dimensional models," *Journal of Sound and Vibration*, V293, pp. 611-625, 2006.

ISBN:978-988-18210-1-0

[7] Moser, W., Duenser, C., Beer, G., "Mapped infinite elements for three-dimensional multi-region boundary element analysis," *International Journal for Numerical Methods in Engineering*, V61, pp. 317-328, 2004.

[8] Ribeiro, D. B., Beer, G., Souza, C. P. G., "Tunnel excavation in rock mass with changing stiffness," *EURO:TUN 2007 Computational Methods in Tunnelling*, Vienna, Austria, Vienna University of Technology, 2007.

[9] Guiggiani, M., Gigante, A., "A general algorithm for multidimensional Cauchy principal value integrals in the boundary element method," *Journal of Applied Mechanics*, V57, pp. 906-915, 1990.

[10] Brebbia, C. A., Dominguez, J., *Boundary Elements: An Introductory Course*, Computational Mechanics Publications, London, 1992.

[11] Davis, P. J., Rabinowitz, P., *Methods of numerical integration*, Academic Press, New York, 1975.

[12] Bettess, P., *Infinite elements*, Penshaw Press, Sunderland, UK, 1992.

[13] Burland, J.B., Broms, B.B., de Mello, V.F.B., "Behavior of foundations and structures," 9^{th} *International Conference on Soil Mechanics and Foundation Engineering*, Tokyo, V2, 1977.