A New and Practical Approach for Calculating the Static Eccentricity of Doubly Asymmetric, Non-proportional, Multi-storey Buildings

B. Rafezy and W.P. Howson*

Abstract— A practical approach for determining the static eccentricity at each floor level in a doubly asymmetric, multi-storey building is presented. The method uses a new and simple flexibility approach to relate the loading sustained by the resisting elements of the structure to the total lateral load on the building. The method is easy to apply and offers considerable advantage over the Plane Frame Method that is commonly used in practice. An example is included to illustrate the method.

Index Terms—**centre of rigidity; centre of shear; static eccentricity; asymmetric buildings; irregular buildings; equivalent static analysis.**

I. Introduction

Establishing the centres of rigidity (CR) and shear (CS) at each floor level in a multi-storey building is fundamental to many problems encountered when dealing with the statics, buckling and dynamics of such structures [1-5] and is an essential pre-requisite in the application of most seismic codes of practice e.g. [6-8]. Naturally, many methods of determining this important parameter have been proposed using a variety of techniques that offer solutions of varying accuracy depending on the assumptions employed. The most popular technique is probably the Plane Frame Method (PFM) that was first presented by Cheung and Tso [9].

PFM is based on the interpretation of the CRs as "load centres" at each floor level. Therefore, if the loading on each resisting structural element at each floor is known under the assumption of no rotational deformation, the load centre at each floor can be obtained by dividing the first moment of the element loads by the total loading at that floor level. Assuming the building is restrained from rotation, the lateral floor displacement in one direction, e.g. the *x* direction, and the inter-storey shear of all elements under loading in the same direction, can readily be obtained by means of a standard plane frame program. i.e. the global stiffness of the structure in this direction is simulated by joining side-by-side, through pin-ended, rigid bars at each floor level, all the resisting elements in the *x*-direction. This enables the inter-storey shear force of each resisting element to be calculated by subjecting the model to the total lateral loading on the original structure. The floor loads for the individual elements then follow directly. The *y* co-ordinate of the load centres, i.e. CRs, is then given by the ratio of the first moment of these floor loads about reference axis *z* to the total floor load at that level. It should be noted that this procedure does not require explicit knowledge of the global structure stiffness matrix. An identical procedure can then be followed to calculate the *x* co-ordinate of the CRs.

Hejal and Chopra [10] extended Cheung and Tso's work [9] to multi-storey buildings with a generalised floor plan comprising plane frames, columns, shear walls and cores. A number of studies [9-13] also identify a class of building in which the lateral stiffness matrices of all resisting frames are proportional and show that the location of the CRs are independent of the lateral forces, which then lie on a vertical line throughout the height of the structure. This class of buildings will be referred to as "proportional buildings".

In the work that follows, CRs are defined as that set of points, one on each floor of a building, through which application of lateral forces would cause no rotation of any of the floors. In addition, the Centre of Shear (CS) of a floor is defined as the location of the resultant of the shear forces due to the various resisting elements at that floor level when the resultant of the applied lateral forces passes through CR. A practical method for locating the CRs, CSs and hence the static eccentricities of doubly-asymmetric, multi-storey buildings is now developed.

II. Proposed method

The method is based on the use of a plane frame computer program and, like PFM, it does not require explicit knowledge of the global structure stiffness matrix. However, the way in which the program is used is completely different from PFM, since its sole function is to generate the flexibility matrix of each resisting element in the plane under consideration. These are subsequently used to form a matrix relationship between the loading on the resisting elements and the total lateral loading on the building. The CRs of the building are then obtained from the fact that they can be interpreted as the load centres at each floor level under the

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assumption of no rotational deformation.

Consider a typical floor plan of a multi-storey building comprising plane resisting elements (frames, columns, shear walls, bracing etc.) that run in two orthogonal directions and which are joined to each other by rigid diaphragms at each floor level. See Fig. 1. The coordinate system Oxy is fixed at an arbitrary point on the plan, with the x and y axes running parallel to the orthogonal planes of resisting elements. It is assumed that the building is subjected to a known set of lateral loads at each floor level, defined by the vectors

$$\mathbf{V}_x^T = \begin{bmatrix} V_{x1} & V_{x2} & V_{x3} & . & . & V_{xn} \end{bmatrix}$$

and

$$\mathbf{V}_y^T = \begin{bmatrix} V_{y1} & V_{y2} & V_{y3} & . & . & V_{yn} \end{bmatrix} \qquad (2a,b)$$

where Vxj and Vyj (j=1,n)) are the resultant lateral loads applied at the jth storey level in the x and y directions, respectively, n is the number of storeys and T represents transpose.

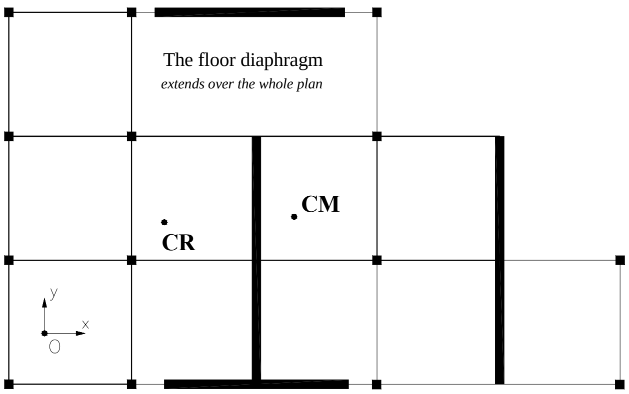

Fig. 1. Floor plan of an asymmetric multi-storey building

comprising resisting elements running in two orthogonal directions Ox and Oy. CM is the Centre of Mass and CR is the Centre of Rigidity of the resisting elements.

In similar fashion, the vectors

$$\mathbf{X}_R^T = \begin{bmatrix} X_{R1} & X_{R2} & . & . & . & X_{Rn} \end{bmatrix}$$

and

$$\mathbf{Y}_R^T = \begin{bmatrix} Y_{R1} & Y_{R2} & . & . & . & Y_{Rn} \end{bmatrix} \qquad (3a,b)$$

are assumed to contain, respectively, the unknown x and y coordinates, X_{Rj} and Y_{Rj} (j=1,n), that define the location of the CR at floor level j. Assuming that the resultant lateral loads at each floor level are now applied through their respective CRs, the building will undergo pure translation in both directions. Since the plane resisting elements have no out of plane stiffnesses, the structure can be analysed in the x and y directions independently.

From the definition of CR, the coordinates X_R and Y_R can be interpreted as defining the load centres at each floor level. Therefore, if the loading on each element of resisting structure at each floor level is known under the assumption of no rotational deformation, the location of the load centre at each floor can be determined by summing the first moment of the element loads at that floor and dividing by the total loading imposed at that level.

The loading on each resisting element can be calculated using equilibrium equations and the knowledge that all elements deflect equally in the x direction, and likewise in the y direction, when the building is subjected to lateral loads applied through the CRs, since the floor diaphragm is rigid in its plane.

Consider the y direction first. The loading on the ith resisting element and the corresponding equation of equilibrium are

$$\mathbf{p}_y^{(i)T} = \begin{bmatrix} p_{y1}^{(i)} & p_{y2}^{(i)} & . & . & . & p_{yn}^{(i)} \end{bmatrix} \quad \text{and}$$

$$\mathbf{V}_y = \sum_{i=1}^{m} \mathbf{p}_y^{(i)} \qquad (4,5)$$

where $p_{yj}^{(i)}$ is the loading of the i[th] element at the j[th] floor and m is the number of resisting elements. See Fig. 2.

Applying lateral loads at the CRs of the building requires that the displacement vector of all resisting elements be equal. This gives

$$\mathbf{d}_y^{(1)} = \mathbf{d}_y^{(2)} \cdots = \mathbf{d}_y^{(m)} = \mathbf{d}_y \quad \text{and}$$

$$\mathbf{d}_y^{(i)T} = \begin{bmatrix} d_{y1}^{(i)} & d_{y2}^{(i)} & . & . & . & d_{yn}^{(i)} \end{bmatrix} \qquad (6,7)$$

where $\mathbf{d}_y^{(i)}$ denotes the displacement vector of resisting element i and $d_{yj}^{(i)}$ gives the deflection of element i at the j[th] floor. See Fig. 2.

Displacement vector $\mathbf{d}_y^{(i)}$ can now be written in terms of the stiffness matrix and lateral loading of the i[th] resisting element as

$$\mathbf{d}_y^{(i)} = \mathbf{k}_y^{(i)^{-1}} \mathbf{p}_y^{(i)} \qquad (8)$$

in which $\mathbf{k}_y^{(i)^{-1}}$ is the inverse of the stiffness matrix of the i[th] element.

The stiffness matrix of a resisting element, however, is not always readily available. An alternative approach is therefore suggested that does not require its explicit use and as a result is more suitable for use in the design context.

Consider the i[th] resisting element subjected to lateral $\mathbf{d}_y^{(i)}$ as shown in Fig. 2. The relationship between $\mathbf{p}_y^{(i)}$ load can be established by using flexibility coefficients, $\mathbf{p}_y^{(i)}$ and , which relate the deflection of the j[th] floor of element i $\delta_{yjk}^{(i)}$ to a lateral unit force applied at level k.

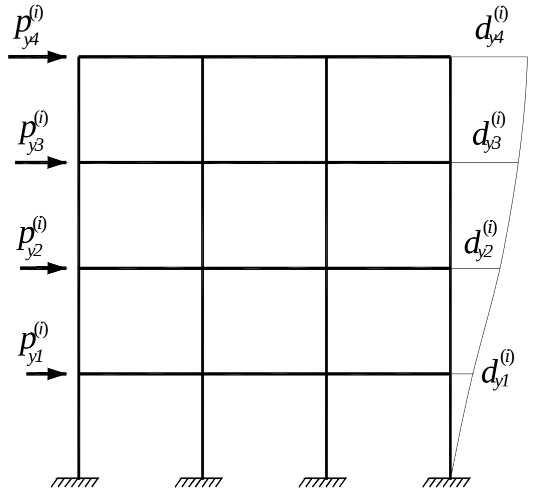

Fig. 2. Resisting element i subjected to lateral loading vector $\mathbf{p}_y^{(i)}$ and the resulting displacement vector $\mathbf{d}_y^{(i)}$.

Since the behaviour of the building is assumed to be linear and elastic, the deflection of the jth floor of element i subjected to $\mathbf{p}_y^{(i)}$ can be written as

$$d_{yj}^{(i)} = \sum_{k=1}^{n} \delta_{yjk}^{(i)} p_{yk}^{(i)} \quad \text{or} \quad \mathbf{d}_y^{(i)} = \boldsymbol{\delta}_y^{(i)} \mathbf{p}_y^{(i)} \tag{9a,b}$$

where $\boldsymbol{\delta}_y^{(i)}$ is the flexibility matrix of element i in which $\delta_{yjk}^{(i)}$ is the coefficient of the matrix located at the intersection of row j and column k. Therefore Eq. (6) enables Eq. (9b) to be written as

$$\boldsymbol{\delta}_y^{(i)} \mathbf{p}_y^{(i)} - \mathbf{d}_y = \mathbf{0} \tag{10}$$

Eqs. (5) and (10) can be combined to give an $n(m+1)$ system of algebraic equations for calculating $\mathbf{p}_y^{(i)}$ and $\mathbf{d}_y$ as

$$\begin{bmatrix} \boldsymbol{\delta}_y^{(1)} & \mathbf{0} & \mathbf{0} & . & . & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \boldsymbol{\delta}_y^{(2)} & \mathbf{0} & . & . & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\delta}_y^{(3)} & . & . & \mathbf{0} & \mathbf{I} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & . & . & \boldsymbol{\delta}_y^{(m)} & \mathbf{I} \\ \mathbf{I} & \mathbf{I} & \mathbf{I} & . & . & \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{p}_y^{(1)} \\ \mathbf{p}_y^{(2)} \\ \mathbf{p}_y^{(3)} \\ . \\ . \\ \mathbf{p}_y^{(m)} \\ -\mathbf{d}_y \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ . \\ . \\ \mathbf{0} \\ \mathbf{V}_y \end{bmatrix} \tag{11}$$

Once the $\mathbf{p}_y^{(i)}$ are determined, the location of each of the centres of rigidity and the corresponding vector $\mathbf{X}_R$ containing all j such locations can be determined as

$$X_{Rj} = \frac{\sum_{i=1}^{m} p_{yj}^{(i)} x_i}{V_{yj}} \quad \text{and} \quad \mathbf{X}_R = \mathbf{V}_{yd}^{-1} \mathbf{P}_y \mathbf{X}_e \tag{12,13}$$

where x_i is the distance of element i from the y axis and

$$\mathbf{V}_{yd} = \begin{bmatrix} V_{y1} & 0 & 0 & . & . & 0 \\ 0 & V_{y2} & 0 & . & . & 0 \\ 0 & 0 & V_{y3} & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & V_{yn} \end{bmatrix}$$

$$\mathbf{P}_y = \begin{bmatrix} p_{y1}^{(1)} & p_{y1}^{(2)} & p_{y1}^{(3)} & . & . & p_{y1}^{(m)} \\ p_{y2}^{(1)} & p_{y2}^{(2)} & p_{y2}^{(3)} & . & . & p_{y2}^{(m)} \\ p_{y3}^{(1)} & p_{y3}^{(2)} & p_{y3}^{(3)} & . & . & p_{y3}^{(m)} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ p_{yn}^{(1)} & p_{yn}^{(2)} & p_{yn}^{(3)} & . & . & p_{yn}^{(m)} \end{bmatrix}$$

$$\mathbf{X}_e = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ x_m \end{bmatrix} \tag{14a,b,c}$$

It can be seen that the location of the centres of rigidity are, in general, load dependent and do not lie on a vertical line through the building. An identical procedure can then be used to determine the y components of the CRs by considering the motion of the structure in the x direction.

III. PROPORTIONAL BUILDINGS

A special case arises when the stiffness matrix of each resisting element is proportional to a datum stiffness matrix. In this case it is straightforward to show that Eq. (13) can be re-written as

$$\mathbf{X}_R = \mathbf{u}\boldsymbol{\alpha}\mathbf{X}_e \tag{15}$$

where u is the $(n \times m)$ unitary matrix whose elements are all equal to 1 and $\boldsymbol{\alpha}$ is the diagonal $(m \times m)$ matrix of proportionality constants. Eq. (15) clearly shows that in this particular case, the location of the CRs are not load dependent and therefore lie on a vertical line through the height of the structure.

III.I. EXAMPLE

The building analysed by Cheung and Tso [9] is now considered. It is located in seismic Zone 2 in Canada and is subjected to seismic lateral loading that has a triangular distribution along the height of the building in the y direction. It is a nine storey, singly asymmetric, wall-frame building

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having uniform rectangular floors of dimension 20m by 10m and a uniform storey height of 3m. It is symmetric about the x axis and comprises two identical uniform walls, W, and two identical uniform frames, F, running in the y direction as shown in Fig. 3. The beams forming the frames are considered to be very stiff and the second moment of area of all columns is assumed to be $I_c = 2.7375 \times 10^{-3}$ m^4. The walls have a uniform cross-section with second moment of area, $I_w = 0.445$ m^4. Young's modulus for all members is taken as E=2×1010 N/m2. It should be noted that although the frame and wall systems are independently proportional, the building is non-proportional since the stiffness matrix of a wall differs from that of a frame.

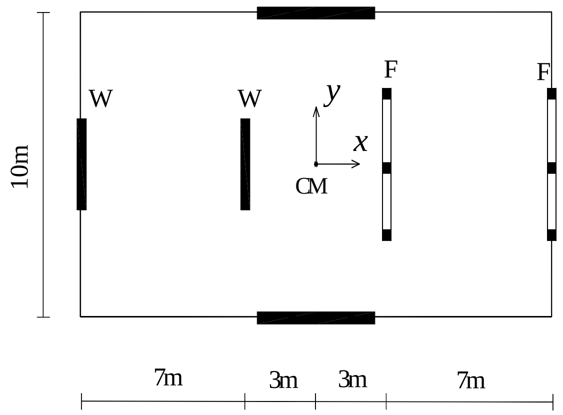

Fig. 3. Floor plan of the building, comprising wall (W) and frame (F) resisting elements.

The second column of Table 1 shows the location of the CRs at each floor calculated by (Cheung and Tso 1986) using PFM and the x-y co-ordinate system shown in Fig. 3, whose origin is at CM of each floor. Columns 3 and 4 show, respectively, the location of the CRs and CSs determined by the proposed method. This again shows that there is very good agreement between the results of the two approaches and indicates that the proposed method may be used with confidence.

Table 1. Location of the CRs and CSs of the building measured from CM in Fig. 3. XR(S) is the vector of x co-ordinates of the CRs(CSs).

Floor j	XR (PFM) (m)	XR (Proposed method) (m)	XS (Proposed method) (m)
9	20.6	20.45	20.42
8	-3.30	-3.42	9.22
7	-1.30	-1.31	6.14
6	-0.70	-0.70	4.77
5	-1.80	-1.82	3.83
4	-5.80	-5.85	2.83
3	-16.50	-16.39	1.46
2	-45.30	-45.42	-0.66
1	-154.9	-155.94	-4.12

The virtual work method (Thomson and Haywood 1986) of computing translations and rotations is used here

IV. Conclusions

A simple and practical method for locating the centres of rigidity of multi-storey buildings has been presented. It enables the static eccentricity to be determined easily, which is particularly important in the application of codified rules and special provisions when implementing the static force procedures of most seismic building codes. The method is based on the use of a plane frame computer program but does not require explicit expressions for the stiffness matrix. The method has the following advantages in comparison with the Plane Frame Method

1. Resisting elements are analyzed separately, so the input file is much smaller and there is no necessity to model pin ended, rigid bars at each floor level.
2. Identical plane elements are analyzed only once, since they all have a unique flexibility matrix $\boldsymbol{\delta}^{(i)}$.
3. The method lends itself to simple data generation and automated solutions.
4. The method developed herein is easily extendable to enable a general static analysis of doubly symmetric structures with rigid diaphragms to be undertaken using the two-dimensional approach.
5. Simple additional modifications to the method can also be made in order to account for the analysis of structures with flexible diaphragms.

It has been shown that the centres of rigidity and centres of shear of multi-storey buildings do not generally coincide. Moreover the locations of each do not generally lie on a vertical line through the height of the structure, but are dependent on the geometric and stiffness characteristics of the building as well as the lateral forces. A particular class of buildings was distinguished, the so called 'proportional buildings', in which the centres of rigidity and centres of shear of the floors are coincident, load independent and lie on a vertical line throughout their height. Buildings belonging to this special class comprise resisting elements that have proportional stiffness matrices along both their principal planes. The proportionality in the x and y directions are independent and it is not necessary that the resisting elements running in the x direction be proportional to those running in the y direction.

Torsional provision in most building codes is based on the evaluation of static eccentricity at each floor level, usually given as the distance between the centres of mass and the centres of rigidity of a building. Since the variation in the location of the centres of shear along the height of the structure is usually less than that of the centres of rigidity, it

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would seem more practical in such cases for code provisions to give their rules based on eccentricity defined as the distance between the centres of mass and centres of shear.

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