Numerical and Analytical Approach of Thermo-Mechanical Stresses in FGM Beams

Fatemeh Farhatnia, Gholam-Ali Sharifi and Saeid Rasouli

Abstract—**In this paper, thermo-mechanical stress distribution has been determined for a three layered composite beam having a middle layer of functionally graded material (FGM), by analytical and numerical methods. Beam is subjected to uniformly transverse distributed loading whereas the uniform temperature gradient arises in it. FGM beams with continuous and smooth grading of metal and ceramics based on power law index are considered for study, whereas Poisson ratio is to be held constant through FGM layer. Analytical solution is based on simple Euler-Bernoulli type beam theory for long, slender beam. Also, the principle of stationary potential function is used to obtain the static finite element equations for the FGM composite beam. By comparing the deduced results with FEM calculations in ANSYS, good agreement has been indicated between them.**

Index Terms— **composite beam, Euler-Bernoulli beam theory, functionally graded material, neutral axis, thermo-mechanical stress.**

I. INTRODUCTION

Functionally graded material (FGM) is a kind of material in which the individual material composition varies continuously along certain directions in a controllable way. In FGM, the best properties of metal and ceramics are combined--the toughness, electrical conductivity and machinability of metals and the low density, high strength, high stiffness, and temperature resistance of ceramics. Hence the use of FGMs has been increasing in various engineering applications; these inhomogeneous solids also have received considerable scientific interest and numerous research papers have been published. In the following, however, only works related to FGM beams will be referenced. Suresh and Mortensen [1], Miyamoto, Kaysser, Rabin, Kawasaki and Ford [2] provided an excellent introduction to fundamentals of FGM. Analytical and numerical studies have been carried out to investigate thermo-mechanical behavior of FGMs [3]-[5]. Most of them have been limited to FGMs with linear compositional gradation. Sankar [6] has also solved the plane elasticity problem of an FGM beam subjected to transverse loading using a Fourier series technique. He assumed an exponential variation of properties. Sankar and Tzeng [7] have obtained a closed-form analytical solution for the thermal stresses in functionally graded beams considering thermo-elastic constants of the beam and the temperature varying exponentially through the beam thickness. Zhu and Sankar [8] assumed that the elastic compliance parameters are some proportional to a polynomial of z, for which exact solution can not be obtained by Fourier series expansion method. Most of Exact methods are applicable only for symmetrical boundary conditions and loadings. Some researchers [9], [10], [11] have found the approximate and semi-analytical methods for obtaining thermo-elastic stresses in FGM beams. Nirmala and Upadhyay [12] a numerical scheme of discretizing the continuous FGM layer (in sublayers) and treating the beam as a discretely graded structure has also been discussed. Appropriate expressions for the solution have been derived for the power law gradation (mth power) of the FGM layer. Chakraboty, Golpalakrishnan and Reddy [13] developed a new beam finite element to study the thermo-elastic behaviour of FGM beam structures. The element was based on the first-order shear deformation theory and it accounts for varying elastic and thermal properties along its thickness. In the present paper, we proposed an analytical solution for deducing thermo-mechanical stresses in a three layered composite beam in plane strain condition. Then we used the principle of stationary potential function to obtain the static finite element equations for the FGM composite beam. The boundary condition is assumed to symmetric and non-symmetric one (simply supported and clamped-free boundary conditions). Effect of temperature rise/fall is considered by augmenting the thermal strain to the mechanical strain, instead of solving the coupled thermo-elastic equations. Power law is taken for the variation of material properties through the depth of the beam. FEM analysis in ANSYS commercial software is carried out to validate the results.

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II. ANALYTICAL FORMULATION AND SOLUTION

A. Stress Analysis under Mechanical Loading

In Figure 1, the dimension of FGM beam of length unit length and thickness h are considered, where the material property varies continuously in z direction. FGM beams have their volume fraction of ceramics V_c defined according to the power law function, based on rule of mixture and the volume fraction of metal V_m is obtained as follows[1]:

$$V_c(z) = ((z + h_3)/2h_3)^m \quad (1)$$
$$V_c(z) = 1 - V_m(z) \quad (2)$$

Where z is the distance from mid–surface and m is the power law index, a positive real number.

V_c is zero for lower layer ($h_1 \le z \le -h_3$) and is unit for upper layer ($h_3 \le z \le h_2$).

ISBN:978-988-18210-1-0

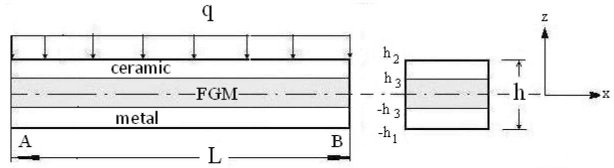


Fig. 1 Three layer composite beam under distributed load

The mathematical modeling for evaluating of properties of FGM, P(z), is obtained in base of law of mixture:

$$P(z)= P_m + (P_c - P_m)V_c(z) \tag{3}$$

The basic assumptions are as follow:

1. The Beam is assumed to be in a state of plane strain, it is normal to the xz plane.
2. Simple Euler-Bernoulli type beam theory is applied.
3. There is no variation in thickness along the length of beam.
4. Poisson's ratio is to be held constant along FG layer.
5. Material properties are independent of temperature gradient.

Then, for a cantilever beam, the displacement field can be written as [6]:

$$\begin{aligned} w(x,z) &= w(x) \\ u(x,z) &= u_0(x) - z\,\frac{dV(x)}{dx} \end{aligned} \tag{4}$$

In above equations, u and w are denoted on horizontal and vertical displacement of beam across the thickness in anywhere. It may be noted that u_0 denoted the displacements of points on the middle surface of the beam along the x direction. We assume that σ_{zz} is negligible. Then the stress-strain relations take the form:

$$\sigma_x(z) = \breve{E}(z)\varepsilon_x \ , \tau_{xz}(z) = \breve{G}(z)\gamma_{xz} \tag{5}$$

Where the plane strain Young modulus is given by: $\breve{E} = \frac{E}{1-v^2}$. The expressions for axial strain and stress can be derived as:

$$\sigma_x(z) = \breve{E}(z).\, \varepsilon_{x0} + z.\breve{E}(z).\kappa \tag{6}$$

ε_{x0} and κ are axial strain in the middle surface and the beam curvature. According to Euler-Bernoulli beam theory, the axial force and bending moment, N and M, are defined as follows:

$$(N,M)=\int_{-h_1}^{h_2} \sigma_x(z)(1,z)dz \tag{7}$$

By substituting the (6) into (7), it may be derived the relation between axial force and moment resultants:

$$(N,M)= \int_{-h_1}^{h_2}\left[\breve{E}(z).\,\varepsilon_{x0} + z.\breve{E}(z).\kappa\right](1,z)dz \tag{8}$$

Since the axial force resultant is zero, the expressions for the deformation take the form:

$$\varepsilon_{x0} = B.\,M(x) \ \ , \ \kappa = D.\,M(x) \tag{9}$$

By integrating of the first equation of differential equilibrium equations, the relation for distributing of shear stress can be derived, whereas shear stress is zero on top and bottom of the height of beam:

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\,\tau_{xz}}{\partial z} = 0 \ \longrightarrow \qquad \tau_{xz}(x,\zeta) = -\int_0^{\zeta}\frac{\partial\sigma_x(z)}{\partial x}\,dz \tag{10}$$

B. *Stress Analysis due to Temperature Gradient*

When this kind of beam is subjected to a uniform temperature change (ΔT), the total strain under a small strain assumption, can be taken as made up of elastic and a thermal part. For a beam under plane stress condition, the only nonzero stress component is σ_x[12]:

$$\sigma_x = E(z)\left[\, \varepsilon_{x0}^T + z.\kappa^T - \alpha(z)\,\Delta T\right] \tag{12}$$

Since it is assumed that only thermal loading is considered here:

$$\sum F_x = 0 \ \ , \ \sum M_x = 0 \tag{13}$$

On the other hand:

$$(N,M) =\int_{-h_1}^{h_2} \sigma_{xx}(1,z)dz \ = 0 \tag{14}$$

Where ε_{x0}^T is the strain at the midplane (z=0) of the FGM layer and κ^T is the laminate curvature due to temperature gradient. After substitution of the values of E(z) and $\alpha(z)$, which depend on the nature of variation V(z) over the thickness, (14) and (12) can be integrated to give,

$$\begin{cases} I_0\varepsilon_{x0}^T + I_1\kappa^T - I_3 = 0 \\ I_1\varepsilon_{x0}^T + I_2\kappa^T - I_4 = 0 \end{cases} \tag{15}$$

By solving the above equations:

$$\varepsilon_{x0}^T = \frac{I_4\ I_1 - I_3\ I_2}{{I_1}^2 - I_0\ I_2} \tag{16-a}$$

$$\kappa^T = \frac{I_3\ I_1 - I_4\ I_0}{{I_1}^2 - I_0\ I_2} \tag{16-b}$$

Where,

$$(I_3, I_4) = \int_{-h_1}^{h_2} E(z)(1,z)\alpha(z)dz \tag{17}$$

III. Numerical Formulation and solution

The most presented exact solutions were developed for symmetrical loadings and boundary conditions. Employing such this solution are complicated and approximately not applicable for complicated geometrical situations. Numerical solutions can overcome to this limitation. In this section, finite element method is carried out to determination the thermo-mechanical stresses in a composite beam which is subjected to both transverse loading and temperature gradient.

The basic assumptions are as follows:

1. Displacement and rotation is very small.
2. Moment of inertia is constant along the element and the effective Young Modulus is varied across the thickness.

Static differential equation is governed on FGM beam under distributed loading, takes the form:

$$\frac{\partial^2}{\partial x^2}\left[E(z)\ I(x)\ \frac{d^2V(x)}{dx^2}\right] - q(x) = 0 \qquad (18)$$

In above equation, I(x), q(x) and V(x) are denoted on moment of Inertia, distributed loading and shear force. Total potential energy, Π, is:

$$\Pi = \hat{U} - \hat{W} \qquad (19)$$

Where $\hat{U}$ is the work done by internal forces or in the other hand, strain energy in a beam. (In here, only the effect of bending moment in internal work is considered):

$$\hat{U} = \frac{1}{2}\int_0^L \breve{E}(z) I(x) \left(\frac{d^2V(x)}{dx^2}\right)^2 dx \qquad (20)$$

and $\hat{W}_{ext}$ is the work done by external loads:

$$\hat{W}_{ext} = \int_0^L q(x)w(x)dx + \sum p_K w_K \qquad (21)$$

By using equations (19), (20) and (21):

$$\Pi = \frac{1}{2}\int_0^L \breve{E}(z) I \left(\frac{d^2V(x)}{dx^2}\right)^2 dx - \int_0^L q(x)w(x)dx - \sum p_k w_K \qquad (22)$$

Where w_K is transverse deflection of beam. p_k is any concentrated loading that may be applied in two ends of the beam element. The FGM is idealized using finite element with two nodes per element. One dimensional Hermitian Cubic polynomial is used for transverse deflection w and its first derivative for slope φ [14]:

$$w(s) = \sum_{j=1}^4 \check{N}_j(s)\, W_j \ , \ \varphi(s) = \sum_{j=1}^4 \check{N}'_j(s)\, W_j \qquad (23)$$

$$\check{N}_1 = 1 - 3\left(\frac{s}{L_e}\right)^2 + 2\left(\frac{s}{L_e}\right)^3 \ , \ \check{N}_2 = \left[\frac{s}{L_e} - 2\left(\frac{s}{L_e}\right)^2 + \left(\frac{s}{L_e}\right)^3\right] L_e$$
$$\check{N}_3 = 3\left(\frac{s}{L_e}\right)^2 - 2\left(\frac{s}{L_e}\right)^3 \ , \ \check{N}_4 = \left[-\left(\frac{s}{L_e}\right)^2 + \left(\frac{s}{L_e}\right)^3\right] L_e \qquad (24)$$

L_e is the length of element. s is the local coordinate. Therefore, the order of interpolation for transverse deflection is one order higher than slope. These shape functions have the required feathers: That is, either the function or its derivative takes the value of unity at one end and both are zero at the other end, due to the boundary values of transverse deflection and slope for two end of beam element [14]:

$$w(0) = w_1 \ , \ w(L_e) = w_2 \ , \ \theta(0) = \theta_1 \ , \ \theta(L_e) = \theta_2 \qquad (25)$$

Using the principal of total potential energy, the first variation of Π (variation of Π with respect to each nodal variation) $\delta\Pi = 0$ leads to different element matrices and load vectors. The element equation takes the final form:

$$\sum_{j=1}^4 U_j \int_0^{L_e} E(z) I\, \check{N}_j''(s)\check{N}_i''(s) ds = \int_0^{L_e} q(s)\check{N}_i(s) ds + p_i \qquad (26)$$

Substituting equations (28) and (25) into equation (26), results in the final form of element equation matrix:

$$\frac{E_e(z)I}{L_e^3}\begin{bmatrix} 12 & 6L_e & -12 & 6L_e \\ 6L_e & 4L_e^2 & -6L_e & 2L_e^2 \\ -12 & -6L_e & 12 & -6L_e \\ 6L_e & 2L_e^2 & -6L_e & 4L_e^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = \qquad (27)$$
$$\frac{qL_e}{2}\begin{Bmatrix} 1 \\ \frac{L_e}{6} \\ 1 \\ \frac{-L_e}{6} \end{Bmatrix} + \begin{Bmatrix} F_1 \\ M_1 \\ F_1 \\ M_2 \end{Bmatrix}$$

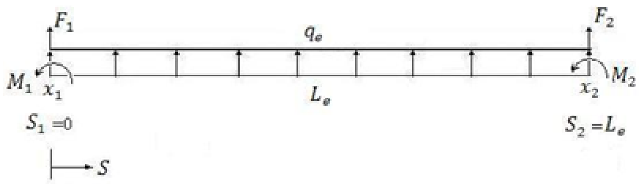


Fig. 2 Free diagram of beam element in local coordinate

In the left side of the above equation, the first sentence is related to uniformly transverse loading and the second sentence is written for the probable concentrated loading that is assumed to be applied on the nodal points, as they were shown in Fig. 2. Based on the formulation discussed, a MATLAB code is developed to consider the accuracy of the numerical analysis.

IV. RESULTS AND CONCLUSION

Regarding the problem undertaken in this study, results were obtained for the linear, quadratic and cubic variations to verify the present analytical and numerical methods for obtaining the thermo-mechanical stresses in a composite beam with an FGM layer of any arbitrary gradation profile. Then for consideration of the accuracy of results, the FEM modelling is carried out using ANSYS. The three layered system of Steel–FGM–Al_2O_3 was assumed. The properties of the metallic (Steel) and ceramic (Al_2O_3) are given in Table 1.

Table 1. Thermo-elastic properties for metallic (Steel) and ceramic (Al_2O_3)phases

material	α ($^\circ c^{-1}$)	υ	E (Gpa)
Al_2O_3	6.9×10^{-6}	.25	390
Steel	14×10^{-6}	.25	210

Using these materials a functionally graded cantilever beam of 0.5 m length subjected to transverse distributed load on upper surface is considered. The topmost material is steel which has a thickness of 0.005 m and bottom layer is alumina of thickness 0.005 m. In between these layers there is a FGM layer of 0.01 m. The beam has unit width.

Two boundary conditions are taken: clamped-free (CF) and both ends are simply supported (SS). First, beam is subjected to a uniform distributed loading (q=100 kN/m) and there is no rise in temperature ($\Delta T = 0$). Next, the effect of thermal loading is studied. In this step, beam is subjected to a temperature gradient ($\Delta T = 100\ ^{o}C$). Effect of temperature rise/fall is considered by augmenting the thermal strain to the mechanical strain.

Fig. 3 and Fig. 4 show the effect of existence and absence of FGM layer. In Fig. 3, we see that in the absence of FGM layer for a beam which is subjected to transverse loading under CF edges, there is discontinuity in stress distribution. Introduction of a small FGM layer smoothens the axial stress about 10 Mpa over .02 m depth which corresponds to an axial stress gradient of 500pa/m.

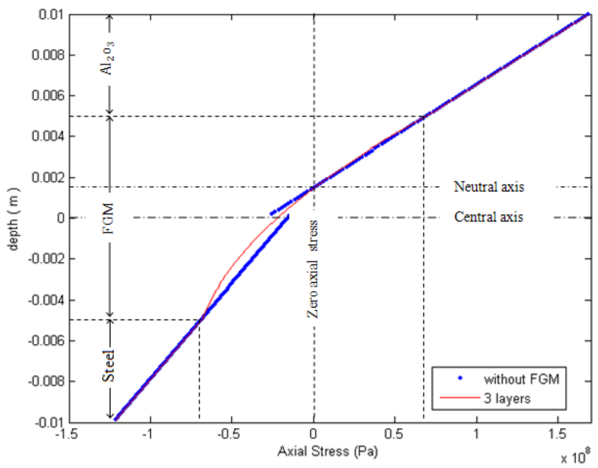


Fig. 3 Depthwise stress distribution for transverse loading in a beam with CF edges for m=1 with and without FG layer

Fig. 4 depicts the depthwise thermal stress distribution, whereas there is no mechanical loading. Introduction of FGM layer has effect in the same way as seen in Fig. 3 smoothens the thermal stress to the tune of about 210 Mpa which corresponds to a thermal stress gradient of 1150 pa/m.

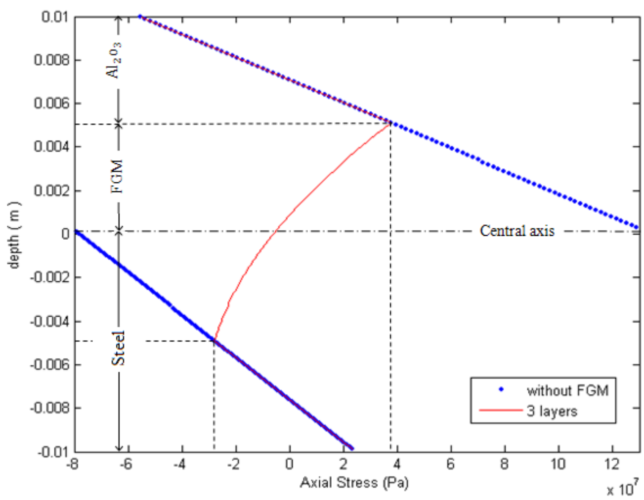


Fig. 4 Depthwise stress distribution for thermal loading in a beam with CF edges for m=1 with and without FG layer

Fig. 5 shows the distribution of the axial stress for Steel-FGM-Al_2O_3 beam with Clamped-Free edges. When a uniform distributed transverse loading is applied.

As shown, the axial stress variation across FGM thickness is not linear, whereas in the metallic and ceramic ones, it is linear. The effect of power law index (m=1, 2, 3) is not considerable. Due to CF boundary condition, applying the transverse loading on the top surface of beam, distributes compressive axial stress in depth of Steel layer and some part of FGM one.

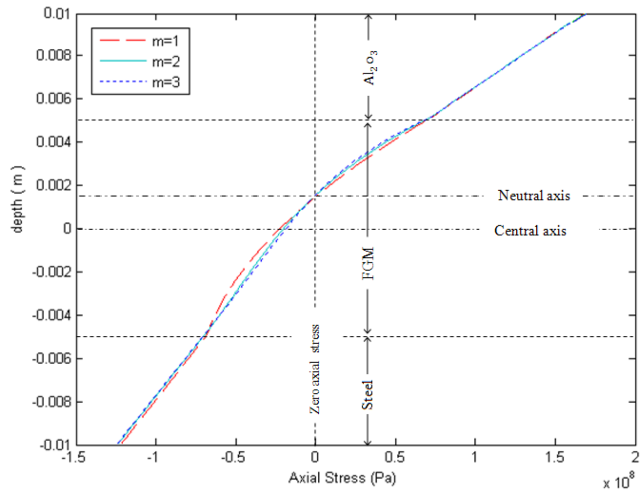


Fig. 5 Depthwise axial stresses distribution for transverse uniform loading in a beam with CF edges at x=.0625 m

Fig. 6 depicts the variation of the shear stress across the thickness of beam. With increasing power law index (m), the tip of shear stress decreases. By the way, it has not considerable effect on the distribution of shear stress.

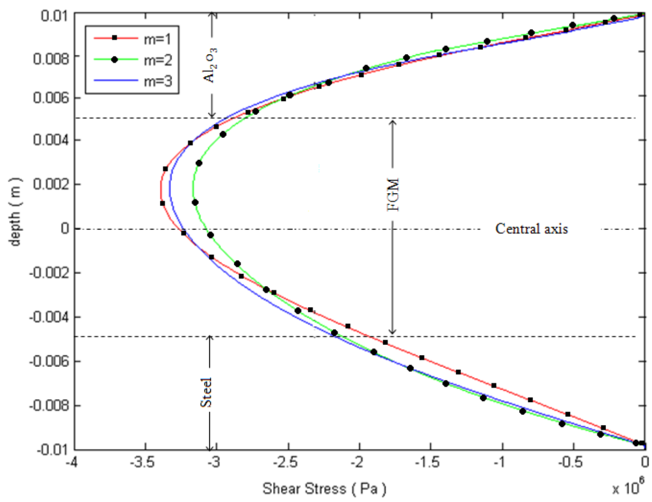


Fig. 6 Depthwise shear stresses distribution for transverse uniform loading in a beam with CF edges at x=.0625 m

Fig. 7 shows the variation of axial stresses for a beam with SS edges. For this kind of boundary condition the axial stresses is less than the one in CF edges. The location of neutral axis is in FGM layer, above the central axis of cross section of beam. In x=0.25, there is no shear stress distribution.

In Fig. 8 and Fig. 9, the variation of axial stress in a Steel–FGM–Al_2O_3 beam, is plotted under transverse uniform distributed loading whereas the beam is exposed to temperature gradient too. As seen, the stress distribution for FGM layer is nonlinear. In a beam with CF edges, the neutral axis is unit(approximately 0.003 m above the central axis of cross section of beam), but for the same beam with SS edges

ISBN:978-988-18210-1-0

there are three location which the axial stress gets zero across the thickness of beam. By introducing the different FGM layer based on power law index (m), the profile of stress distribution is changed somehow.

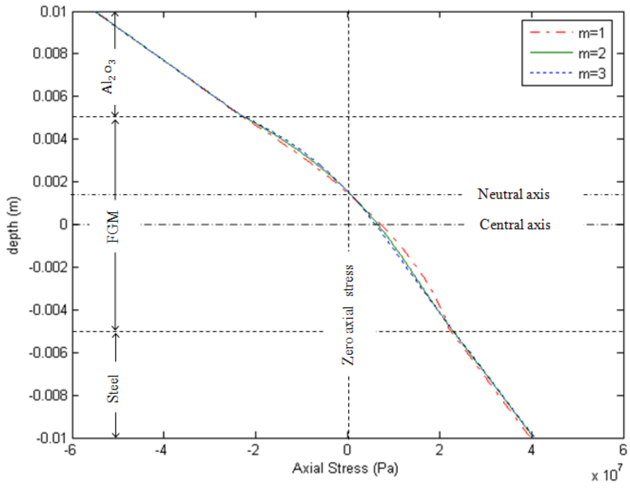


Fig. 7 Depthwise axial stresses distribution for transverse uniform loading in a beam with SS edges at x=.25 m

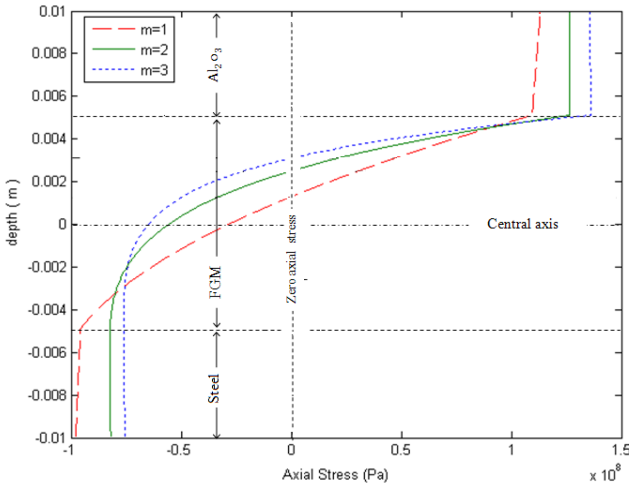


Fig. 8 Depthwise axial stresses distribution for thermal and transverse uniform loading in a beam with CF edges at x=.0625 m

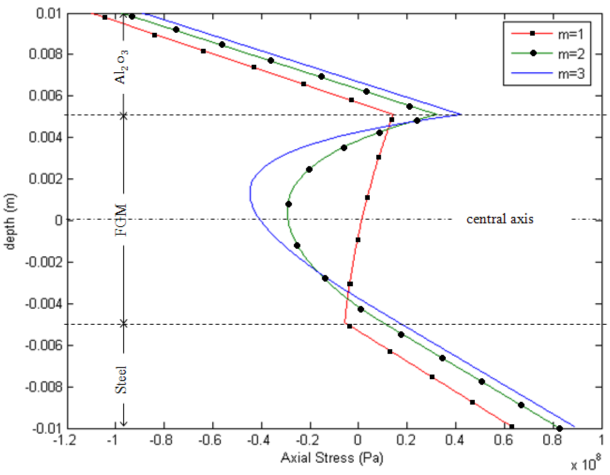


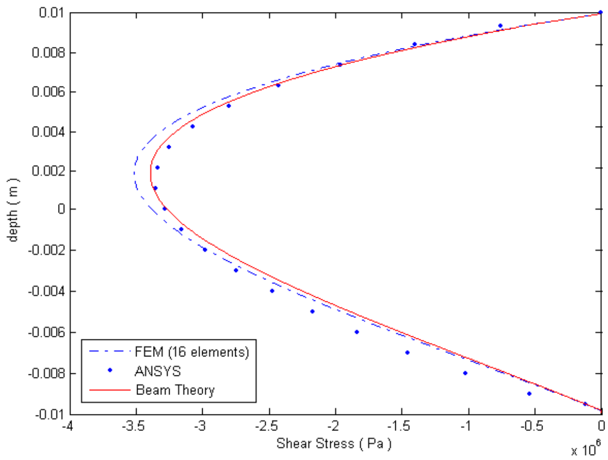
Fig. 9 Depthwise axial stresses distribution for transverse uniform loading in a beam with SS edges at x=.25 m

For comparison the results of analytical method with the other methods, Fig. 10(a) and 10(b) were plotted. In numerical method using FEM based on presented formulation in section III, 16 beam elements were used. Table 2 gives the convergence study for FEM formulation that was presented here. The beam with SS edges under uniform distributed transverse loading is undertaken. It is observed from table 2 that convergence is obtained for 16 elements.

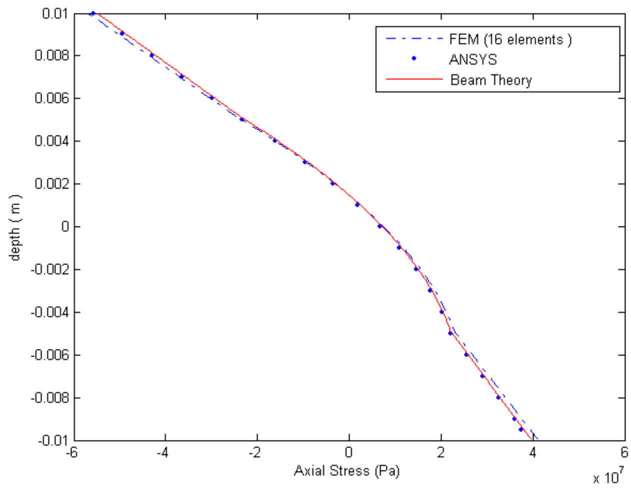
Table 2. Convergence study for FGM beam

σ_x (Mpa) / No. of Elements	z= -0.01 (m)	z= 0.01 (m)
2	13	-19
4	24	-33
6	32	-44
12	36	-49
16	41.1	-55.2
32	41.5	-55.5

The obtained results were compared with FEM model using ANSYS that gave good agreement between three models.



(a)



(b)

Fig. 10 Comparison between present analytical and numerical model with ANSYS FEM calculation for stress distribution across the thickness under transverse uniform loading in FGM beam for m=1 (a) shear stress (b) axial stress

It can be seen from these two figures that there is no practically considerable difference, between stress profiles obtained analytically and from FEM model and ANSYS results. Therefore, from the results the following may be concluded:

- Introduction of a FGM layer, smoothen the stress distribution and solves the problem of discontinuity in stress distribution in border between the two layers with apparent difference of properties.
- Different values of power index (m), has considerable effect on profile of thermo-mechanical stress distribution.
- Stress distribution in a FGM beam across the inhomogeneous layer is nonlinear.
- Studies from plots of thermo-mechanical stress reveals that, for FGM beams the neutral plane location is influenced by the power law index (m).

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