Proceedings of the World Congress on Engineering 2009 Vol II
WCE 2009, July 1 - 3, 2009, London, U.K.

Nonlinear Resonances of a Semi-Infinite Cable on a Unilateral Elastic Substrate

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Abstract—**We study the occurrence of backbones in the nonlinear oscillations of semi-infinite cables resting on an elastic substrate reacting in compression only, and subjected to a constant distributed load and to a small harmonic displacement applied to the finite boundary. The moving boundary problem, which arises because the position of the points where the system detaches from the substrate is not known in advance, is solved by the multiple scales expansion technique.**

Keywords: J-lay problem, Wave Equation, Moving Boundary Problems, Multiple time scales expansions, Backbones

1 Introduction

In this work, we continue our investigations [8, 9, 10] of a moving boundary problem for the wave equation, which arises, e.g., in the modeling of the J-lay technique for marine pipelines or cables, or in marine moorings [6]. The mechanical system under consideration consists of a semi-infinite cable resting on a (unilateral) elastic substrate reacting in compression only, subject to a constant distributed load and to a harmonic displacement applied to the finite boundary, which induces nonlinear forced oscillations. With regard to the J-lay problem, this model describes only the laid part and the first part of the suspended span, which are divided by the so-called Touch-Down Point (TDP) (Fig. 1). Since the position of the TDP is an additional unknown, which depends upon the solution itself, the resulting dynamics is governed by a nonlinear moving boundary problem [1].

The nonlinear dynamics of beams and cables resting on unilateral elastic foundations have been investigated by various authors in the past (see, e.g. [7, 12]), and we quote [5] for a general overview of the interactions between structures and foundations.

Although in the statical regime exact solutions can easily be found, even for large displacements (see, e.g. [10, 11]), in the dynamical regime an exact, analytical solution of the nonlinear model equations is unattainable even for small displacements; therefore, we resort to an approximate solution by using asymptotic analysis [3, 4]. The first-order solution was presented in [8], and the second-order solution in [9]. In those papers, however, typical nonlinear effects such as the bending of the resonant curves ("backbones") were not present. In order to detect these effects, a different scaling must be used for the external excitation [4]. In this paper, we analyze the behaviour of the system near a primary resonance, which leads to the occurrence of backbones.

In our perturbation expansion, the zero-order terms correspond to the static solution obtained in the absence of a time-dependent excitation applied at the boundary. The first-order terms give the primary resonances of the system and their relation to the wave propagation toward infinity [2]. In particular, two different regimes, below and above a certain critical (cutoff) excitation frequency, with very different wave properties [2], were identified in [8] and in [9]. The second-order terms give information on the nonlinear coupling of the linear modes, while the third-order terms contain the information on the bending of the resonant curves ("backbones").

This paper is organized as follows: in section 2 we introduce the mathematical model, in section 3 our perturbative solution is presented and in section 4 we state our conclusions.

2 The mathematical model

The profile of the cable is represented by the function $u(x,t)$, where $0 \leq x < +\infty$ is the space variable and $t \geq 0$ the time. A constant downward load acts on the whole cable, while a restoring elastic force is present only on the portion of the spatial domain where the solution $u(x,t)$ is negative. This describes the action of the elastic substrate (e.g., a Winkler soil) that acts in compression only, and represents the unique source of nonlinearity in the model (Fig. 1).

We assume that there exists only one point of the domain, $x = c(t)$, called Touch-Down-Point (TDP), where the profile function vanishes, namely $u(c(t),t) = 0$; in particular, we suppose that $u(x,t) > 0$ for $0 \leq x < c(t)$ and $u(x,t) < 0$ for $c(t) < x < \infty$, which is justified for mo-

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ISBN:978-988-18210-1-0

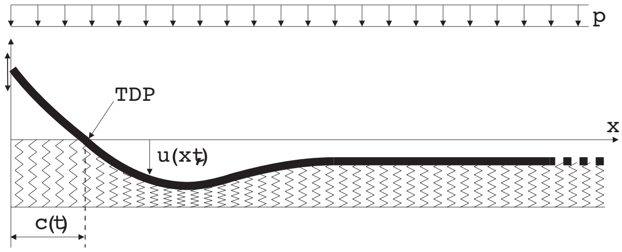

Figure 1: The mechanical system

tions in the vicinity of the static solution, $u(x,t) \equiv u_S(x)$, which exhibits only one TDP at $x = c(t) \equiv c_0$ [8].

In this work, we shall look for time-dependent solutions that correspond to small oscillations about the static solution, induced by a harmonic displacement applied at the $x = 0$ boundary. The TDP $x = c(t)$ then exhibits an oscillating behaviour as well, which is described an amplification factor [8, 9]. The dimensionless governing equations are given by [9]

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + 1 = 0, \qquad 0 < x < c(t), \tag{1}$$

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u + 1 = 0, \qquad x > c(t). \tag{2}$$

where the constant $+1$ term in equation (2) represents the external constant load applied to the system. The boundary condition at $x = 0$ corresponds to small harmonic excitation about a constant average value U_0. In [8] and [9] we chose $u(0,t) = U_0\,[1 + \varepsilon \sin(\omega t)]$, where ε is a smallness parameter which represents the amplitude of the external excitation and ω is the imposed external frequency. In the present work, a different scaling of the external amplitude w.r. to ε will be chosen (see secion 3). At the infinite end of the system, we require that $u(x,t)$ be bounded; moreover, we assume that, whenever the equations support traveling-wave solutions, terms corresponding to waves returning from $+\infty$ are not present, so that only "outgoing" waves (traveling to the right) are admitted [2]. Finally, we impose the continuity of the solution and of its first spatial derivative (continuity condition) and the vanishing of the solution (vanishing condition) at $x = c$, with $c = c_0$ for static solutions and $c = c(t)$ for time-dependent solutions. The static solution, $u_S(x)$, is obtained by switching off the time derivatives in equations (1)-(2) and setting $\varepsilon = 0$ in the boundary condition. We will discuss it in the next section; here, we only anticipate the expression for the static TDP c_0, which is an important parameter in our analysis and is given by [8]

$$c_0 = \sqrt{1 + 2U_0} - 1. \tag{3}$$

This expression shows that there is a one-to-one correspondence between c_0 and U_0; for this reason, we will use either c_0 or U_0, whichever is more convenient, as a governing parameter in our analysis. The other one is the excitation frequency ω.

3 Perturbative solution

3.1 Expansion

We will recover the bending of the resonant curves by using multiple time scales analysis on the system (1)-(2) with the boundary condition

$$u(0,t) = U_0\,(1 + \varepsilon^3 \sin\omega\, t). \tag{4}$$

To this aim, we introduce two additional time scales, $\tau = \varepsilon t$ (intermediate time scale) and $T = \varepsilon^2 t$ (long time scale) and expand the unknown functions according to

$$\begin{aligned} u(x,t) &= u_0(x) + \varepsilon\, u_1(x,t,\tau,T) && (5)\\ &\quad + \varepsilon^2\, u_2(x,t,\tau,T) + \varepsilon^3\, u_3(x,t,\tau,T) + \mathcal{O}(\varepsilon^4) \\ v(x,t) &= v_0(x) + \varepsilon\, v_1(x,t,\tau,T) && (6)\\ &\quad + \varepsilon^2\, v_2(x,t,\tau,T) + \varepsilon^3\, v_3(x,t,\tau,T) + \mathcal{O}(\varepsilon^4) \\ c(t) &= c_0 + \varepsilon\, c_1(t,\tau,T) && (7)\\ &\quad + \varepsilon^2\, c_2(t,\tau,T) + \varepsilon^3\, c_3(t,\tau,T) + \mathcal{O}(\varepsilon^4), \end{aligned}$$

where we have indicated with $v(x,t)$ the solution $u(x,t)$ for $x > c(t)$, keeping the symbol u for $0 < x < c(t)$.The continuity and vanishing conditions at the TDP $x = c(t)$ are

$$u(c(t),t) = v(c(t),t) = 0 \tag{8}$$

$$\frac{\partial u}{\partial x}(c(t),t) = \frac{\partial v}{\partial x}(c(t),t). \tag{9}$$

With the expansion given by (5)-(7) we have for the time derivatives:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \varepsilon \frac{\partial u_1}{\partial t} + \varepsilon^2 \left(\frac{\partial u_1}{\partial \tau} + \frac{\partial u_2}{\partial t} \right) \\ &+ \varepsilon^3 \left(\frac{\partial u_1}{\partial T} + \frac{\partial u_2}{\partial \tau} + \frac{\partial u_3}{\partial t} \right) + \mathcal{O}(\varepsilon^4) && (10)\\ \frac{\partial^2 u}{\partial t^2} &= \varepsilon \frac{\partial^2 u_1}{\partial t^2} + \varepsilon^2 \left(2\,\frac{\partial^2 u_1}{\partial t \partial \tau} + \frac{\partial^2 u_2}{\partial t^2} \right) \\ &+ \varepsilon^3 \left(\frac{\partial^2 u_1}{\partial \tau^2} + 2\,\frac{\partial^2 u_1}{\partial t \partial T} + 2\,\frac{\partial^2 u_2}{\partial t \partial \tau} + \frac{\partial^2 u_3}{\partial t^3} \right) \\ &+ \mathcal{O}(\varepsilon^4) && (11) \end{aligned}$$

Similar asymptotic expansions can be obtained for the continuity conditions (8)-(9) by expanding the functions u and v and their derivatives at the TDP in a Taylor series about the static value c_0. For the sake of conciseness, we do not report these expressione here. After substituting into (1)-(2) and into the continuity conditions we obtain the usual hierarchy of equations to all orders in ε.

3.2 Zero-order solution

To $\mathcal{O}(\varepsilon^0)$ the equations of the hierarchy are

$$u_0''(x) = 1 \tag{12}$$

$$v_0''(x) - v_0(x) = 1, \tag{13}$$

ISBN:978-988-18210-1-0

with the boundary condition $u_0(0) = U_0$ and the continuity conditions

$$u_0(c_0) = v_0(c_0) = 0 \quad (14)$$
$$u_0'(c_0) = v_0'(c_0). \quad (15)$$

This leads to the stationary solution, which was already outlined in [8] and [9], and is given by

$$u_0(x) = (x - c_0)\left(x - \frac{U_0}{c_0}\right), \quad v_0(x) = e^{c_0 - x} - 1$$

with c_0 given by (3).

3.3 First-order solution

To $\mathcal{O}(\varepsilon)$ we have

$$\frac{\partial^2 u_1}{\partial t^2} - \frac{\partial^2 u_1}{\partial x^2} = 0 \quad (16)$$
$$\frac{\partial^2 v_1}{\partial t^2} - \frac{\partial^2 v_1}{\partial x^2} + v_1 = 0, \quad (17)$$

with the boundary condition $u_1(0, t, \tau, T) = 0$. The continuity conditions, thanks to the zero-order equations (12) and (13) and continuity conditions (14) and (15), are given by

$$u_1(c_0, t) = v_1(c_0, t) \quad (18)$$
$$\frac{\partial u_1}{\partial x}(c_0, t) = \frac{\partial v_1}{\partial x}(c_0, t) \quad (19)$$
$$c_1 = u_1(c_0, t) \quad (20)$$

Since we are interested in steady-state oscillations, [8, 9], we seek solutions of the form

$$\begin{aligned} u_1(x, t, \tau, T) &= f_{10}(x, \tau, T) + f_{11}(x, \tau, T)\cos\Omega t \\ &\quad + g_{11}(x, \tau, T)\sin\Omega t \\ v_1(x, t, \tau, T) &= h_{10}(x, \tau, T) + h_{11}(x, \tau, T)\cos\Omega t \\ &\quad + k_{11}(x, \tau, T)\sin\Omega t, \end{aligned}$$

with boundary conditions $f_{10}(0, \tau, T) = f_{11}(0, \tau, T) = g_{11}(0, \tau, T) = 0$ and $h_{10}, h_{11}, k_{11} \to 0$ as $x \to +\infty$. After substituting these expansions into (16) and (17) and after equating the coefficients of the trigonometric functions there introduced, we obtain a set of equations which can be easily solved (here, primes indicate derivatives w.r. to x and $\nu = \sqrt{1 - \Omega^2}$):

$$\begin{aligned} f_{10} &= A_{10}(\tau, T)\, x & h_{10} &= C_{10}(\tau, T)\, e^{-x} \\ f_{11} &= A_{11}(\tau, T)\, \sin\Omega x & h_{11} &= C_{11}(\tau, T)\, e^{-\nu x} \\ g_{11} &= B_{11}(\tau, T)\, \sin\Omega x & k_{11} &= D_{11}(\tau, T)\, e^{-\nu x}. \end{aligned}$$

We have chosen to work with the case $\Omega < 1$ (corresponding to the 1-subcritical regime introduced in [9]), so that ν is real and positive. The matching conditions (18) and (19) then give

$$\begin{aligned} A_{10}\, c_0 &= C_{10}\, e^{-c_0} \\ A_{10} &= -C_{10}\, e^{-c_0} \end{aligned}$$

which are solved by $A_{10} = C_{10} = 0$, and

$$\begin{aligned} A_{11}\, \sin\Omega c_0 &= C_{11} e^{-\nu c_0} \\ \Omega\, A_{11} \cos\Omega c_0 &= -\nu C_{11} e^{-\nu c_0} \end{aligned}$$

which have nonvanishing solutions only if

$$\delta(\Omega) \equiv \nu \sin\Omega c_0 + \Omega\cos\Omega c_0 = 0. \quad (21)$$

Equation (21) is the dispersion relation which gives the first-order resonances and which was already obtained in [8] and [9]. In this case we have $C_{11} = A_{11}\, e^{\nu c_0} \sin\Omega c_0$ and it is easy to see that we also have $D_{11} = B_{11}\, e^{\nu c_0} \sin\Omega c_0$. By collecting all these results we obtain for the first-order solution:

$$\begin{aligned} u_1(x, t, \tau, T) &= s_1(t)\, \sin\Omega x \\ v_1(x, t, \tau, T) &= s_1(t)\, e^{\nu(c_0 - x)} \sin\Omega c_0 \\ c_1(t, \tau, T) &= s_1(t)\, \sin\Omega c_0 \end{aligned}$$

where $s_1(t) = A_{11}(\tau, T)\cos\Omega t + B_{11}(\tau, T)\sin\Omega t$, with A_{11} and B_{11} as yet undetermined.

3.4 Second-order solution

To $\mathcal{O}(\varepsilon^2)$ we have

$$\frac{\partial^2 u_2}{\partial t^2} - \frac{\partial^2 u_2}{\partial x^2} = -2\,\frac{\partial^2 u_1}{\partial t \partial \tau}$$
$$\frac{\partial^2 v_2}{\partial t^2} - \frac{\partial^2 v_2}{\partial x^2} + v_2 = -2\,\frac{\partial^2 v_1}{\partial t \partial \tau},$$

with the boundary condition $u_2(0, t, \tau, T) = 0$. The continuity conditions, thanks to the zero- and first-order equations and continuity conditions, are given by

$$u_2(c_0, t) = v_2(c_0, t) \quad (22)$$
$$\begin{aligned} &\frac{\partial u_2}{\partial x}(c_0, t) + \frac{c_1^2}{2} + c_1\, \frac{\partial^2 u_1}{\partial x^2}(c_0, t) \\ &\quad = \frac{\partial v_2}{\partial x}(c_0, t) + c_1\, \frac{\partial^2 v_1}{\partial x^2}(c_0, t) \end{aligned} \quad (23)$$
$$c_2 = u_2(c_0, t) + \frac{c_1^2}{2} + c_1\, \frac{\partial u_1}{\partial x}(c_0, t). \quad (24)$$

We again seek for solutions of the form

$$\begin{aligned} u_2(x, t, \tau, T) &= f_{20}(x, \tau, T) \\ &\quad + f_{21}(x, \tau, T)\cos\Omega t + g_{21}(x, \tau, T)\sin\Omega t \\ &\quad + f_{22}(x, \tau, T)\cos 2\,\Omega t + g_{22}(x, \tau, T)\sin 2\,\Omega t \\ v_2(x, t, \tau, T) &= h_{20}(x, \tau, T) \\ &\quad + h_{21}(x, \tau, T)\cos\Omega t + k_{21}(x, \tau, T)\sin\Omega t \\ &\quad + h_{22}(x, \tau, T)\cos 2\,\Omega t + k_{22}(x, \tau, T)\sin 2\,\Omega t \end{aligned}$$

with boundary conditions $f_{20}(0, \tau, T) = f_{21}(0, \tau, T) = g_{21}(0, \tau, T) = f_{22}(0, \tau, T) = g_{22}(0, \tau, T) = 0$ and $h_{20}, h_{21}, k_{21}, h_{22}, k_{22} \to 0$ as $x \to +\infty$. By proceeding in the same way as for the first-order solution, we find

$$\begin{aligned} f_{20} &= A_{20}(\tau, T)\, x & h_{20} &= C_{20}(\tau, T)\, e^{-x} \\ f_{22} &= A_{22}(\tau, T)\, \sin 2\,\Omega x & h_{22} &= C_{22}(\tau, T)\, e^{-\mu x} \\ g_{22} &= B_{22}(\tau, T)\, \sin 2\,\Omega x & k_{22} &= D_{22}(\tau, T)\, e^{-\mu x}, \end{aligned}$$

where $\mu = \sqrt{1 - 4\,\Omega^2}$, and

$$\begin{aligned} f_{21} &= A_{21}(\tau, T)\,\sin\Omega x + A_p\,x\,\cos\Omega x \\ g_{21} &= B_{21}(\tau, T)\,\sin\Omega x + B_p\,x\,\cos\Omega x \\ h_{21} &= C_{21}(\tau, T)\,e^{-\nu x} + C_p\,x\,e^{-\nu x} \\ k_{21} &= D_{21}(\tau, T)\,e^{-\nu x} + D_p\,x\,e^{-\nu x} \end{aligned}$$

with

$$\begin{aligned} A_p &= -2\,(\partial B_{11}/\partial\tau) & C_p &= (\Omega/\nu)\,A_p\,e^{\nu c_0}\,\sin\Omega c_0 \\ B_p &= 2\,(\partial A_{11}/\partial\tau) & D_p &= (\Omega/\nu)\,B_p\,e^{\nu c_0}\,\sin\Omega c_0. \end{aligned}$$

Here, we have assumed $\Omega < 1/2$, so that μ is real and positive (this corresponds to the 2-subcritical regime introduced in [9]). After equating constant terms in the matching conditions (22) and (23) we obtain, after few steps,

$$A_{20} = \frac{A_{11}^2 + B_{11}^2}{4\,(1 + c_0)}\,\sin^2\Omega c_0 \tag{25}$$

$$C_{20} = \frac{A_{11}^2 + B_{11}^2}{4\,(1 + c_0)}\,c_0\,e^{c_0}\,\sin^2\Omega c_0. \tag{26}$$

After equating the $\cos\Omega t$ and $\sin\Omega t$ terms we have the non homogeneous system

$$\begin{aligned} &A_{21}\sin\Omega c_0 - C_{21}e^{-\nu c_0} \qquad (27) \\ &\quad = -2\,\frac{c_0}{\nu}\,(\Omega\sin\omega c_0 + \nu\cos\Omega c_0)\,\frac{\partial B_{11}}{\partial\tau} \\ &A_{21}\Omega\cos\Omega c_0 + C_{21}\,\nu\,e^{-\nu c_0} \qquad (28) \\ &\quad = -\frac{2}{\nu}\,(\Omega\sin\omega c_0 + \nu\cos\Omega c_0)\,\frac{\partial B_{11}}{\partial\tau} \end{aligned}$$

for the unknowns A_{21} and C_{21}. The homogeneous system associated with (27) and (28) admits non trivial solutions (thanks to the first-order dispersion relation (21)) and therefore the right hand side of the non homogeneous system must be orthogonal to the solution of the associated adjoint homogeneous system. It can be easily seen that this implies

$$\frac{\partial B_{11}}{\partial\tau} = 0 \qquad \frac{\partial A_{11}}{\partial\tau} = 0$$

which shows also that the coefficients A_{20} and C_{20} depend only upon T and that $A_p = B_p = C_p = D_p = 0$. As is customary in the multiple scale technique [3, 4], we neglect the homogeneous solutions at all orders higher than the first. This gives $A_{21} = C_{21} = B_{21} = D_{21} = 0$, from which it follows that and therefore $f_{21} = g_{21} = h_{21} = k_{21} = 0$.

Finally, by equating the $\cos 2\,\Omega t$ and $\sin 2\,\Omega t$ terms in the matching conditions (22) and (23) we obtain:

$$A_{22} = \frac{A_{11}^2 - B_{11}^2}{4\,\delta(2\,\Omega)}\,\sin^2\Omega c_0 \tag{29}$$

$$C_{22} = \frac{A_{11}^2 - B_{11}^2}{4\,\delta(2\,\Omega)}\,e^{\mu c_0}\sin 2\,\Omega c_0\,\sin^2\Omega c_0 \tag{30}$$

$$B_{22} = \frac{A_{11}\,B_{11}}{2\,\delta(2\,\Omega)}\,\sin^2\Omega c_0 \tag{31}$$

$$D_{22} = \frac{A_{11}\,B_{11}}{2\,\delta(2\,\Omega)}\,e^{\mu c_0}\sin 2\,\Omega c_0\,\sin^2\Omega c_0 \tag{32}$$

which shows that also the coefficients A_{22}, C_{22}, B_{22} and D_{22} depend only upon T. The presence of the function $\delta(2\,\Omega)$ at the denominators is consistent with the occurrence of second order superharmonic resonances, as was found in [9]. The second-order solution can then be written as

$$u_2(x, t, \tau, T) = A_{20}(T)\,x + s_2(t)\,\sin 2\,\Omega x \tag{33}$$

$$\begin{aligned} v_2(x, t, \tau, T) &= C_{20}(T)\,e^{-x} \\ &\quad + s_2(t)\,e^{\mu(c_0 - x)}\,\sin 2\Omega c_0 \end{aligned} \tag{34}$$

$$\begin{aligned} c_2(t, \tau, T) &= u_2(c_0, t, \tau, T) + \frac{c_1^2}{2} \\ &\quad + c_1\,\frac{\partial u_1}{\partial x}(c_0, t, \tau, T) \end{aligned} \tag{35}$$

with $s_2(t) = A_{22}(T)\cos 2\,\Omega t + B_{22}(T)\sin 2\,\Omega t$ and A_{20}, C_{20}, A_{22}, B_{22}, C_{22} and D_{22} given by (25), (26), (29), (31), (30) and (32).

3.5 Third-order solution

To $\mathcal{O}(\varepsilon^3)$ we have

$$\begin{aligned} &\frac{\partial^2 u_3}{\partial t^2} - \frac{\partial^2 u_3}{\partial x^2} \\ &\qquad = -\left(\frac{\partial^2 u_1}{\partial\tau^2} + 2\,\frac{\partial^2 u_1}{\partial t\partial T}\right) - 2\,\frac{\partial^2 u_2}{\partial t\partial\tau} \\ &\frac{\partial^2 v_3}{\partial t^2} - \frac{\partial^2 v_3}{\partial x^2} + v_3 \\ &\qquad = -\left(\frac{\partial^2 v_1}{\partial\tau^2} + 2\,\frac{\partial^2 v_1}{\partial t\partial T}\right) - 2\,\frac{\partial^2 v_2}{\partial t\partial\tau}, \end{aligned}$$

with the boundary condition $u_3(0, t, \tau, T) = U_0\,\sin\omega t$. The continuity conditions, due to the zero-, first- and second-order equations and continuity conditions, become

$$\begin{aligned} &u_3(c_0, t) + c_1\,\frac{\partial u_2}{\partial x}(c_0, t) + \frac{c_1^2}{2}\,\frac{\partial^2 u_1}{\partial x^2}(c_0, t) + \frac{c_1^3}{6} \\ &\quad = v_3(c_0, t) + c_1\,\frac{\partial v_2}{\partial x} + \frac{c_1^2}{2}\,\frac{\partial^2 v_1}{\partial x^2}(c_0, t) \end{aligned} \tag{36}$$

$$\begin{aligned} &\frac{\partial u_3}{\partial x}(c_0, t) + c_1\,\frac{\partial^2 u_2}{\partial x^2}(c_0, t) + \frac{c_1^2}{2}\,\frac{\partial^3 u_1}{\partial x^3}(c_0, t) \\ &\quad + c_2\,\frac{\partial^2 u_1}{\partial x^2}(c_0, t) + \frac{c_1^3}{6} + c_1\,c_2 \\ &\quad = \frac{\partial v_3}{\partial x}(c_0, t) + c_1\,\frac{\partial^2 v_2}{\partial x^2}(c_0, t) \\ &\quad + \frac{c_1^2}{2}\,\frac{\partial^3 v_1}{\partial x^3}(c_0, t) + c_2\,\frac{\partial^2 v_1}{\partial x^2}(c_0, t) \end{aligned} \tag{37}$$

$$\begin{aligned} c_3 &= u_3(c_0, t) + c_1\,\frac{\partial u_2}{\partial x}(c_0, t) \\ &\quad + \frac{c_1^2}{2}\,\frac{\partial^2 u_1}{\partial x^2}(c_0, t) + c_2\,\frac{\partial u_1}{\partial x}(c_0, t). \end{aligned} \tag{38}$$

Since we study oscillations at a frequency near a primary resonance, say Ω, we put $\omega = \Omega + \varepsilon^2 \sigma$, where σ is a detuning parameter [4]. We then have for the boundary condition

$$u_3(0,t,\tau,T) = U_0 \sin(\Omega + \varepsilon^2 \sigma)t = U_0 \sin(\Omega t + \sigma T). \quad (39)$$

Again, we seek for solutions of the form

$$\begin{aligned} u_3(x,t,\tau,T) &= f_{30}(x,\tau,T) \\ &+ f_{31}(x,\tau,T)\cos\Omega t + g_{31}(x,\tau,T)\sin\Omega t \\ &+ f_{32}(x,\tau,T)\cos 2\Omega t + g_{32}(x,\tau,T)\sin 2\Omega t \\ &+ f_{33}(x,\tau,T)\cos 3\Omega t + g_{33}(x,\tau,T)\sin 3\Omega t \\ v_3(x,t,\tau,T) &= h_{30}(x,\tau,T) \\ &+ h_{31}(x,\tau,T)\cos\Omega t + k_{31}(x,\tau,T)\sin\Omega t \\ &+ h_{32}(x,\tau,T)\cos 2\Omega t + k_{32}(x,\tau,T)\sin 2\Omega t \\ &+ h_{33}(x,\tau,T)\cos 3\Omega t + k_{33}(x,\tau,T)\sin 3\Omega t \end{aligned}$$

with boundary conditions $f_{30}(0,\tau,T) = f_{32}(0,\tau,T) = g_{32}(0,\tau,T) = f_{33}(0,\tau,T) = g_{33}(0,\tau,T) = 0$, $f_{31}(0,\tau,T) = U_0 \sin\sigma T$, $g_{31}(0,\tau,T) = U_0 \cos\sigma T$ and $h_{30}, h_{31}, k_{31}, h_{32}, k_{32}, h_{33}, k_{33} \to 0$ as $x \to +\infty$. By proceeding as before, we have

$$\begin{aligned} f_{30} &= A_{30}(\tau,T)\, x \\ h_{30} &= B_{30}(\tau,T)\, e^{-x} \\ f_{31} &= \left[A_{31}(\tau,T) + \frac{B'_{11}(T)}{2\Omega} \right] \sin\Omega x \\ &\quad + (U_0 \sin\sigma T - x\, B'_{11}(T)) \cos\Omega x \\ h_{31} &= C_{31}(\tau,T)\, e^{-\nu x} \\ &\quad - \frac{1+2\nu x}{2\nu^2}\, \Omega \sin\Omega c_0\, B'_{11}(T) e^{\nu(c_0 - x)} \\ g_{31} &= \left[B_{31}(\tau,T) - \frac{A'_{11}(T)}{2\Omega} \right] \sin\Omega x \\ &\quad + (U_0 \cos\sigma T + x\, A'_{11}(T)) \cos\Omega x \\ k_{31} &= D_{31}(\tau,T)\, e^{-\nu x} \\ &\quad + \frac{1+2\nu x}{2\nu^2}\, \Omega \sin\Omega c_0\, A'_{11}(T) e^{\nu(c_0 - x)}. \end{aligned}$$

The remaining functions are not needed for our analysis and we omit them. The coefficients A_{30}, B_{30}, A_{31}, B_{31}, C_{31} and D_{31} are to be determined by equating the zero- and first-order Fourier components in the matching conditions (36) and (37). The matching equations are rather long and we made use of a symbolic manipulation program. The matching of the zero component gives $f_{30} = h_{30} = 0$. The four equations which express the matching of the first component (the $\cos\Omega t$ and $\sin\Omega t$ terms) can be cast in the form of two linear nonhomogeneous systems for the unknowns A_{31}, C_{31}, B_{31} and D_{31}:

$$\alpha_{11} A_{31} + \beta_{11} C_{31} = \gamma_{11} \quad (40)$$
$$\alpha_{12} A_{31} + \beta_{12} C_{31} = \gamma_{12} \quad (41)$$

and

$$\alpha_{21} B_{31} + \beta_{21} D_{31} = \gamma_{21} \quad (42)$$
$$\alpha_{22} B_{31} + \beta_{22} D_{31} = \gamma_{22} \quad (43)$$

where the α's and β's are coefficients which depend only upon c_0 and Ω. It can be shown that

$$\begin{aligned} \alpha_{21}\beta_{22} - \beta_{21}\alpha_{22} &= \frac{e^{-\nu c_0}}{4}\,\delta(\Omega) = 0 \\ \alpha_{11}\beta_{12} - \beta_{11}\alpha_{12} &= \frac{e^{-\nu c_0}}{4}\,\delta(\Omega) = 0, \end{aligned}$$

where the last equalities follow from (21). Therefore, the associated homogeneous systems admit nontrivial solutions and we must require that the nonhomogeneous terms in (40)-(43) be orthogonal to the solution of the adjoint homogeneous promlem. This gives the consistency conditions

$$\gamma_{11} + \nu\,\gamma_{12} = 0 \quad (44)$$
$$\gamma_{21} + \nu\,\gamma_{22} = 0 \quad (45)$$

After some algebraic steps, which we carried out with a symbolic manipulation program, (44) and (45) can be cast in the form of a system of coupled ordinary differential equations for A_{11} and B_{11} as functions of T:

$$B'_{11}(T) = \Delta \sin\sigma T - \Lambda\,(A_{11}^3 + A_{11} B_{11}^2) \quad (46)$$
$$A'_{11}(T) = -\Delta \cos\sigma T + \Lambda\,(A_{11}^2 B_{11} + B_{11}^3) \quad (47)$$

where Δ and Λ are constants which depend upon c_0 and the chosen primary resonant frequency Ω. These two equations can be solved by transforming into polar variables by setting

$$\begin{aligned} A_{11}(T) &= \Gamma(T) \cos\Theta(T) \\ B_{11}(T) &= \Gamma(T) \sin\Theta(T). \end{aligned}$$

Substituting into (46) and (47) we obtain after few steps

$$\begin{aligned} &-\Delta \sin(\sigma T + \Theta(T)) + \Lambda\Gamma^3(T) + \Gamma(T)\,\Theta'(T) = 0 \\ &\Delta \cos(\sigma T + \Theta(T)) + \Gamma'(T) = 0. \end{aligned}$$

We introduce $\chi(T) = \sigma T + \Theta(T)$ and look for solutions which correspond to steady-state oscillations, namely $\Gamma = \Gamma_0 = const.$ and $\chi = \chi_0 = const.$ We then obtain the system

$$\begin{aligned} &\Delta \cos\chi_0 = 0 \\ &-\Delta \sin\chi_0 - \sigma\Gamma_0 + \Lambda\Gamma_0^3 = 0 \end{aligned}$$

This gives two solutions, $\chi_0 = \pm\pi/2$, for which $\sin\chi_0 = \pm 1$. In turn, this gives two values for σ as function of Γ_0, namely

$$\sigma_1 = \frac{-\Delta + \Lambda\Gamma_0^3}{\Gamma_0} \quad (48)$$
$$\sigma_2 = \frac{\Delta + \Lambda\Gamma_0^3}{\Gamma_0} \quad (49)$$

Equations (48) and (49) are the desired expressions for the bending of the resonance curves ("backbone") near primary resonances in the case $\Omega < 1/2$.

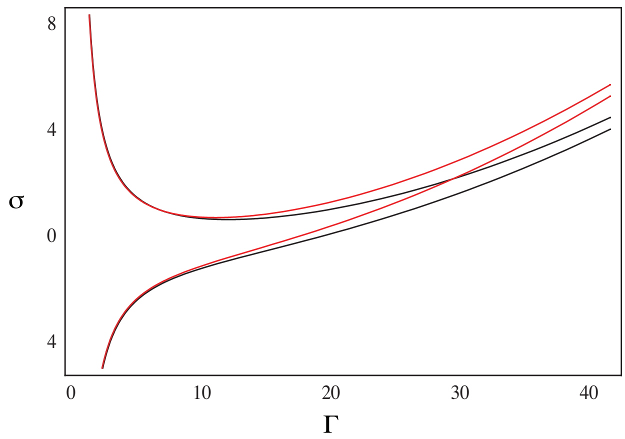

Figure 2: The backbones for $c_0 = 15$ and $\Omega = 0.196$ (black line) and $\Omega = 0.392$ (red line).

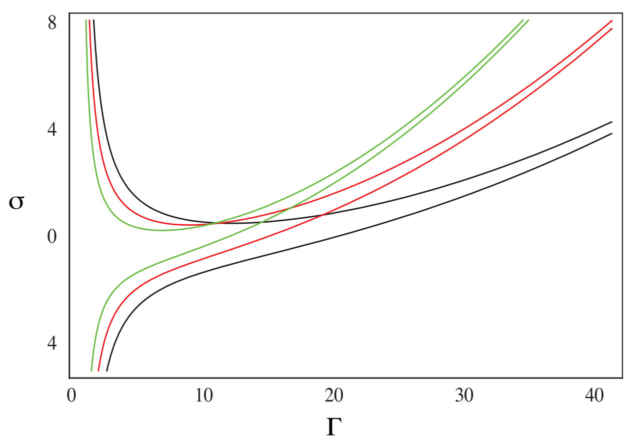

Figure 3: The backbones for the lowest primary resonant frequency for $c_0 = 15$ (black line), 10 (red line) and 6 (green line).

In Figure (2) we show σ_1 and σ_2 as functions of Γ_0 for $c_0 = 15$; in this case, the dispersion relation (21) gives two resonant frequencies below $\Omega = 1/2$, namely $\Omega = 0.196$ (black lines) and $\Omega = 0.392$ (red lines). Figure (2) shows σ_1 and σ_2 as functions of Γ_0 for the lowest resonant frequency corresponding to three different values of c_0: $c_0 = 15$, with $\Omega = 0.196$ (black lines), $c_0 = 10$, with $\Omega = 0.285$ (red lines) and $c_0 = 6$, with $\Omega = 0.446$ (green lines). The bending appears to be larger at higher frequencies (for the same value of c_0) and at lower values of c_0 (for the first resonant frequency).

4 Conclusions and Future Work

We have analyzed the occurrence of nonlinear resonances for a system governed by the wave equation in a semi-infinite domain. The system is subjected to a harmonic displacement applied to the finite boundary, which induces nonlinear forced oscillations. A small parameter ε is introduced, which measures the deviation of the amplitude of the forcing term with respect to a constant average value. In two earlier papers, [8] and [9], we have analyzed the first- and second-order perturbative solutions, obtaining the resonant response of the system. In this work, we have adopted a different scaling of the external excitation with respect to ε in order to describe the bending of the resonant curves (backbones) near a primary resonant frequency.

As a material for future work, we leave some numerical comparisons and a stability analysis of the solutions.

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