Implementation of Boundary Equations in the Finite Transfer Method

Lazaro Gimena, Pedro Gonzaga and Faustino N. Gimena

Abstract— **The Finite Transfer Method used to solve a system of linear ordinary differential equations is extended by adding the boundary equations involved in the problem. A Runge-Kutta scheme could be chosen, for example, to obtain Finite Transfer expressions. The use of a recurrence strategy in these equations permits one to relate different points in the domain where boundary equations could be defined. A final algebraic system of equations is annotated and solved. The method could be applied to determine the structural behaviour of a spatially curved beam element. An example is given to show the procedure exposed.**

Index Terms— **Finite Transfer Method, differential system, boundary equations, curved beam, Frenet-Serret formulas, transfer matrix.**

I. Introduction

The problem to solving a system of linear ordinary differential equations (ODE) with boundary conditions can be approached by using analytic or numerical strategies. Since it is not always possible to use exact methods, approximate procedures have been resorted to [1]. In last decades, several numerical methods have arisen to solve these boundary value problems; see for example, the Shooting Method [2], Finite Differences [3], Finite Element Analysis [4] and the Boundary Element [5] methods.

There exists much literature on modelling arbitrary curved beam elements [6], [7]. Traditionally, the laws governing the mechanical behaviour of a curved warped beam (applying the Euler-Bernuolli and Timoshenko theories) are defined by static equilibrium and kinematics [8], [9] or dynamic motion equations [10]. Some authors present this definition by means of compact energy equations [11], [12], [13]. These interpretations have permitted to reach accurate results, for some types of beams: for example, a circular arch element loaded in plane [14], [15], [16], [17], [18] and loaded perpendicular to its plane [19], parabolic and elliptical beams loaded in plane [20], [21], [22] or a helix uniformly loaded [23].

Manuscript received April 21, 2008.
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In this paper, the Finite Transfer Method (FTM) [24] is followed and applied to a system of differential equations, obtaining an incremental equation based on the transfer matrix. Fourth order Runge-Kutta approximation is adopted. Using the preceding finite expression as a recurrence scheme, both extremes are related, reaching a system of algebraic equations with constant dimension p regardless of the number of intervals.

The establishment of the problem is completed when the p boundary equations are incorporated. A final algebraic system of $2p$ order is reached and solved. Once values at the initial point are known, values at any point of the domain can be obtained.

The authors apply the FTM on the arbitrary curved beam model, by means of a unique system of twelve ordinary differential equations with boundary conditions [25].

An example is given to show the procedure exposed.

II. The Differential Problem

Let's define the system of p ODE of first order, which represents the differential problem to solve:

$$\begin{array}{lllll} \frac{dx_1}{dt}+a_{11}x_1+ & a_{12}x_2 & +\ldots+ & a_{1p}x_p & =b_1 \\ a_{21}x_1 & +\frac{dx_2}{dt}+a_{22}x_2 & +\ldots+ & a_{2p}x_p & =b_2 \\ \vdots & & \ddots & \vdots & \vdots \\ a_{p1}x_1 & + \quad a_{p2}x_2 & +\ldots+ & \frac{dx_p}{dt}+a_{pp}x_p & =b_p \end{array} \quad (1)$$

In vector notation it can be written as

$$\frac{d\mathbf{x}(t)}{dt}=\left[\mathbf{A}(t)\right]\mathbf{x}(t)+\mathbf{b}(t), \quad (2)$$

where

$$\mathbf{x}(t)=\left\{x_1(t),x_2(t),\ldots,x_p(t)\right\}^T$$

is the vector of the unknown functions,

$$\left[\mathbf{A}(t)\right]=-\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1p} \\ a_{21} & a_{22} & \ldots & a_{2p} \\ \vdots & & \ddots & \vdots \\ a_{p1} & a_{p2} & \ldots & a_{pp} \end{bmatrix}$$

is the matrix of variable coefficients and

$$\mathbf{b}(t)=\left\{b_1,b_2,\ldots,b_p\right\}^T$$

is the independent vector term.

Full definition of the problem is complete when adding p boundary equations in the domain.

They can be expressed as follows:

$$[\mathbf{B_I}]\mathbf{x}(t_\mathbf{I})+[\mathbf{B_{II}}]\mathbf{x}(t_\mathbf{II})=\mathbf{b_{I,II}} \tag{3}$$

Where $[\mathbf{B_I}]$, $[\mathbf{B_{II}}]$ and $\mathbf{b_{I,II}}$ are known.

A particular case of the above expression is when boundary equations are all given at an initial point, meaning initial conditions, thus $[\mathbf{B_{II}}]=[\mathbf{0}]$; therefore, boundary equations would be:

$$[\mathbf{B_I}]\mathbf{x}(t_\mathbf{I})=\mathbf{b_I} \tag{4}$$

The structure of these sets of equations is linear resulting ,when FTM is applied, in a linear system of algebraic equations.

III. FTM WITH BOUNDARY EQUATIONS

In this part of the paper, the establishment of the FTM with boundary equations is carried out. Fourth order Runge-Kutta approximation will be used [24].

A. *Finite Transfer Equation of fourth order RK approximation*

Applying the fourth order approximation:

$$\frac{d\mathbf{x}(t)}{dt}\cong\frac{\Delta\tilde{\mathbf{x}}(t_i)}{\Delta t}=\frac{\tilde{\mathbf{x}}(t_{i+1})-\tilde{\mathbf{x}}(t_i)}{\Delta t}=\frac{\mathbf{k}_1+2\mathbf{k}_2+2\mathbf{k}_3+\mathbf{k}_4}{6}, \tag{5}$$

where

$$\mathbf{k}_1=[\mathbf{A}_i]\tilde{\mathbf{x}}(t_i)+\mathbf{b}_i$$

$$\mathbf{k}_2=\left[\mathbf{A}_{i+1/2}\right][\tilde{\mathbf{x}}(t_i)+\mathbf{k}_1\,\Delta t/2]+\mathbf{b}_{i+1/2}$$

$$\mathbf{k}_3=\left[\mathbf{A}_{i+1/2}\right][\tilde{\mathbf{x}}(t_i)+\mathbf{k}_2\,\Delta t/2]+\mathbf{b}_{i+1/2}$$

$$\mathbf{k}_4=[\mathbf{A}_{i+1}][\tilde{\mathbf{x}}(t_i)+\mathbf{k}_3\Delta t]+\mathbf{b}_{i+1}$$

and approximated functions $\mathbf{x}(t_{i+1})\cong\tilde{\mathbf{x}}(t_{i+1})$; $\mathbf{x}(t_i)\cong\tilde{\mathbf{x}}(t_i)$.

Therefore, the *Finite Transfer Equation* that relates two consecutive point of the domain is:

$$\begin{aligned}\tilde{\mathbf{x}}(t_{i+1}) &= \Big[[\mathbf{I}]+\Big[[\mathbf{A}_{i+1}]+4\left[\mathbf{A}_{i+1/2}\right]+[\mathbf{A}_i]\Big]\Delta t/6+\\ &+\Big[[\mathbf{A}_{i+1}]\left[\mathbf{A}_{i+1/2}\right]+\left[\mathbf{A}_{i+1/2}\right]^2+\left[\mathbf{A}_{i+1/2}\right][\mathbf{A}_i]\Big]\Delta t^2/6+\\ &+\Big[[\mathbf{A}_{i+1}]\left[\mathbf{A}_{i+1/2}\right]^2+\left[\mathbf{A}_{i+1/2}\right]^2[\mathbf{A}_i]\Big]\Delta t^3/12+\\ &+[\mathbf{A}_{i+1}]\left[\mathbf{A}_{i+1/2}\right]^2[\mathbf{A}_i]\Delta t^4/24\Big]\tilde{\mathbf{x}}(t_i)+\\ &+\left(\mathbf{b}_{i+1}+4\mathbf{b}_{i+1/2}+\mathbf{b}_i\right)\Delta t/6+\\ &+\left([\mathbf{A}_{i+1}]\mathbf{b}_{i+1/2}+\left[\mathbf{A}_{i+1/2}\right]\mathbf{b}_{i+1/2}+\left[\mathbf{A}_{i+1/2}\right]\mathbf{b}_i\right)\Delta t^2/6+\\ &+\left([\mathbf{A}_{i+1}]\left[\mathbf{A}_{i+1/2}\right]\mathbf{b}_{i+1/2}+\left[\mathbf{A}_{i+1/2}\right]^2\mathbf{b}_i\right)\Delta t^3/12+\\ &+[\mathbf{A}_{i+1}]\left[\mathbf{A}_{i+1/2}\right]^2\mathbf{b}_i\,\Delta t^4/24=\\ &=[\mathbf{A_T}(t_i)]\tilde{\mathbf{x}}(t_i)+\mathbf{b_T}(t_i)\end{aligned} \tag{6}$$

B. *Recurrence scheme*

Using the above *Finite Transfer Equation*, a recurrence scheme could be applied to write the expression of the functions at a general point t_{i+1} in terms of the initial point $t_\mathbf{I}$

$$\begin{aligned}\tilde{\mathbf{x}}(t_{i+1}) &= \left[\prod_{j=0}^{j=i}\left[\mathbf{A_T}(t_j)\right]\right]\tilde{\mathbf{x}}(t_\mathbf{I})+\sum_{j=0}^{j=i}\left[\prod_{k=j+1}^{k=i}[\mathbf{A_T}(t_k)]\right]\mathbf{b_T}(t_j)=\\ &=\left[\mathbf{A_T}(t_\mathbf{I},t_{i+1})\right]\tilde{\mathbf{x}}(t_\mathbf{I})+\mathbf{b_T}(t_\mathbf{I},t_{i+1})\end{aligned} \tag{7}$$

with $\mathbf{x}(t_\mathbf{I})\cong\tilde{\mathbf{x}}(t_\mathbf{I})$; which represent the *General Solution*.

Establishing n intervals, the two end points **I** and **II** of the domain are related by next equation, where the *boundary equations* could be applied (see Fig. 1).

$$\begin{aligned}\tilde{\mathbf{x}}(t_\mathbf{II}) &= \left[\prod_{j=0}^{j=n-1}\left[\mathbf{A_T}(t_j)\right]\right]\tilde{\mathbf{x}}(t_\mathbf{I})+\sum_{j=0}^{j=n-1}\left[\prod_{k=j+1}^{k=n-1}[\mathbf{A_T}(t_k)]\right]\mathbf{b_T}(t_j)=\\ &=\left[\mathbf{A_T}(t_\mathbf{I},t_\mathbf{II})\right]\tilde{\mathbf{x}}(t_\mathbf{I})+\mathbf{b_T}(t_\mathbf{I},t_\mathbf{II})\end{aligned} \tag{8}$$

with $\mathbf{x}(t_\mathbf{II})\cong\tilde{\mathbf{x}}(t_\mathbf{II})$.

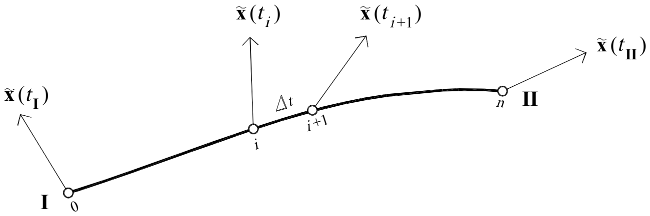

Figure 1. Approximated function and variables in the domain.

C. *Boundary conditions*

Let's implement the *boundary equations* to define and complete the extended problem.

Assuming that *boundary equations* (Eq. 3) are applied at the approximation functions $\mathbf{x}(t_{i+1})\cong\tilde{\mathbf{x}}(t_{i+1})$; $\mathbf{x}(t_i)\cong\tilde{\mathbf{x}}(t_i)$:

$$[\mathbf{B_I}]\tilde{\mathbf{x}}(t_\mathbf{I})+[\mathbf{B_{II}}]\tilde{\mathbf{x}}(t_\mathbf{II})=\mathbf{b_{I,II}} \tag{9}$$

A new algebraic system of $2p$ order is reached:

$$\begin{bmatrix}\left[\mathbf{A_T}(t_\mathbf{I},t_\mathbf{II})\right] & -[\mathbf{I}]\\ [\mathbf{B_I}] & [\mathbf{B_{II}}]\end{bmatrix}\begin{bmatrix}\tilde{\mathbf{x}}(t_\mathbf{I})\\ \tilde{\mathbf{x}}(t_\mathbf{II})\end{bmatrix}=\begin{bmatrix}-\mathbf{b_T}(t_\mathbf{I},t_\mathbf{II})\\ \mathbf{b_{I,II}}\end{bmatrix} \tag{10}$$

Solutions at the extremes **I** y **II** are directly obtained:

$$\begin{bmatrix}\tilde{\mathbf{x}}(t_\mathbf{I})\\ \tilde{\mathbf{x}}(t_\mathbf{II})\end{bmatrix}=\begin{bmatrix}\left[\mathbf{A_T}(t_\mathbf{I},t_\mathbf{II})\right] & -[\mathbf{I}]\\ [\mathbf{B_I}] & [\mathbf{B_{II}}]\end{bmatrix}^{-1}\begin{bmatrix}-\mathbf{b_T}(t_\mathbf{I},t_\mathbf{II})\\ \mathbf{b_{I,II}}\end{bmatrix} \tag{11}$$

D. *Initial conditions*

A particular case is when *boundary conditions* are given at the initial point (Eq. 4), so that the approximated function is:

$$[\mathbf{B_I}]\tilde{\mathbf{x}}(t_\mathbf{I})=\mathbf{b_I} \tag{12}$$

An algebraic system of $2p$ equations is written in matricial form:

$$\begin{bmatrix}\left[\mathbf{A_T}(t_\mathbf{I},t_{i+1})\right] & -[\mathbf{I}]\\ [\mathbf{B_I}] & [\mathbf{0}]\end{bmatrix}\begin{bmatrix}\tilde{\mathbf{x}}(t_\mathbf{I})\\ \tilde{\mathbf{x}}(t_{i+1})\end{bmatrix}=\begin{bmatrix}-\mathbf{b_T}(t_\mathbf{I},t_{i+1})\\ \mathbf{b_I}\end{bmatrix} \tag{13}$$

Solution will be expressed for the point i+1 in function of the values given at the initial point **I**, as follows:

$$\begin{bmatrix}\tilde{\mathbf{x}}(t_\mathbf{I})\\ \tilde{\mathbf{x}}(t_{i+1})\end{bmatrix}=\begin{bmatrix}\left[\mathbf{A_T}(t_\mathbf{I},t_{i+1})\right] & -[\mathbf{I}]\\ [\mathbf{B_I}] & [\mathbf{0}]\end{bmatrix}^{-1}\begin{bmatrix}-\mathbf{b_T}(t_\mathbf{I},t_{i+1})\\ \mathbf{b_I}\end{bmatrix} \tag{14}$$

E. *General solution*

Once values at the initial point of the problem are known, either of the above problems of boundary or initial conditions, applying the *Finite Transfer Equation General Solution* (Eq. 7), values at any point i+1 can easily be determined:

$$\tilde{\mathbf{x}}(t_{i+1}) = \left[\prod_{j=0}^{j=i}\left[\mathbf{A_T}(t_j)\right]\right]\tilde{\mathbf{x}}(t_\mathbf{I})+\sum_{j=0}^{j=i}\left[\prod_{k=j+1}^{k=i}[\mathbf{A_T}(t_k)]\right]\mathbf{b_T}(t_j) \tag{15}$$

In the limit, when the increment tends to cero $\Delta t\rightarrow 0$, the above expression tends to the analytical solution [24]:

$$\mathbf{x}(t)=\exp\left(\int_{t_\mathbf{I}}^{t}[\mathbf{A}]dt\right)\left[\mathbf{x}(t_\mathbf{I})+\int_{t_\mathbf{I}}^{t}\exp\left(-\int_{t_\mathbf{I}}^{t}[\mathbf{A}]dt\right)\mathbf{b}\,dt\right] \tag{Z}$$

ISBN:978-988-18210-1-0

IV. DIFFERENTIAL SYSTEM FOR SPATIALLY CURVED BEAMS

A curved beam is generated by a plane cross section whose centroid sweeps through all the points of an axis curve. The vector radius $\mathbf{r} = \mathbf{r}(s)$ expresses this curved line, where s (arc length of the centroid line) is the independent variable.

The reference system used to represent the intervening known and unknown functions is the Frenet frame. Its unit vectors tangent $\mathbf{t}$, normal $\mathbf{n}$ and binormal $\mathbf{b}$ are:

$$\mathbf{t} = D\mathbf{r}\ ;\ \mathbf{n} = D^2\mathbf{r}\big/\left|D^2\mathbf{r}\right|\ ;\ \mathbf{b} = \mathbf{t}\wedge\mathbf{n}$$

where, $D = d/ds$ is the derivative with respect to the parameter s.

The natural equations of the centroid line are expressed by the flexion curvature $\chi = \sqrt{D^2\mathbf{r}\cdot D^2\mathbf{r}}$ and the torsion curvature $\tau = D\mathbf{r}\wedge\left(D^2\mathbf{r}\cdot D^3\mathbf{r}\right)/\left(D^2\mathbf{r}\cdot D^2\mathbf{r}\right)$.

The Frenet-Serret formulas are [26].

$$\begin{aligned} D\mathbf{t} &= \chi\mathbf{n} \\ D\mathbf{n} &= -\chi\mathbf{t} + \tau\mathbf{b} \\ D\mathbf{b} &= -\tau\mathbf{n} \end{aligned} \tag{16}$$

Assuming the habitual principles and hypotheses (Euler-Bernoulli and Timoshenko classical beam theories) and considering the stresses associated with the normal cross-section (σ, τ_n, τ_b), the geometric characteristics of the section are: area $A(s)$, shearing coefficients $\alpha_n(s)$, $\alpha_{nb}(s)$, $\alpha_{bn}(s)$, $\alpha_b(s)$, and moments of inertia $I_t(s)$, $I_n(s)$, $I_b(s)$, $I_{nb}(s)$. Longitudinal $E(s)$ and transversal $G(s)$ elasticity moduli give the elastic properties of the material.

Applying equilibrium and kinematics laws to an infinitesimal element of the curve, the system of differential equations governing the structural behaviour of a spatially curved beam can be obtained [25] (Equation 17 in this page at the top of next column).

The first six rows of the system (Eq. 17) represent the equilibrium equations.

The functions involved in the equilibrium equation are:

Internal forces

$$N\,\mathbf{t} + V_n\,\mathbf{n} + V_b\,\mathbf{b} = \int_A \sigma\, dA\,\mathbf{t} + \int_A \tau_n\, dA\,\mathbf{n} + \int_A \tau_b\, dA\,\mathbf{b}$$

Internal moments

$$T\,\mathbf{t} + M_n\,\mathbf{n} + M_b\,\mathbf{b} = \int_A \left(\tau_b n - \tau_n b\right) dA\,\mathbf{t} + \int_A \sigma b\, dA\,\mathbf{n} - \int_A \sigma n\, dA\,\mathbf{b}$$

Load force $q_t\mathbf{t} + q_n\mathbf{n} + q_b\mathbf{b}$

Load moment $m_t\mathbf{t} + m_n\mathbf{n} + m_b\mathbf{b}$

The last six rows of the system (Eq. 17) represent the kinematics equations.

Rotations $\theta_t\,\mathbf{t} + \theta_n\,\mathbf{n} + \theta_b\,\mathbf{b}$

Displacements $u\,\mathbf{t} + v\,\mathbf{n} + w\,\mathbf{b}$

Rotation load $\varTheta_t\,\mathbf{t} + \varTheta_n\,\mathbf{n} + \varTheta_b\,\mathbf{b}$

Displacement load $\varDelta_t\,\mathbf{t} + \varDelta_n\,\mathbf{n} + \varDelta_b\,\mathbf{b}$

$$\begin{aligned}
&DN - \chi V_n + q_t = 0 \\
&\chi N + DV_n - \tau V_b + q_n = 0 \\
&\tau V_n + DV_b + q_b = 0 \\
&DT - \chi M_n \quad (20 \quad + m_t = 0 \\
&-V_b + \chi T + DM_n - \tau M_b + m_n = 0 \\
&V_n + \tau M_n + DM_b + m_b = 0 \\
&-\frac{T}{GI_t} + D\theta_t - \chi\theta_n - \varTheta_t = 0 \\
&-\frac{M_n I_b}{E\left[I_n I_b - I_{nb}^2\right]} - \frac{M_b I_{nb}}{E\left[I_n I_b - I_{nb}^2\right]} + \chi\theta_t + D\theta_n - \tau\theta_b - \varTheta_n = 0 \\
&-\frac{M_n I_{nb}}{E\left[I_n I_b - I_{nb}^2\right]} - \frac{M_b I_n}{E\left[I_n I_b - I_{nb}^2\right]} + \tau\theta_n + D\theta_b - \varTheta_b = 0 \\
&-\frac{N}{EA} + Du - \chi v - \varDelta_t = 0 \\
&-\frac{\alpha_n V_n}{GA} - \frac{\alpha_{nb} V_b}{GA} - \theta_b + \chi u + Dv - \tau w - \varDelta_n = 0 \\
&-\frac{\alpha_{bn} V_n}{GA} - \frac{\alpha_b V_b}{GA} + \theta_n + \tau v + Dw - \varDelta_b = 0
\end{aligned}$$

Equation 17. Differential System for Spatially Curved Beams.

The above differential system (Eq. 17) can be expressed in the vector form as follows:

$$\frac{d\mathbf{e}(s)}{ds} = \left[\mathbf{T}(s)\right]\mathbf{e}(s) + \mathbf{q}(s) \tag{18}$$

Where

$$\mathbf{e}(s) = \left\{N,\ V_n,\ V_b,\ T,\ M_n,\ M_b,\ \theta_t,\ \theta_n,\ \theta_b,\ u,\ v,\ w\right\}^T$$

is the state vector $\mathbf{e}(s)$ of internal forces and deflections at a point s of the beam element, named *effect at the section*,

$$[\mathbf{T}(s)] = \begin{bmatrix}
0 & \chi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\chi & 0 & \tau & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\tau & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \chi & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -\chi & 0 & \tau & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & -\tau & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{GI_t} & 0 & 0 & 0 & \chi & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{I_b}{E\left[I_n I_b - I_{nb}^2\right]} & \frac{I_{nb}}{E\left[I_n I_b - I_{nb}^2\right]} & -\chi & 0 & \tau & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{I_{nb}}{E\left[I_n I_b - I_{nb}^2\right]} & \frac{I_n}{E\left[I_n I_b - I_{nb}^2\right]} & 0 & -\tau & 0 & 0 & 0 & 0 \\
\frac{1}{EA} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \chi & 0 \\
0 & \frac{\alpha_n}{GA} & \frac{\alpha_{nb}}{GA} & 0 & 0 & 0 & 0 & 0 & 1 & -\chi & 0 & \tau \\
0 & \frac{\alpha_{bn}}{GA} & \frac{\alpha_b}{GA} & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -\tau & 0
\end{bmatrix}$$

is the *Infinitesimal Transfer Matrix*, and

$$\mathbf{q}(s) = \left\{-q_t,\ -q_n,\ -q_b,\ -m_t,\ -m_n,\ -m_b,\ \varTheta_t,\ \varTheta_n,\ \varTheta_b,\ \varDelta_t,\ \varDelta_n,\ \varDelta_b\right\}^T$$

is the applied load.

Therefore, twelve boundary equations are needed to solve this structural problem, and they can be expressed as (see Eq. 3):

$$\left[\mathbf{B_I}\right]\mathbf{e}(s_\mathbf{I}) + \left[\mathbf{B_{II}}\right]\mathbf{e}(s_\mathbf{II}) = \mathbf{b_{I,II}} \tag{19}$$

For the particular case where the supports are fixed in both ends, the set of boundary equations will be:

$$\begin{bmatrix} [0] & [\mathbf{I}] \\ [0] & [0] \end{bmatrix}\mathbf{e}(s_\mathbf{I}) + \begin{bmatrix} [0] & [0] \\ [0] & [\mathbf{I}] \end{bmatrix}\mathbf{e}(s_\mathbf{II}) = \mathbf{0} \tag{20}$$

Other example, when the initial support is clamped and the other end is free with a punctual load $\mathbf{Q}$ applied, the set will be:

$$\begin{bmatrix} [0] & [\mathbf{I}] \\ [0] & [0] \end{bmatrix}\mathbf{e}(s_\mathbf{I}) + \begin{bmatrix} [0] & [0] \\ [\mathbf{I}] & [0] \end{bmatrix}\mathbf{e}(s_\mathbf{II}) = \left\{\mathbf{0}, \mathbf{Q}\right\}^T \tag{21}$$

being $\mathbf{Q}=\left\{Q_t^{\mathbf{II}},Q_n^{\mathbf{II}},Q_b^{\mathbf{II}},M_t^{\mathbf{II}},M_n^{\mathbf{II}},M_b^{\mathbf{II}}\right\}$.

A. Applying the FTM of fourth order RK

The approximation of the differential system (Eq. 18) is given by [27]:

$$\frac{d\mathbf{e}(s)}{dt}\cong\frac{\Delta\tilde{\mathbf{e}}(s_i)}{\Delta s}=\frac{\tilde{\mathbf{e}}(s_{i+1})-\tilde{\mathbf{e}}(s_i)}{\Delta s}=\frac{\mathbf{k}_1+2\mathbf{k}_2+2\mathbf{k}_3+\mathbf{k}_4}{6} \qquad (22)$$

being

$$\mathbf{k}_1=\left[\mathbf{T}_i\right]\tilde{\mathbf{e}}(s_i)+\mathbf{q}_i$$

$$\mathbf{k}_2=\left[\mathbf{T}_{i+1/2}\right]\left[\tilde{\mathbf{e}}(s_i)+\mathbf{k}_1\,\Delta s/2\right]+\mathbf{q}_{i+1/2}$$

$$\mathbf{k}_3=\left[\mathbf{T}_{i+1/2}\right]\left[\tilde{\mathbf{e}}(s_i)+\mathbf{k}_2\,\Delta s/2\right]+\mathbf{q}_{i+1/2}$$

$$\mathbf{k}_4=\left[\mathbf{T}_{i+1}\right]\left[\tilde{\mathbf{e}}(s_i)+\mathbf{k}_3\Delta s\right]+\mathbf{q}_{i+1}$$

Assuming that approximated functions are:

$\mathbf{e}(s_{i+1})\cong\tilde{\mathbf{e}}(s_{i+1})$; $\mathbf{e}(s_i)\cong\tilde{\mathbf{e}}(s_i)$

Thus, the *Finite Transfer Equation* (Eq. 6) of four order is:

$$\begin{aligned}\tilde{\mathbf{e}}(s_{i+1}) =\ &\Big[[\mathbf{I}]+\Big[[\mathbf{T}_{i+1}]+4\left[\mathbf{T}_{i+1/2}\right]+[\mathbf{T}_i]\Big]\Delta s/6+\\ &+\Big[[\mathbf{T}_{i+1}]\left[\mathbf{T}_{i+1/2}\right]+\left[\mathbf{T}_{i+1/2}\right]^2+\left[\mathbf{T}_{i+1/2}\right][\mathbf{T}_i]\Big]\Delta s^2/6+\\ &+\Big[[\mathbf{T}_{i+1}]\left[\mathbf{T}_{i+1/2}\right]^2+\left[\mathbf{T}_{i+1/2}\right]^2[\mathbf{T}_i]\Big]\Delta s^3/12+\\ &+[\mathbf{T}_{i+1}]\left[\mathbf{T}_{i+1/2}\right]^2[\mathbf{T}_i]\Delta s^4/24\Big]\tilde{\mathbf{e}}(s_i)+\\ &+\left(\mathbf{q}_{i+1}+4\mathbf{q}_{i+1/2}+\mathbf{q}_i\right)\Delta s/6+\\ &+\Big([\mathbf{T}_{i+1}]\mathbf{q}_{i+1/2}+\left[\mathbf{T}_{i+1/2}\right]\mathbf{q}_{i+1/2}+\left[\mathbf{T}_{i+1/2}\right]\mathbf{q}_i\Big)\Delta s^2/6+\\ &+\Big([\mathbf{T}_{i+1}]\left[\mathbf{T}_{i+1/2}\right]\mathbf{q}_{i+1/2}+\left[\mathbf{T}_{i+1/2}\right]^2\mathbf{q}_i\Big)\Delta s^3/12+\\ &+[\mathbf{T}_{i+1}]\left[\mathbf{T}_{i+1/2}\right]^2\mathbf{q}_i\,\Delta s^4/24=\\ &=[\mathbf{T}_\mathbf{T}(s_i)]\tilde{\mathbf{e}}(s_i)+\mathbf{q}_\mathbf{T}(s_i)\end{aligned} \qquad (23)$$

Applying the recurrence scheme (see Eq. 7):

$$\begin{aligned}\tilde{\mathbf{e}}(s_{i+1}) &=\left[\prod_{j=0}^{j=i}\left[\mathbf{T}_\mathbf{T}(s_j)\right]\right]\tilde{\mathbf{e}}(s_\mathbf{I})+\sum_{j=0}^{j=i}\left[\prod_{k=j+1}^{k=i}\left[\mathbf{T}_\mathbf{T}(s_k)\right]\right]\mathbf{q}_\mathbf{T}(s_j)= \qquad (24)\\ &=\left[\mathbf{T}_\mathbf{T}\left(s_\mathbf{I},s_{i+1}\right)\right]\tilde{\mathbf{e}}(s_\mathbf{I})+\mathbf{q}_\mathbf{T}\left(s_\mathbf{I},s_{i+1}\right)\end{aligned}$$

with $\mathbf{e}(s_\mathbf{I})\cong\tilde{\mathbf{e}}(s_\mathbf{I})$.

Establishing n intervals, the two end points **I** and **II** of the curved line can be related (see Eq. 8):

$$\begin{aligned}\tilde{\mathbf{e}}(s_\mathbf{II}) &=\left[\prod_{j=0}^{j=n-1}\left[\mathbf{T}_\mathbf{T}(s_j)\right]\right]\tilde{\mathbf{e}}(s_\mathbf{I})+\sum_{j=0}^{j=n-1}\left[\prod_{k=j+1}^{k=n-1}\left[\mathbf{T}_\mathbf{T}(s_k)\right]\right]\mathbf{q}_\mathbf{T}(s_j)= \qquad (25)\\ &=\left[\mathbf{T}_\mathbf{T}\left(s_\mathbf{I},s_\mathbf{II}\right)\right]\tilde{\mathbf{e}}(s_\mathbf{I})+\mathbf{q}_\mathbf{T}\left(s_\mathbf{I},s_\mathbf{II}\right)\end{aligned}$$

Here $\left[\mathbf{T}\left(s_\mathrm{I},s_\mathrm{II}\right)\right]$ is the Transfer Matrix and $\mathbf{q}_\mathbf{T}\left(s_\mathrm{I},s_\mathrm{II}\right)$ the load transfer vector [28],

With the approximated value at the final end $\mathbf{e}(s_\mathbf{II})\cong\tilde{\mathbf{e}}(s_\mathbf{II})$.

Finally, *boundary equations* are applied to solve the problem:

$$[\mathbf{B}_\mathbf{I}]\tilde{\mathbf{e}}(s_\mathbf{I})+[\mathbf{B}_\mathbf{II}]\tilde{\mathbf{e}}(s_\mathbf{II})=\mathbf{b}_\mathbf{I,II} \qquad (26)$$

An algebraic system of twenty four equations is reached and can be written in matricial form as follows:

$$\begin{bmatrix}\left[\mathbf{T}_\mathbf{T}\left(s_\mathbf{I},s_\mathbf{II}\right)\right] & -[\mathbf{I}]\\ [\mathbf{B}_\mathbf{I}] & [\mathbf{B}_\mathbf{II}]\end{bmatrix}\begin{bmatrix}\tilde{\mathbf{e}}(s_\mathbf{I})\\ \tilde{\mathbf{e}}(s_\mathbf{II})\end{bmatrix}=\begin{bmatrix}-\mathbf{q}_\mathbf{T}\left(s_\mathbf{I},s_\mathbf{II}\right)\\ \mathbf{b}_\mathbf{I,II}\end{bmatrix} \qquad (27)$$

Therefore, the values at both ends **I** y **II** are determined:

$$\begin{bmatrix}\tilde{\mathbf{e}}(s_\mathbf{I})\\ \tilde{\mathbf{e}}(s_\mathbf{II})\end{bmatrix}=\begin{bmatrix}\left[\mathbf{T}_\mathbf{T}\left(s_\mathbf{I},s_\mathbf{II}\right)\right] & -[\mathbf{I}]\\ [\mathbf{B}_\mathbf{I}] & [\mathbf{B}_\mathbf{II}]\end{bmatrix}^{-1}\begin{bmatrix}-\mathbf{q}_\mathbf{T}\left(s_\mathbf{I},s_\mathbf{II}\right)\\ \mathbf{b}_\mathbf{I,II}\end{bmatrix} \qquad (28)$$

Once values at the initial point $\tilde{\mathbf{e}}(s_\mathbf{I})$ are known, general solution (see Eq. 15) at any point i+1 is directly written as:

$$\tilde{\mathbf{e}}(s_{i+1})=\left[\prod_{j=0}^{j=i}\left[\mathbf{T}_\mathbf{T}(s_j)\right]\right]\tilde{\mathbf{e}}(s_\mathbf{I})+\sum_{j=0}^{j=i}\left[\prod_{k=j+1}^{k=i}\left[\mathbf{T}_\mathbf{T}(s_k)\right]\right]\mathbf{q}_\mathbf{T}(s_j) \qquad (29)$$

V. EXAMPLE.

A. Bending in a beam. General solution.

For simplicity, the example to be considered here is a particular case of the full structural problem of the curved beam, given in Eq. 17, but the procedure to solve the whole problem is the same exposed formerly.

The beam will be straight and the intervening coefficients constant along the axis line. Load will produce flexion effects.

Figure 2 shows these considerations.

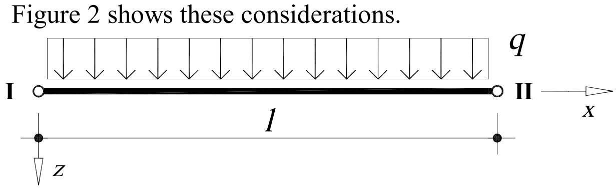

Figure 2. Beam with flexion load.

The differential system will be particularized as:

$$\begin{aligned}&\frac{dV_z}{dx} & & & +q_z &=0\\ -V_z &+\frac{dM_y}{dx} & & & +m_y &=0\\ &-\frac{M_y}{EI_y} &+\frac{d\theta_y}{dx} & & -\Theta_y &=0\\ & &+\theta_y &+\frac{dw}{dx} & -\Lambda_z &=0\end{aligned} \qquad (30)$$

Annotating the above system in matricial form (see Eq. 18):

$$\frac{d}{dx}\begin{bmatrix}V_z\\ M_y\\ \theta_y\\ w\end{bmatrix}=\begin{bmatrix}0&0&0&0\\ 1&0&0&0\\ 0&\frac{1}{EI_y}&0&0\\ 0&0&-1&0\end{bmatrix}\begin{bmatrix}V_z\\ M_y\\ \theta_y\\ w\end{bmatrix}+\begin{bmatrix}-q\\ 0\\ 0\\ 0\end{bmatrix} \qquad (31)$$

Applying the *Finite Transfer Equation* (Eq. 23), yields:

$$\begin{bmatrix}V_z\\ M_y\\ \theta_y\\ w\end{bmatrix}_{i+1}=\begin{bmatrix}1&0&0&0\\ \Delta x&1&0&0\\ \frac{\Delta x^2}{2EI_y}&\frac{\Delta x}{EI_y}&1&0\\ -\frac{\Delta x^3}{6EI_y}&-\frac{\Delta x^2}{2EI_y}&-\Delta x&1\end{bmatrix}\begin{bmatrix}V_z\\ M_y\\ \theta_y\\ w\end{bmatrix}_i+\begin{bmatrix}-q\Delta x\\ -q\frac{\Delta x^2}{2}\\ -q\frac{\Delta x^3}{6EI_y}\\ q\frac{\Delta x^4}{24EI_y}\end{bmatrix}$$

With the recurrence scheme (Eq. 24), it is obtained:

$$\begin{bmatrix}V_z\\ M_y\\ \theta_y\\ w\end{bmatrix}_{i+1}=\begin{bmatrix}1&0&0&0\\ (i+1)\Delta x&1&0&0\\ \frac{(i+1)^2\Delta x^2}{2EI_y}&\frac{(i+1)\Delta x}{EI_y}&1&0\\ -\frac{(i+1)^3\Delta x^3}{6EI_y}&-\frac{(i+1)^2\Delta x^2}{2EI_y}&-(i+1)\Delta x&1\end{bmatrix}\begin{bmatrix}V_z\\ M_y\\ \theta_y\\ w\end{bmatrix}_\mathrm{I}+\begin{bmatrix}-q(i+1)\Delta x\\ -q\frac{(i+1)^2\Delta x^2}{2}\\ -q\frac{(i+1)^3\Delta x^3}{6EI_y}\\ q\frac{(i+1)^4\Delta x^4}{24EI_y}\end{bmatrix}$$

Relation between the end points **I** and **II** (see Eq. 25) of the unknown functions are established:

$$\begin{bmatrix} V_z \\ M_y \\ \theta_y \\ w \end{bmatrix}_{\mathbf{II}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ n\Delta x & 1 & 0 & 0 \\ \frac{n^2\Delta x^2}{2EI_y} & \frac{n\Delta x}{EI_y} & 1 & 0 \\ -\frac{n^3\Delta x^3}{6EI_y} & -\frac{n^2\Delta x^2}{2EI_y} & -n\Delta x & 1 \end{bmatrix} \begin{bmatrix} V_z \\ M_y \\ \theta_y \\ w \end{bmatrix}_{\mathbf{I}} + \begin{bmatrix} -qn\Delta x \\ -q\frac{n^2\Delta x^2}{2} \\ -q\frac{n^3\Delta x^3}{6EI_y} \\ q\frac{n^4\Delta x^4}{24EI_y} \end{bmatrix}$$

B. Clamped-clamped

Let's consider the case when both support are fixed, as shown in the next figure:

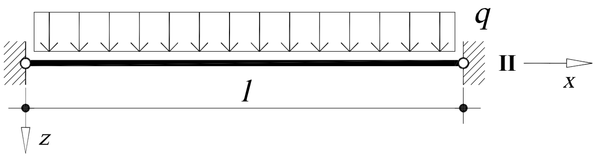

Figure 3. Clamped-clamped flexion beam.

Following the procedure given in Eq. 20, the boundary equations are:

$$\begin{bmatrix} 0\,0\,1\,0 \\ 0\,0\,0\,1 \\ 0\,0\,0\,0 \\ 0\,0\,0\,0 \end{bmatrix} \begin{bmatrix} V_z \\ M_y \\ \theta_y \\ w \end{bmatrix}_{\mathbf{I}} + \begin{bmatrix} 0\,0\,0\,0 \\ 0\,0\,0\,0 \\ 0\,0\,1\,0 \\ 0\,0\,0\,1 \end{bmatrix} \begin{bmatrix} V_z \\ M_y \\ \theta_y \\ w \end{bmatrix}_{\mathbf{II}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The final algebraic system (see Eq. 27X) to be solved is composed in this case by eight linear equations:

$$\left[\begin{array}{cccc:cccc} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ n\Delta x & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ \frac{n^2\Delta x^2}{2EI_y} & \frac{n\Delta x}{EI_y} & 1 & 0 & 0 & 0 & -1 & 0 \\ -\frac{n^3\Delta x^3}{6EI_y} & -\frac{n^2\Delta x^2}{2EI_y} & -n\Delta x & 1 & 0 & 0 & 0 & -1 \\ \hdashline 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} \begin{bmatrix} V_z \\ M_y \\ \theta_y \\ w \end{bmatrix}_{\mathbf{I}} \\ \hdashline \begin{bmatrix} V_z \\ M_y \\ \theta_y \\ w \end{bmatrix}_{\mathbf{II}} \end{array}\right] = \left[\begin{array}{c} qn\Delta x \\ q\frac{n^2\Delta x^2}{2} \\ q\frac{n^3\Delta x^3}{6EI_y} \\ -q\frac{n^4\Delta x^4}{24EI_y} \\ \hdashline 0 \\ 0 \\ 0 \\ 0 \end{array}\right]$$

Solution (see Eq. 28) at both ends **I** and **II** will be:

$$\left[\begin{array}{c} \begin{bmatrix} V_z \\ M_y \\ \theta_y \\ w \end{bmatrix}_{\mathbf{I}} \\ \hdashline \begin{bmatrix} V_z \\ M_y \\ \theta_y \\ w \end{bmatrix}_{\mathbf{II}} \end{array}\right] = \left[\begin{array}{cccc:cccc} 0 & 0 & \frac{6EI_y}{n^2\Delta x^2} & \frac{12EI_y}{n^3\Delta x^3} & \frac{6EI_y}{n^2\Delta x^2} & -\frac{12EI_y}{n^3\Delta x^3} & \frac{6EI_y}{n^2\Delta x^2} & \frac{12EI_y}{n^3\Delta x^3} \\ 0 & 0 & -\frac{2EI_y}{n\Delta x} & -\frac{6EI_y}{n^2\Delta x^2} & -\frac{4EI_y}{n\Delta x} & \frac{6EI_y}{n^2\Delta x^2} & -\frac{2EI_y}{n\Delta x} & -\frac{6EI_y}{n^2\Delta x^2} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hdashline -1 & 0 & \frac{6EI_y}{n^2\Delta x^2} & \frac{12EI_y}{n^3\Delta x^3} & -\frac{6EI_y}{n^2\Delta x^2} & -\frac{12EI_y}{n^3\Delta x^3} & \frac{6EI_y}{n^2\Delta x^2} & \frac{12EI_y}{n^3\Delta x^3} \\ 0 & -1 & \frac{4EI_y}{n\Delta x} & \frac{6EI_y}{n^2\Delta x^2} & \frac{2EI_y}{n\Delta x} & -\frac{6EI_y}{n^2\Delta x^2} & \frac{4EI_y}{n\Delta x} & \frac{6EI_y}{n^2\Delta x^2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} qn\Delta x \\ q\frac{n^2\Delta x^2}{2} \\ q\frac{n^3\Delta x^3}{6EI_y} \\ -q\frac{n^4\Delta x^4}{24EI_y} \\ \hdashline 0 \\ 0 \\ 0 \\ 0 \end{array}\right]$$

Thus, initial values are: $\tilde{\mathbf{e}}(s_{\mathbf{I}}) = \left\{ \frac{qn\Delta x}{2}, -q\frac{n^2\Delta x^2}{12}, 0, 0 \right\}^T$

Once initial values are known $\tilde{\mathbf{e}}(s_{\mathbf{I}})$, solution at any point $i+1$ will be given by:

$$\begin{bmatrix} V_z \\ M_y \\ \theta_y \\ w \end{bmatrix}_{i+1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ (i+1)\Delta x & 1 & 0 & 0 \\ \frac{(i+1)^2\Delta x^2}{2EI_y} & \frac{(i+1)\Delta x}{EI_y} & 1 & 0 \\ -\frac{(i+1)^3\Delta x^3}{6EI_y} & -\frac{(i+1)^2\Delta x^2}{2EI_y} & -(i+1)\Delta x & 1 \end{bmatrix} \begin{bmatrix} \frac{qn\Delta x}{2} \\ -q\frac{n^2\Delta x^2}{12} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -q(i+1)\Delta x \\ -q\frac{(i+1)^2\Delta x^2}{2} \\ -q\frac{(i+1)^3\Delta x^3}{6EI_y} \\ q\frac{(i+1)^4\Delta x^4}{24EI_y} \end{bmatrix}$$

C. Clamped-free with a punctual load.

Let's consider the case when initial support is fixed and the other is free with a punctual load, as shown in the figure 4:

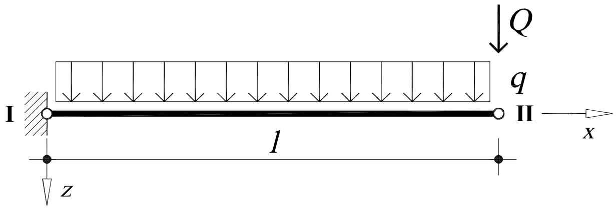

Figure 4. Clamped-free with punctual load, flexion beam.

Following the procedure given in Eq. 21, the boundary equations are:

$$\begin{bmatrix} 0\,0\,1\,0 \\ 0\,0\,0\,1 \\ 0\,0\,0\,0 \\ 0\,0\,0\,0 \end{bmatrix} \begin{bmatrix} V_z \\ M_y \\ \theta_y \\ w \end{bmatrix}_{\mathbf{I}} + \begin{bmatrix} 0\,0\,0\,0 \\ 0\,0\,0\,0 \\ 1\,0\,0\,0 \\ 0\,1\,0\,0 \end{bmatrix} \begin{bmatrix} V_z \\ M_y \\ \theta_y \\ w \end{bmatrix}_{\mathbf{II}} = \begin{bmatrix} 0 \\ 0 \\ Q \\ 0 \end{bmatrix}$$

The final algebraic system (see Eq. 27) to be solved is composed in this case by eight linear equations:

$$\left[\begin{array}{cccc:cccc} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ n\Delta x & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ \frac{n^2\Delta x^2}{2EI_y} & \frac{n\Delta x}{EI_y} & 1 & 0 & 0 & 0 & -1 & 0 \\ -\frac{n^3\Delta x^3}{6EI_y} & -\frac{n^2\Delta x^2}{2EI_y} & -n\Delta x & 1 & 0 & 0 & 0 & -1 \\ \hdashline 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right] \left[\begin{array}{c} \begin{bmatrix} V_z \\ M_y \\ \theta_y \\ w \end{bmatrix}_{\mathbf{I}} \\ \hdashline \begin{bmatrix} V_z \\ M_y \\ \theta_y \\ w \end{bmatrix}_{\mathbf{II}} \end{array}\right] = \left[\begin{array}{c} qn\Delta x \\ q\frac{n^2\Delta x^2}{2} \\ q\frac{n^3\Delta x^3}{6EI_y} \\ -q\frac{n^4\Delta x^4}{24EI_y} \\ \hdashline 0 \\ 0 \\ Q \\ 0 \end{array}\right]$$

ISBN:978-988-18210-1-0

Solution (see Eq. 28) at both ends **I** and **II** will be:

$$
\begin{bmatrix} \begin{bmatrix} V_z \\ M_y \\ \theta_y \\ w \end{bmatrix}_{\mathrm{I}} \\ \begin{bmatrix} V_z \\ M_y \\ \theta_y \\ w \end{bmatrix}_{\mathrm{II}} \end{bmatrix} = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -n\Delta x & 1 & 0 & 0 & 0 & 0 & -n\Delta x & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -\dfrac{n^2\Delta x^2}{2EI_y} & \dfrac{n\Delta x}{EI_y} & -1 & 0 & 1 & 0 & -\dfrac{n^2\Delta x^2}{2EI_y} & \dfrac{n\Delta x}{EI_y} \\ \dfrac{n^3\Delta x^3}{3EI_y} & -\dfrac{n^2\Delta x^2}{2EI_y} & 0 & -1 & -n\Delta x & 1 & \dfrac{n^3\Delta x^3}{3EI_y} & -\dfrac{n^2\Delta x^2}{2EI_y} \end{array}\right] \left[\begin{array}{c} qn\Delta x \\ q\dfrac{n^2\Delta x^2}{2} \\ q\dfrac{n^3\Delta x^3}{6EI_y} \\ -q\dfrac{n^4\Delta x^4}{24EI_y} \\ \hline 0 \\ 0 \\ Q \\ 0 \end{array}\right]
$$

Thus, initial values are $\tilde{\mathbf{e}}(s_{\mathrm{I}})=\left\{qn\Delta x+Q, -q\dfrac{n^2\Delta x^2}{2}-Qn\Delta x, 0, 0\right\}^T$

Once initial value is known $\tilde{\mathbf{e}}(s_{\mathrm{I}})$ solution at any point $i+1$ will be given by:

$$
\begin{bmatrix} V_z \\ M_y \\ \theta_y \\ w \end{bmatrix}_{i+1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ (i+1)\Delta x & 1 & 0 & 0 \\ \dfrac{(i+1)^2\Delta x^2}{2EI_y} & \dfrac{(i+1)\Delta x}{EI_y} & 1 & 0 \\ -\dfrac{(i+1)^3\Delta x^3}{6EI_y} & -\dfrac{(i+1)^2\Delta x^2}{2EI_y} & -(i+1)\Delta x & 1 \end{bmatrix} \begin{bmatrix} qn\Delta x+Q \\ -q\dfrac{n^2\Delta x^2}{2}-Qn\Delta x \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -q(i+1)\Delta x \\ -q\dfrac{(i+1)^2\Delta x^2}{2} \\ -q\dfrac{(i+1)^3\Delta x^3}{6EI_y} \\ q\dfrac{(i+1)^4\Delta x^4}{24EI_y} \end{bmatrix}
$$

VI. CONCLUSIONS

Boundary Equations are added to extend the presented Finite Transfer Method (FTM) that solves systems of linear ODE's. Applying a proper numerical approximation, Finite Transfer Equations are obtained. The fourth order Runge-Kutta scheme offers accurate results. The use of a recurrence strategy permits obtaining the General Solution that relates unknown functions at different point of the domain where boundary equations could be applied.

These boundary equations are notated in matricial form and incorporated to the algebraic system. The dimension of the resultant algebraic system is always constant and equal to the double of the number of functions in the system, regardless of the intervals adopted, without the need of defining a mesh. The showed method is general, consistent and easy to implement in a software application.

The FTM could solve either initial or boundary conditions.

The procedure is suitable to determine the structural behaviour of the classical problem of an arbitrary curved beam element. Normally this problem is formulated in a compact energy equation form, but here the research is approached in an extended system of differential equations.

With this approach, there is no need to distinguish between statically determinate or indeterminate beams, no need to define reactions in the extremes and no need of extra formulation (virtual work principle, Castigliano's theorems, or energy formulation). An example of a beam under bending moment effect is presented to show the different steps of the procedure exposed.

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