A Weighted Ostrowski- Grüss Type Inequality and Applications

Ather Qayyum *

Abstract—**In this paper, we establish a weighted Ostrowski - Grüss type inequality for twice differentiable mappings in terms of lower and upper bounds of the second derivative. Applications in numerical integration are also given.**

Keywords: ***Ostrowski inequality, Grüss inequality, weight function, numerical integration***

1 INTRODUCTION

In 1938, Ostrowski first proclaimed his inequality for differentiable mappings. Many years ago, Newton-Cotes type quadrature rules have been examined extensively. Dragomir and Wang [9], have investigated the mid-point, trapezoid and Simpson rules with the aim of obtaining bounds on the quadrature rules in terms of a variety of Lebescue spaces involving, at most, the first derivative. In addition, the current approach of obtaining the bounds, for a particular quadrature rule, have depended on the use of peano kernel. The general approach in the past has involved the assumption of bounded derivatives of degree greater than one. The partitioning is halved until the desired accuracy is obtained [4]. The work by Dragomir and Wang [9] aims at obtaining a priori estimates of the partition required in order to obtain a particular bound on the error. In [6], the authors emphasized, with the help of the modern theory of inequalities and by the use of peano kernels, the methods and some results of obtaining bounds for quadrature rules consisting of, at most, three points and depending on the second derivative.

In this paper, our aim is to generalize the results obtained in [6], by the use of weighted peano Kernel [5]. In this way, we can get a wide variety of results.

In 1938, Ostrowski [3, p.468] proved the following integral inequality:

Theorem 1.1 Let $f : I \subseteq \rightarrow \Re \rightarrow \Re$ be a differentiable mapping on I^0 (I^0 is the interior of I), and let $a, b \in I^0$ with $a < b$. If $f' : (a, b) \rightarrow \Re$ is bounded on (a, b), i.e

$$\|f'\|_\infty = \sup_{t\in[a,b]} |f'(t)| < \infty$$

then we have the inequality

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right] (b-a)\|f'\|_\infty$$

for all $x \in [a, b]$. The constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one.

The integral inequality that establishes a connection between the integral of the product of two functions and the product of the integrals is known in the literature as the Grüss inequality (see for example [2, p.296]). The inequality is as follows:

Theorem 1.2 Let $f,\ g : [a, b] \rightarrow \Re$ be integrable functions such that $\varphi \leq f(x) \leq \Phi$ and $\gamma \leq g(x) \leq \Gamma$, for all $x \in [a, b]$,

where $\varphi,\ \Phi,\ \gamma,\ \Gamma$ are constants. Then we have

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx.\frac{1}{b-a}\int_a^b g(x)dx\right| \leq \frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma). \quad (1)$$

where the constant $\frac{1}{4}$ is sharp.

In [8], S. S. Dragomir and S. Wang proved the following Ostrowski type inequality in terms of lower and upper bounds of the first derivative.

Theorem 1.3 Let $f : [a, b] \rightarrow \Re$ be continuous on $[a, b]$ and differentiable on (a, b), and the first derivative satisfies the condition:

$$\gamma \leq f'(x) \leq \Gamma \quad \text{for all} \quad x \in [a, b],$$

*The author is grateful to University of Ha'il for providing an opportunity and research facility to complete this work. Mailing address: Department of Mathematics, University of Ha'il, Saudi Arabia. Email: ather@uoh.edu.sa.

Then

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt - \left(\frac{f(b)-f(a)}{b-a}\right)\right.$$
$$\left.\left(x - \frac{a+b}{2}\right)\right| \leq \frac{1}{4}\,(b-a)\,(\Gamma - \gamma)\,, \qquad (2)$$

for all $x \in [a,b]$.

In [6, $p.25$], S. S. Dragomir and N. S. Barnett, proved the following inequality.

Theorem 1.4 Let $f : [a,b] \to \Re$ be continuous on $[a,b]$ and twice differentiable function on (a,b), whose second derivative $f'' : (a,b) \to \Re$ is bounded on (a,b). Then we have the inequality

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt - \left(\frac{f(b)-f(a)}{b-a}\right)\right.$$
$$\left.\left(x - \frac{a+b}{2}\right)\right| \leq \frac{1}{2}\left\{\left[\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2} + \frac{1}{4}\right]^2 + \frac{1}{12}\right\}$$
$$(b-a)^2\,\|f\,''\|_\infty \qquad (3)$$

for all $x \in [a,b]$.

2 MAIN RESULTS

In this section we review the weighted version [1] of (3) and in section 3 apply the result in numerical integration.

We assume that the weight function $w : (a,b) \longrightarrow [0,\infty)$ is integrable, non negative and

$$\int_a^b w(t)dt < \infty.$$

The domain of w may be finite or infinite and w may vanish at the boundary points. We denote

$$m(a,b) = \int_a^b w(t)dt.$$

We now give our main theorem.

Theorem 2.1 Let w be as defined above and $f : [a,b] \to \Re$ be continuous on $[a,b]$ and twice differentiable function on (a,b) whose second derivative $f'' : (a,b) \to \Re$ is bounded on (a,b). Then we have the inequality

$$\left| f(x) - \frac{1}{m\,(a,b)} w\,(x)\,(b-a)\left(x - \frac{a+b}{2}\right) f\,'\,(x)\right.$$
$$\left. - \frac{1}{m\,(a,b)}\int_a^b f(t)w(t)dt\right| \leq \frac{1}{2}\;\|f\,''\|_{w,\infty}$$
$$\left\{\left(\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2} + \frac{1}{4}\right)^2 + \frac{1}{12}\right\}\frac{(b-a)^4}{m^2\,(a,b)} \qquad (4)$$

for all $x \in [a,b]$.

Proof The following weighted integral inequality is proved in [1].

$$f(x) = \frac{1}{m(a,b)}\int_a^b P_w(x,t)f\,'(t)dt + \frac{1}{m(a,b)}\int_a^b f(t)w(t)dt \qquad (5)$$

for all $x \in [a,b]$.

In [6, $p.24$], N. S Bernett, P. Cerone, S. S. Dragomir, J. Roumeliotis and A. Sofo used the peano kernel $P : [a,b]^2 \longrightarrow \Re$ is given by

$$P(x,t) = \begin{cases} t-a & \text{if} \quad t \in [a,x] \\ t-b & \text{if} \quad t \in (x,b] \end{cases}$$

where $t \in [a,b]$.

but here we use the weighted peano Kernel, $P(.,.) : [a,b]^2 \longrightarrow \Re$ is given by

$$P_w(x,t) = \begin{cases} \int_a^t w(u)du & \text{if} \quad t \in [a,x] \\ \int_b^t w(u)du & \text{if} \quad t \in (x,b] \end{cases}$$

for all $t \in [a,b]$.

Applying the identity (5) for $f\,'(.)$, we can state

$$f\,'(t) = \frac{1}{m(a,b)}\int_a^b P_w(t,s)f\,''(s)ds + \frac{1}{m(a,b)}\int_a^b f\,'(s)\,w(s)ds.$$

Substituting $f'(t)$ in the right hand side of (5), we obtain

$$\begin{aligned} f(x) &= \frac{1}{m(a,b)}\int_a^b P_w(x,t)\left[\frac{1}{m(a,b)}\int_a^b P_w(t,s)\right. \\ &\quad \left. f''(s)ds+\frac{1}{m(a,b)}\int_a^b f'(s)w(s)ds\right]dt \\ &\quad +\frac{1}{m(a,b)}\int_a^b f(t)w(t)dt \\ &= \frac{1}{m^2(a,b)}\int_a^b\int_a^b P_w(x,t)P_w(t,s)f''(s)dsdt+ \\ &\quad \frac{1}{m^2(a,b)}\int_a^b P_w(x,t)dt\int_a^b f'(s)w(s)ds+ \\ &\quad \frac{1}{m(a,b)}\int_a^b f(t)w(t)dt. \end{aligned} \tag{6}$$

Now by using First Mean Value Theorem for Integration, we get

$$\int_a^b f'(s)w(s)ds = f'(x)m(a,b)$$

and

$$\int_a^b P_w(x,t)dt = w(x)(b-a)\left(x-\frac{a+b}{2}\right).$$

Thus (6) becomes

$$\begin{aligned} f(x) &= \frac{1}{m(a,b)}w(x)(b-a)\left(x-\frac{a+b}{2}\right)f'(x)+ \\ &\quad \frac{1}{m(a,b)}\int_a^b f(t)w(t)dt+\frac{1}{m^2(a,b)}\int_a^b\int_a^b P_w(x,t) \\ &\quad P_w(t,s)f''(s)dsdt. \end{aligned} \tag{7}$$

for all $x\in[a,b]$.

Now using the identity (7), we get

$$\begin{aligned} &\left|f(x)-\frac{1}{m(a,b)}w(x)(b-a)\left(x-\frac{a+b}{2}\right)f'(x)-\right. \\ &\left.\frac{1}{m(a,b)}\int_a^b f(t)w(t)dt\right| \\ &\leq \frac{1}{m^2(a,b)}\int_a^b\int_a^b |P_w(x,t)|\ |P_w(t,s)|\ |f''(s)|dsdt. \end{aligned} \tag{8}$$

By using Second Mean Value Theorem for Integration, we get

$$\int_a^b |P_w(t,s)|ds = \frac{1}{2}w(t)\left[(t-a)^2+(b-t)^2\right]$$

and

$$\begin{aligned} A \; : &= \int_a^b |P_w(x,t)|\left[\frac{1}{2}w(t)\left((t-a)^2+(b-t)^2\right)\right]dt \\ &= \frac{1}{2}w(t)\left[\int_a^x\left\{\int_a^t w(u)du\right\}\left((t-a)^2+(b-t)^2\right)dt\right. \\ &\quad \left.+\int_x^b\left\{\int_t^b w(u)du\right\}\left((t-a)^2+(b-t)^2\right)dt\right] \\ &= \frac{1}{2}w(t)\left[\int_a^x w(t)(t-a)\left((t-a)^2+(b-t)^2\right)\right. \\ &\quad \left.+dt\int_x^b w(t)(b-t)\left((t-a)^2+(b-t)^2\right)dt\right] \\ &= \frac{1}{2}(w(t))^2\left[\int_a^x\left((t-a)^3+(t-a)(b-t)^2\right)dt\right. \\ &\quad \left.+\int_x^b\left((b-t)^3+(b-t)(b-t)^2\right)dt\right]. \end{aligned}$$

Note that

$$\begin{aligned} \int_a^x (t-a)^3dt &= \frac{(x-a)^4}{4}, \\ \int_x^b (b-t)^3dt &= \frac{(x-b)^4}{4}, \end{aligned}$$

$$\begin{aligned} \int_a^x (t-a)(b-t)^2dt &= -\frac{1}{3}(x-a)(b-x)^3-\frac{1}{12} \\ &\quad (b-x)^4+\frac{1}{12}(b-a)^4, \end{aligned}$$

and

$$\begin{aligned} \int_x^b (b-t)(t-a)^2dt &= -\frac{1}{3}(b-x)(x-a)^3-\frac{1}{12} \\ &\quad (x-a)^4+\frac{1}{12}(b-a)^4. \end{aligned}$$

Consequently, we have

$$A=\frac{1}{12}(w(t))^2 \left[\begin{array}{c} (x-a)^4+(b-x)^4-2(x-a)(b-x)^3 \\ -2(b-x)(x-a)^3+(b-a)^4 \end{array}\right].$$

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Now observe that

$$(x-a)^4+(b-x)^4=\left[(x-a)^2+(b-x)^2\right]^2-2(x-a)^2(b-x)^2$$

and

$$-2(x-a)(b-x)^3-2(b-x)(x-a)^3=2(x-a)(b-x)\left[(x-a)^2+(b-x)^2\right]$$

Then

$$\begin{aligned} B &= 12A=\left[(x-a)^2+(b-x)^2-(x-a)(b-x)\right]^2 \\ &\quad -3(x-a)^2(b-x)^2+(b-a)^4. \end{aligned}$$

However, a simple calculation shows that

$$(x-a)^2+(b-x)^2=\frac{1}{2}(b-a)^2+2\left(x-\frac{a+b}{2}\right)^2$$

and as

$$(x-a)^2+(b-x)^2+2(x-a)(b-x)=(b-a)^2,$$

we get

$$2(x-a)(b-x)=(b-a)^2-\left[(x-a)^2+(b-x)^2\right],$$

i.e

$$\begin{aligned} (x-a)(b-x) &= \frac{1}{2}(b-a)^2-\frac{1}{2}\left[(x-a)^2+(b-x)^2\right] \\ &= \frac{1}{4}(b-a)^2-\left(x-\frac{a+b}{2}\right)^2. \end{aligned}$$

Consequently

$$B=6\left(x-\frac{a+b}{2}\right)^2+3(b-a)^2\left(x-\frac{a+b}{2}\right)^2+\frac{7}{8}(b-a)^4,$$

and then

$$A=\frac{1}{12}\left[6\left(x-\frac{a+b}{2}\right)^2+3(b-a)^2\left(x-\frac{a+b}{2}\right)^2+\frac{7}{8}(b-a)^4\right].$$

Now using the inequality (8) and simple algebraic manipulations, we get

$$\begin{aligned} &\left|f(x)-\frac{1}{m(a,b)}w(x)(b-a)\left(x-\frac{a+b}{2}\right)f'(x)-\frac{1}{m(a,b)}\int_a^b f(t)w(t)dt\right| \leq \frac{1}{12m^2(a,b)} \\ &\left[\begin{array}{c} 6\left(x-\frac{a+b}{2}\right)^2+3(b-a)^2\left(x-\frac{a+b}{2}\right)^2 \\ +\frac{7}{8}(b-a)^4 \end{array}\right] \\ &\sup_{t\in[a,b]}(w(t))^2\ |f''(t)| \\ &= \frac{1}{2}\|f''\|_{w,\infty}\left[\left\{\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}+\frac{1}{4}\right\}^2+\frac{1}{12}\right]\frac{(b-a)^4}{m^2(a,b)}. \end{aligned}$$

Remark 2.2 The inequality (4) is a modified inequality. For $w(t)=1$, in (4), we get (3). If we put another value of $w(t)$, it may result in some other inequalities.

Corollary 2.3 Let f be as in Theorem (2.1), then we have perturbed mid-point inequality:

$$\begin{aligned} &\left|f\left(\frac{a+b}{2}\right)-\frac{1}{m(a,b)}\int_a^b f(t)w(t)dt\right| \\ &\leq \frac{7}{96}\frac{(b-a)^4}{m^2(a,b)}\ \|f''\|_{w,\infty} \end{aligned} \tag{9}$$

Corollary 2.4 Let f be as in Theorem (2.1), then we have the following perturbed trapezoid inequality:

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2}+\frac{1}{2m(a,b)}(b-a)\left(\frac{a-b}{2}\right)[w(b)\right. \\ &\left. f'(b)-w(a)f'(a)]-\frac{1}{m(a,b)}\int_a^b f(t)w(t)dt\right| \\ &\leq \frac{1}{6}\frac{(b-a)^4}{m^2(a,b)}\ \|f''\|_{w,\infty} \end{aligned} \tag{10}$$

Proof Put in (4), $x=a$ and $x=b$ to get

$$\left|f(a)-\frac{1}{m(a,b)}w(a)(b-a)\left(\frac{a-b}{2}\right)f'(a)-\frac{1}{m(a,b)}\int_a^b f(t)w(t)dt\right| \leq \frac{1}{6}\frac{(b-a)^4}{m^2(a,b)}\ \|f''\|_{w,\infty}$$

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and

$$\left| f(b) + \frac{1}{m(a,b)} w(b)(b-a)\left(\frac{a-b}{2}\right) f'(b) - \frac{1}{m(a,b)} \int_a^b f(t)w(t)dt \right| \leq \frac{1}{6}\frac{(b-a)^4}{m^2(a,b)} \|f''\|_{w,\infty}$$

respectively.

Summing the above two inequalities, using the triangle inequality and dividing by 2, we get (10).

3 Applications in Numerical Integration.

Let $I_n : a = x_0 < x_1 < x_2 < .... < x_{n-1} < x_n = b$ be a division of the interval $[a,b]$, $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, 1, ..., n-1)$ a sequence of intermediate points $h_i := x_{i+1} - x_i$ $(i = 0, 1, ...n-1)$. We have the following quadrature formula:

Theorem 3.1 *Let* $f : [a,b] \to \Re$ be continuous on $[a,b]$ and a twice differentiable on (a,b), whose second derivative $f'' : (a,b) \to \Re$ is bounded on (a,b), then we have the quadrature formula for all $x \in (a,b)$

$$\int_a^b w(t) f(t)dt = A(f, \xi, I_n) + R(f, \xi, I_n), \tag{11}$$

where

$$A(f, I, w, \xi_i) = \sum_{i=0}^{n-1} m(x_i, x_{i+1}) f(\xi_i) - \sum_{i=0}^{n-1} \left[w(\xi_i) \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right) h_i f'(\xi_i) \right] \tag{12}$$

and the remainder $R_G(f, \xi, I_h)$ satisfies the estimation

$$|R(f, f', \xi, I_n)| \leq \frac{1}{2} \|f''\|_{w,\infty} \sum_{i=0}^{n-1} \left\{ \left(\frac{\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2}{(h_i)^2} + \frac{1}{4} \right)^2 + \frac{1}{12} \right\} \frac{h_i^4}{m(x_i, x_{i+1})}, \tag{13}$$

for all ξ_i as above.

Proof

Apply Theorem (2.1) on the interval $[x_i, x_{i+1}]$, $(i = 0, 1, ...n-1)$, to obtain

$$\left| f(\xi_i) m(x_i, x_{i+1}) - w(\xi_i)(h_i)\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right) f'(\xi_i) - \int_{x_i}^{x_{i+1}} f(t)w(t)dt \right| \leq \frac{1}{2} \|f''\|_{w,\infty} \left\{ \left(\frac{\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2}{(h_i)^2} + \frac{1}{4} \right)^2 + \frac{1}{12} \right\} \frac{h_i^4}{m(x_i, x_{i+1})}$$

for all $\xi_i \in [x_i, x_{i+1}]$, where $h_i = x_{i+1} - x_i$, $(i = 0, 1, ...n-1)$.

Summing over i from 0 to $(n-1)$ and using the generalized triangular inequality, we get the desired inequality (13).

Remark 3.2

If we choose $\xi_i = \frac{x_i + x_{i+1}}{2}$, we recapture the weighted mid-point quadrature formula

$$\int_a^b w(t) f(t)dt = A_M(f, \xi, I_n) + R_M(f, \xi, I_n)$$

where,

$$A_M(f, \xi, I_n) = \sum_{i=0}^{n-1} m(x_i, x_{i+1}) f\left(\frac{x_i + x_{i+1}}{2}\right)$$

and the remainder $|R_M(f, \xi, I_n)|$ satisfies the estimation:

$$|R(f, \xi, I_n)| \leq \frac{11}{96} \sum_{i=0}^{n-1} \frac{h_i^4}{m(x_i, x_{i+1})} \|f''\|_{w,\infty}.$$

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