An Efficient Technique to Treat Singularities when Applying BEM with Quadratic Boundary Elements to the Problem of Compressible Fluid Flow

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***Abstract*— This paper is focused on solving the problem of the 2D compressible subsonic fluid flow around an obstacle using BEM with higher order boundary elements, with special consideration regarding to the treatment of singulatities. The 2D problem of a compressible fluid flow around an obstacle is equivalent with a singular boundary integral, obtained in terms of primary variables of the problem, the components of the velocity field. This singular boundary integral equation is solved by using quadratic isoparametric boundary elements. The problem is finally reduced to a linear system of equations. Aspects regarding the evaluation of the matrix coefficients are presented and a special attention is given to the treatment of integrals of singular kernels. A method based on the definition of the Cauchy Principal Value of an integral is developed. The method described is implemented into a computer code made in MathCAD and numerical results are obtained for different types of obstacles. We validate the computer through an analytical checking, made by comparing the numerical results with the exact solutions that exist in some particular cases, which are in very good agreement.**

***Index Terms*—boundary element method, compressible fluid flow, linear boundary elements, singular boundary integral equation, singular kernels.**

I. INTRODUCTION

The 2D problem of the compressible fluid flow around an obstacle, that represents in fact the problem of an infinite span airfoil in a subsonic flow, is solved in this paper using a BEM with quadratic boundary elements.

This problem has been studied by many authors by using different types of numerical methods, as finite differences, finite elements, Galerkin collocation methods, and other techniques, but mostly as considering the incompressible case only, and using the potential or stream function as initial unknowns of the problem. Even when BEM was applied the velocity and pressure field are found after finding the potential or stream function, through a differentiation technique, and therefore new errors are introduced at this stage.

The Boundary Element Method (BEM) is a powerful numeric technique used to solve many kinds of problems of continuum mechanics with boundary values, which consists in two big steps [1], [2], [3]. First, a boundary integral formulation equivalent with the mathematical model of the problem must be obtained, and then this boundary integral, that is usually a singular one, must be solved.

In paper [4] a boundary integral formulation in terms of velocity field is deduced and so, directly solving this singular boundary integral we can find the perturbation velocity, without using a differentiation technique, so with less errors. In a such approach errors arise only at the discretization stage, and when evaluating matrix coefficients using numerical integration. We propose in this paper a solution based on higher order boundary elements for the singular boundary integral equation, to ensure a global continuity for the unknown function.

The problem to solve is so reduced to a linear system of equations, the unknowns being the nodal values of the functions to be found.

The calculation of the matrix coefficients requires several evaluations of integrals with singular and non-singular kernels. An efficient and accurate method of computing the non-singular integrals is to employ Gaussian integration schemes, but the treatment of the singular integrals is more difficult to carry out.

An integral whose integrand reaches an infinite value at one or more points in the domain of integration, named singular integral, is in general, defined by eliminating a small space including the singularity, and then taking the limit as this small space disappears. It is said that integral can converges, and in this case, it is said to exist. A singular integral can be understood in the sense of Cauchy Principal Value or in Hadamard sense [5].

Integral of singular kernels evaluation is one of the most important and difficult step in solving problems with BEM and has a big influence on the numerical solutions accuracy. Their evaluation needs a special attention and has a great practical importance, because they lead to the dominant and concentrated near the diagonal matrix coefficients, and they are very important for a well-conditioned behavior of the matrix.

For the bi dimensional steady subsonic ideal

Manuscript received March 23, 2009.
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ISBN:978-988-18210-1-0

compressible fluid flow around a body the boundary integral equation, obtained by applying an indirect technique with a distribution of sources on the boundary, has the form (see [4]):

$$\left(n_x^{0\,2}+\beta^2 n_y^{0\,2}\right)f(\bar{x}_0)+ +\frac{1}{\pi}\int_C' f(\bar{x})\frac{(x-x_0)n_x^0+\beta^2(y-y_0)n_y^0}{|\bar{x}-\bar{x}_0|^2}ds = 2\beta n_x^0 \quad (1),$$

where n_x^0, n_y^0 are the components of the normal unit vector outward the fluid (inward the body) in the point $\bar{x}_0$, $\beta=\sqrt{1-M^2}$ (for the subsonic flow, M= Mach number), and f is the unknown function, the intensity of the sources, presumed to satisfy a hölder condition. The sign " ' " denotes the Cauchy principal value of the integral.

II. Quadratic Isoparametric Boundary Elements

When we use quadratic isoparametric boundary elements to solve the singular boundary integral equation (1), the geometry and the unknown, have local a quadratic variation, on each of the boundary elements obtained after the discretization of the boundary. In paper [6] constant and linear boundary elements are used, and in [7] also quadratic ones, but the treatment of singularities was carried out in a different manner.

So we consider that the boundary is divided into N unidimensional quadratic boundary elements, noted L_i, each of them with three nodes: two extreme nodes and an interior one. Considering that the discrete equation is satisfied in every node, we obtain:

$$\left(n_x^{j\,2}+\beta^2 n_y^{j\,2}\right)f(\bar{x}_j)+ +\frac{1}{\pi}\sum_{i=1}^{N}\int_{L_i} f(\bar{x})\frac{(x-x_j)n_x^j+\beta^2(y-y_j)n_y^j}{|\bar{x}-\bar{x}_j|^2}ds = 2\beta n_x^j \quad (2)$$

For describing the geometry and the behavior of the unknown f, on a boundary element, we use a quadratic model, with the same set of basic functions, noted N_1, N_2, N_3. Using a local system of coordinates (the intrinsic system) we have:

$$\left(n_x^{j\,2}+\beta^2 n_y^{j\,2}\right)f(\bar{x}_j)+\frac{1}{\pi}\sum_{i=1}^{N}\left(\sum_{l=1}^{3}a_{ij}^l f_l^i\right)=2\beta n_x^j \quad (3)$$

where

$$a_{ij}^l=\int_{-1}^{1}N_l\frac{\left([N]\{x^i\}-x_j\right)n_x^j+\beta^2\left([N]\{y^i\}-y_j\right)n_y^j}{\left|[N]\{\bar{x}\}-\bar{x}_j\right|^2}J(\xi)d\xi \quad (4)$$

$[N]$ is a matrix with a single line, $[N]=(N_1\ N_2\ N_3)$,

$N_1(\xi)=\frac{\xi(\xi-1)}{2}$, $N_2(\xi)=1-\xi^2$, $N_3(\xi)=\frac{\xi(\xi+1)}{2}$, $\xi\in[-1,1]$

$\{x^i\},\{y^i\}$ the column matrices made with the global coordinates of the nodes of the boundary element L_i.

There are used two systems of notation: a global and a local one (global- f_j is the value of f for the node number j, $j=\overline{1,2N}$ -and local- $f_l^i, l=\overline{1,3}, i=\overline{1,N}$ is the value for the node number l of element i).

Returning to the global system of notation, we obtaine the following linear algebric system:

$$[A]\{f\}=\{B\},\ A\in M_{2N}(R),\ \{f\},\{B\}\in R^{2N}$$
$$B_j=2\pi\beta n_x^j. \quad (11)$$

$\{f\}$ being the column matrix made with the nodal values of the unknown function

After solving this system, which has usually a big number of unknowns and equations, using therefore a preconditionary technique, we find the intensity nodal values and then the components of the other fields of interest. But, first we have to evaluate matrix coefficients in order to obtain the system.

III. The Expressions of the Coefficients of Matrix A and the Evaluation of the Nonsingular Integrals

For getting the matrix $[A]$ we need to evaluate the integrals that appeare. One of them are usual integrals, but the other are singular integrals.
Denoting by:

$$m_i=x_1^i+x_3^i-2x_2^i,\ n_i=x_3^i-x_1^i,\ u_{ij}=x_2^i-x_j,$$

$$M_i=y_1^i+y_3^i-2y_2^i,\ N_i=y_3^i-y_1^i,\ U_{ij}=y_2^i-y_j,$$

$$a_i=\frac{m_i^2+M_i^2}{4}\quad aa_i=\frac{n_i^2+N_i^2}{4}\quad b_i=\frac{m_in_i+M_iN_i}{2},$$

$$c_{ij}=aa_i+m_iu_{ij}+M_iU_{ij},\ d_{ij}=n_iu_{ij}+N_iU_{ij},$$

$$e_{ij}=u_{ij}^2+U_{ij}^2,\quad i,j=\overline{1,2N}$$

$$J(\xi)=\sqrt{4a_i\xi^2+2b_i\xi+aa_i},$$

we get for $\bar{x}\in L_i$,

$$\left\|\bar{x}-\bar{x}_j\right\|^2=a_i\xi^4+b_i\xi^3+c_{ij}\xi^2+d_{ij}\xi+e_{ij}\underset{not}{=}N_{ij}(\xi)=N_{ij} \quad (7)$$

Then, denoting by

$$I_{ij}^k=\int_{-1}^{1}\frac{\xi^k}{N_{ij}(\xi)}J(\xi)d\xi,\quad i,j=\overline{1,2N},\ k=0,1,2,3,4, \quad (8)$$

we obtaine the following expressions for the coefficients:

$$a_{ij}^1=\frac{m_i n_x^j+\beta^2 M_i n_y^j}{4}I_{ij}^4+\frac{(n_i-m_i)n_x^j+\beta^2(N_i-M_i)n_y^j}{4}I_{ij}^3+$$

$$+\frac{(2u_{ij}-n_i)n_x^j+\beta^2(2U_{ij}-N_i)n_y^j}{4}I_{ij}^2-\frac{u_{ij}n_x^j+\beta^2 U_{ij}n_y^j}{2}I_{ij}^1$$

$$a_{ij}^2=-\frac{m_i n_x^j+\beta^2 M_i n_y^j}{2}I_{ij}^4-\frac{n_i n_x^j+\beta^2 N_i n_y^j}{2}\left(I_{ij}^3-I_{ij}^1\right)-$$

$$-\frac{(2u_{ij}-m_i)n_x^j+\beta^2(2U_{ij}-M_i)n_y^j}{4}I_{ij}^2+\left(u_{ij}n_x^j+\beta^2 U_{ij}n_y^j\right)I_{ij}^0$$

$$a_{ij}^3=\frac{m_i n_x^j+\beta^2 M_i n_y^j}{4}I_{ij}^4+\frac{(n_i+m_i)n_x^j+\beta^2(N_i+M_i)n_y^j}{4}I_{ij}^3+$$

$$+\frac{(2u_{ij}+n_i)n_x^j+\beta^2(2U_{ij}+N_i)n_y^j}{4}I_{ij}^2+\frac{u_{ij}n_x^j+\beta^2 U_{ij}n_y^j}{2}I_{ij}^1. \tag{9}$$

For $j\neq 2i-1,2i,2i+1$ this integrals are nonsingular integrals $(\forall)\ l=1,2,3$, and they can be evaluated with usual numerical integration techniques, or with a math software, using a computer.

IV. A method based on Cauchy Principal Value of an integral for singular integrals evaluation

We give special attention to the treatment of singular integrals. Errors due to them have a high influence on the numerical solutions accuracy as results for example from papers [8], [9]. If we apply a truncation method which consists of isolating the singularity while using a Gauss quadrature method, for evaluating the singular integrals I_{ij}^k, $j=2i-1,2i,2i+1$, the numerical results are not satisfactory and large errors appear.

Another method we have applied, is based on the definition of Cauchy Principal Value of an integral, and allows us to obtain very small errors between the numerical and the exact solution of the problem, when the latter exists.

We start applying this method using the following expressions for coefficients $a_{ij}^l, l=1,2,3$:

$$a_{ij}^1=\frac{1}{4}\int_{-1}^{1}\frac{A_{ij}\xi^4+B_{ij}\xi^3+C_{ij}\xi^2+D_{ij}\xi}{a_i\xi^4+b_i\xi^3+c_{ij}\xi^2+d_{ij}\xi+e_{ij}}\sqrt{4a_i\xi^2+2b_i\xi+aa_i}\,d\xi$$

equivalent with

$$a_{ij}^1=\frac{1}{4}\int_{-1}^{1}\frac{P(\xi)}{Q(\xi)}\sqrt{4a_i\xi^2+2b_i\xi+aa_i}\,d\xi\ ,$$

where

$$P(\xi)=A_{ij}\xi^4+B_{ij}\xi^3+C_{ij}\xi^2+D_{ij}\xi,$$

$$Q(\xi)=a_i\xi^4+b_i\xi^3+c_{ij}\xi^2+d_{ij}\xi+e_{ij} \tag{13}$$

For $j=2i-1$, the integrals have singularities at the first node of element number *i*, node corresponding to the intrinsic coordinate (-1).

After some manipulations, we can write that:

$$P(\xi)=(\xi+1)P_1(\xi)\ ,\ P_1(-1)\neq 0, P_1(\xi)\neq 0$$
$$Q(\xi)=(\xi+1)^2 Q_1(\xi)\ \ Q_1(-1)\neq 0, Q_1(\xi)\neq 0,\ \forall\xi\in[-1,1].$$

We deduce that:

$$a_{ij}^1=\frac{1}{4}\int_{-1}^{1}\frac{P_1(\xi)}{Q_1(\xi)(\xi+1)}\sqrt{4a_i\xi^2+2b_i\xi+aa_i}\,d\xi\cdot$$

We try to separate the above integrals into two parts, one regular and the other with a singularity. We consider for this the following relation:

$$P_1(\xi)=C(\xi)(\xi+1)+r, r\in R.$$

We further obtain relation:

$$a_{ij}^1=\frac{1}{4}\int_{-1}^{1}\left(\frac{C(\xi)}{Q_1(\xi)}+\frac{r}{Q_1(\xi)(\xi+1)}\right)\sqrt{4a_i\xi^2+2b_i\xi+aa_i}\,d\xi$$

And we finally deduce that:

$$a_{ij}^1=\frac{1}{4}\int_{-1}^{1}\frac{A_{ij}\xi^2+(B_{ij}-2A_{ij})\xi}{a_i\xi^2+(b_i-2a_i)\xi+c_{ij}-2b_i+3a_i}\sqrt{4a_i\xi^2+2b_i\xi+aa_i}\,d\xi+$$
$$+\frac{1}{4}\int_{-1}^{1}\frac{C_{ij}-2B_{ij}+3A_{ij}}{(a_i\xi^2+(b_i-2a_i)\xi+c_{ij}-2b_i+3a_i)(\xi+1)}\sqrt{4a_i\xi^2+2b_i\xi+aa_i}\,d\xi$$

So, as we can see, only the second integral still has a weakly singularity, and only for its evaluation we use the method that isolates the singularity. As shown in paper [10] this method offers good results for improper integrals numerical evaluations, integrals in which the integrand doesn't quickly oscillate near the singularity. We get:

$$a_{ij}^1=\frac{1}{4}\int_{-1}^{1}\frac{A_{ij}\xi^2+(B_{ij}-2A_{ij})\xi}{a_i\xi^2+(b_i-2a_i)\xi+c_{ij}-2b_i+3a_i}\sqrt{4a_i\xi^2+2b_i\xi+aa_i}\,d\xi+$$
$$+\frac{1}{4}\int_{-1+\varepsilon}^{1}\frac{C_{ij}-2B_{ij}+3A_{ij}}{(a_i\xi^2+(b_i-2a_i)\xi+c_{ij}-2b_i+3a_i)(\xi+1)}\sqrt{4a_i\xi^2+2b_i\xi+aa_i}\,d\xi \tag{14}$$

Applying the same idea we obtain, in this case, for the other coefficients the following expressions:

$$a_{ij}^2=\int_{-1}^{1}\frac{-A_{ij}\xi^2+(2A_{ij}-B_{ij})\xi-3A_{ij}+2B_{ij}-C_{ij}}{a_i\xi^2+(b_i-2a_i)\xi+c_{ij}-2b_i+3a_i}\sqrt{4a_i\xi^2+2b_i\xi+aa_i}\,d\xi$$

$$a_{ij}^3=\frac{1}{4}\int_{-1}^{1}\frac{A_{ij}\xi^2+(B_{ij}-2A_{ij})\xi}{a_i\xi^2+(b_i-2a_i)\xi+c_{ij}-2b_i+3a_i}\sqrt{4a_i\xi^2+2b_i\xi+aa_i}\,d\xi \tag{15}$$

For $j=2i$, case in which the singularity arises when $\xi=0$, doing like before, we have the following expressions for $a_{ij}^l, l = 1,2,3$:

$$a_{ij}^1 = \frac{1}{4}\int_{-1}^{1}\frac{\left(A_{ij}\xi^2 + B_{ij}\xi + C_{ij}\right)\xi^2}{\left(a_i\xi^2 + b_i\xi + c_{ij}\right)\xi^2}\sqrt{4a_i\xi^2 + 2b_i\xi + aa_i}\,d\xi =$$

.

$$= \frac{1}{4}\int_{-1}^{1}\frac{A_{ij}\xi^2 + B_{ij}\xi + C_{ij}}{a_i\xi^2 + b_i\xi + c_{ij}}\sqrt{4a_i\xi^2 + 2b_i\xi + aa_i}\,d\xi$$

$$a_{ij}^2 = \int_{-1}^{1}\frac{-A_{ij}\xi^2 - B_{ij}\xi - C_{ij}}{a_i\xi^2 + b_i\xi + c_{ij}}\sqrt{4a_i\xi^2 + 2b_i\xi + aa_i}\,d\xi +$$

$$+\left(\int_{\varepsilon}^{1} + \int_{-1}^{-\varepsilon}\right)\left(\frac{B_{ij}}{\left(a_i\xi^2 + b_i\xi + c_{ij}\right)\xi}\sqrt{4a_i\xi^2 + 2b_i\xi + aa_i}\,d\xi\right)$$

$$a_{ij}^3 = \frac{1}{4}\int_{-1}^{1}\frac{A_{ij}\xi^2 + B_{ij}\xi + C_{ij}}{a_i\xi^2 + b_i\xi + c_{ij}}\sqrt{4a_i\xi^2 + 2b_i\xi + aa_i}\,d\xi \,. \tag{16}$$

For $j = 2k+1$, with singularity for $\xi = 1$, analogous, we have:

$$a_{ij}^1 = \frac{1}{4}\int_{-1}^{1}\frac{A_{ij}\xi^2 + \left(B_{ij} + 2A_{ij}\right)\xi}{a_i\xi^2 + \left(b_i + 2a_i\right)\xi + c_{ij} + 2b_i + 3a_i}\sqrt{4a_i\xi^2 + 2b_i\xi + aa_i}\,d\xi$$

$$a_{ij}^2 = \int_{-1}^{1}\frac{-A_{ij}\xi^2 - \left(2A_{ij} + B_{ij}\right)\xi - 3A_{ij} - 2B_{ij} - C_{ij}}{a_i\xi^2 + \left(b_i + 2a_i\right)\xi + c_{ij} + 2b_i + 3a_i}\sqrt{4a_i\xi^2 + 2b_i\xi + aa_i}\,d\xi$$

$$a_{ij}^3 = \frac{1}{4}\int_{-1}^{1}\frac{A_{ij}\xi^2 + \left(B_{ij} + 2A_{ij}\right)\xi}{a_i\xi^2 + \left(b_i + 2a_i\right)\xi + c_{ij} + 2b_i + 3a_i}\sqrt{4a_i\xi^2 + 2b_i\xi + aa_i}\,d\xi +$$

$$+\frac{1}{4}\int_{-1}^{1-\varepsilon}\frac{C_{ij} + 2B_{ij} + 3A_{ij}}{\left(a_i\xi^2 + \left(b_i + 2a_i\right)\xi + c_{ij} + 2b_i + 3a_i\right)\left(\xi - 1\right)}\sqrt{4a_i\xi^2 + 2b_i\xi + aa_i}\,d\xi \tag{17}$$

The coefficients of the velocity's components are evaluated in a same manner.

After solving system (11) and finding the nodal values of the intensities we can compute the components of the velocity on the boundary and then the local pressure coefficient:

$$C_p = \frac{2}{\gamma M^2}\left\{\left[1 + \frac{\gamma - 1}{2}M^2\left(1 - \left(\frac{1+u}{\beta}\right)^2 - v^2\right)\right]^{\frac{\gamma}{\gamma-1}} - 1\right\}$$

if $M \neq 0$, or in case of an incompressible fluid flow with relation:

$$C_p = -u^2 - v^2 - 2u$$

For showing the effectiveness of the method proposed in this paper, especially as regarding the singular coefficients evaluation, we made a computer code in MATHCAD. Numerical results can be obtained for different types of obstacles with a smooth boundary. For making the analytical checking we consider a particular case - an incompressible ideal fluid flow ($\beta = 1$) and a circular obstacle. In this case the problem has an exact solution. The analytical expressions for the dimensionless components of the velocity and the local pressure coefficient are given by the following relations, see [11]:

$$u = -\cos 2\theta \,,\ v = -\sin 2\theta \,,\ cp = -1 + 2\cos 2\theta \,.$$

The computer code ofers the numerical results presented in Fig.1, where a comparison between the numerical solution and the analytical one is made too

We have choosen for ε the value 0.09, and 20 nodes for the boundary discretization.

As we can see from the graphic below, the error between the numerical and the exact solution is very small indeed.

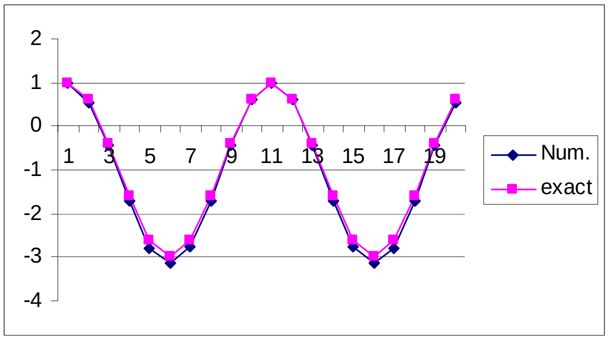

Fig.1. Numerical and exact solution for $\beta = 1$ and a circular obstacle - 20 nodes on the boundary

We have also considered another smooth obstacle, the elliptical obstacle, and an incompressible ideal fluid flow, because in this case the problem has an exact solution too

In [11] the problem of an incompressible fluid flow around an elliptical object is exactly solved. The expression of the perturbed fluid velocity is obtained using the complex potential, given by the following expression:

$$f(z) = \frac{U_\infty}{2}\left[\left(z + \sqrt{z^2 - c^2}\right)e^{-i\alpha} + \left(z - \sqrt{z^2 - c^2}\right)\frac{(a+b)e^{i\alpha}}{a-b}\right]$$

$x = a\cos t,\ y = b\sin t$, $z = x + iy$, and $c^2 = a^2 - b^2$.

The components of the velocity field are in this case:

$$u(z) = \frac{1}{U_\infty}\mathrm{Re}\left(\frac{df}{dz}\right),\ v(z) = -\frac{1}{U_\infty}\mathrm{Im}\left(\frac{df}{dz}\right)$$

We consider an elliptical profile with a=2 and b=1.

Another computer code in MATHCAD gives us the solution for this case. These computer codes can be run for any number of nodes used for the boundary discretization.

In Fig.2, a comparison between the numerical solution and the analytical one is made. We have choosen for ε the value 0.001, and 32 nodes for the boundary discretization.

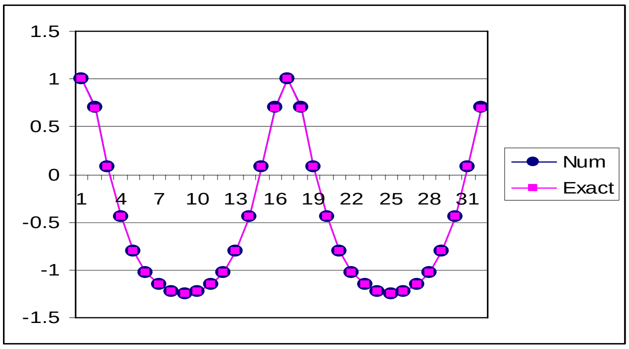

Fig.2. Numerical and exact solution for $\beta = 1$ and an elliptical obstacle - 32 nodes on the boundary.

As we can see, for the elliptical obstacle, the numerical results are in very good agreement with the exact ones too.

V. Conclusions

As we have shown in this paper when applying BEM to solve problems of 2D compressible fluid flow good numerical results can be obtained even when using a small number of nodes for the boundary discretization if a good technique for singular coefficients evaluation is applied.

The Boundary Element Method (BEM) is an efficient numerical technique, which can be used to solve the problem of a 2D compressible fluid flow around obstacles, and generally for solving boundary value problems for systems of partial differential equations.

The principal advantage of the BEM over other numerical methods is the ability to reduce the problem dimension by one, leading to improved computational efficiency.

The equivalent boundary integral formulation is usually a singular boundary integral equation, and after solving it the numerical solution of the problem can be found.

The type of boundary elements used to solve the boundary integral equations plays an important role in applying BEM, because the accuracy of the numerical solution is affected by the approximation models brought into solving through them, but numerical solutions accuracy depends also on other factors too, like the treatment of singularities, which is one of the most important sources of errors in applying BEM. If suitable techniques as quadrature schemes, changes of coordinates, or other regularization techniques are used we can obtain a high degree of accuracy.

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ISBN:978-988-18210-1-0