Modelling Traffic Flow with Constant Speed using the Galerkin Finite Element Method

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Abstract—At macroscopic level, traffic can be described as a continuum flow. Lighthill Witham and Richards (LWR) have developed a traffic flow model based on the fluid dynamics continuity equation, which is known as the first order LWR traffic flow model. The resulting first order partial differential equation (PDE) can be analytically solved for some special cases, given initial and boundary conditions, and numerically using for example the finite element method (FEM).

This paper makes use of the Galerkin FEM to solve the LWR model with constant speed. The road is divided into a number of road segments (elements) using the Galerkin FEM. Each element consists of two nodes. Each node has one degree of freedom (d.o.f.), namely the traffic density. The FEM provides a solution for the degrees of freedom, i.e. traffic densities of each node. The resulting simultaneous equations are solved at different time steps using the Euler backward time-integration algorithm.

In Belgium and also in the Netherlands, there is a special technique that can be used in order to prevent traffic jams and increasing safety in situations with high volume of cars on the roads, i.e. block driving. It is a technique where cars drive in groups by order of the police when the roads are crowded. In this paper block driving is used as a practical example of the LWR model with constant speed. Thereby, it is simulated using the Galerkin FEM and the results are compared with the analytical solution. The FEM gives good results providing that: the road segments and time steps are small enough. A road with length 5000 m, constant speed of 25 m/s, segment length of 100 m and time steps of 1 s gives good results for the studied case. At points of traffic density rate discontinuities, depending on the road segment size and time step size, the Galerkin FEM is accurate and requires reasonable computational effort.

From the research work carried out in this paper, it is found that the Galerkin FEM is suitable for modelling traffic flow at macroscopic level. The element size and time step size are important parameters in determining the convergence of the solution in case of discontinuities in traffic density rate. Although this paper considers the case of constant speed, the technique can be extended in the future to include the case of non-constant speed, i.e. speed as a function of traffic density.

Index Terms—Macroscopic traffic flow, Galerkin finite element method, LWR model, block driving.

I. INTRODUCTION

There has been a little bit research done in the literature concerned with macroscopic traffic flow modelling using the Galerkin finite element method (FEM). In [1], [2], and [3] a Galerkin FEM type is used to solve the macroscopic Lighthill Witham and Richards (LWR) [4], [5] traffic flow model in conjunction with Greenshields' flow-density relationship [6]. A wavelet-Galerkin FEM is used in [7] to solve the macroscopic non-constant speed LWR traffic model. A discontinuous Galerkin FEM is used in [8] for solving red-and-green light models for the traffic flow.

This paper presents a numerical solution (using a Galerkin FEM [9]) of the LWR traffic flow model with constant speed. The validation of results is done by using the analytical method of characteristics [10].

The first section describes the most important parts of the macroscopic traffic flow theory. The LWR model is presented in details. A description of the analytical solution (using the method of characteristics) and the numerical solution (using the Galerkin FEM in combination with the Euler backward time-integration algorithm [9]) can be found in the next sections. The last section provides a numerical example, namely a block driving simulation.

II. TRAFFIC FLOW THEORY

There are several traffic flow models, which can be mostly divided in 4 categories: macroscopic, mesoscopic, microscopic and submicroscopic (listed with growing level of details).

The first order LWR traffic flow model is a macroscopic continuum model.

A. Macroscopic traffic flow parameters

The most important macroscopic traffic flow parameters are:

- Density (k): expressed in vehicles per kilometre (veh/km)
- Flow (q): expressed in vehicles per hour (veh/h)
- Speed (u): expressed in kilometres per hour (km/h)

B. Fundamental relation

The unique relation between the three macroscopic traffic flow parameters density, flow and speed is:

$$q = k \cdot u \qquad (1)$$

C. Fundamental diagrams

Beside the fundamental relation, there are also experimental relations between the traffic flow parameters.
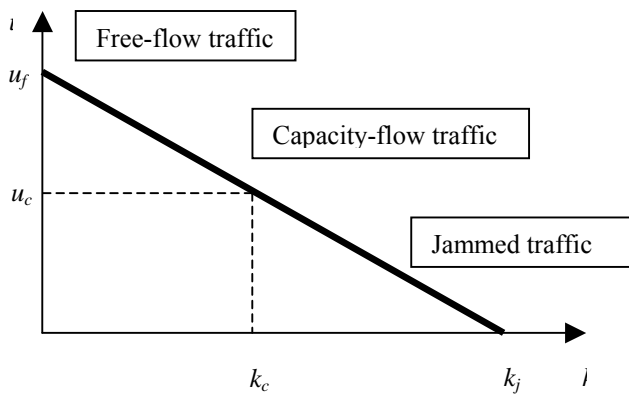
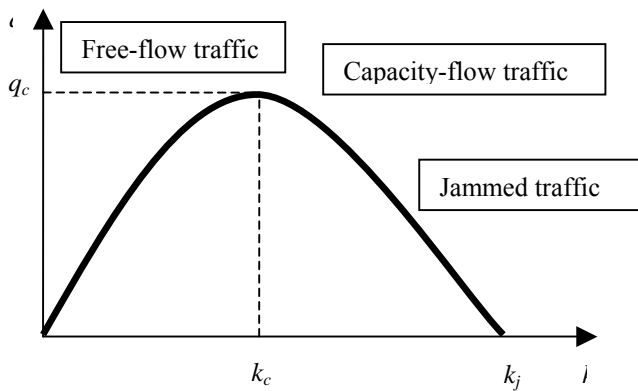Figure 1: The fundamental *u-k* diagram according to Greenshields

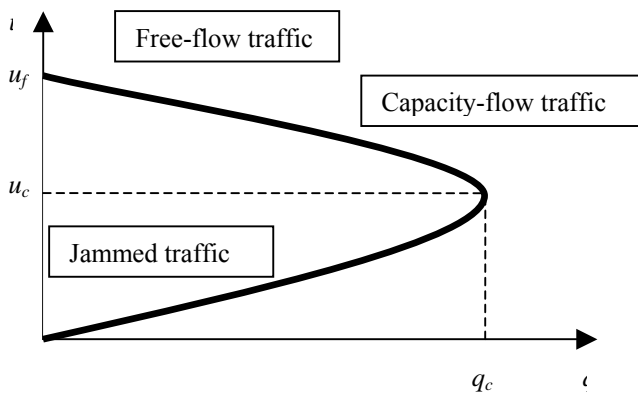Figure 2: The fundamental *q-k* diagram according to Greenshields

Figure 3: The fundamental *u-q* diagram according to Greenshields

These relationships are called fundamental diagrams. They are obtained from measurements.

The fundamental diagrams are:

- Speed-density relationship:

$$u = u_f \cdot (1 - \frac{k}{k_j}) \qquad (2)$$

- Speed-flow relationship:

$$q = u_f \cdot k \cdot (1 - \frac{k}{k_j}) \qquad (3)$$

- Flow-density relationship:

$$q = k_j \cdot u \cdot (1 - \frac{u}{u_f}) \qquad (4)$$

Fig. 1, 2, and 3 give graphical overviews of the fundamental diagrams according to Greenshields [6].

Traffic can have different regimes (characterized by variables related to the traffic state):

- Free-flow traffic is characterized by a low density (high speed), which results in a free-flow speed u_f. Mostly u_f is the maximum allowed speed.
- Capacity-flow traffic is characterized by a maximum flow which is called the capacity flow q_c.
- Jammed traffic is characterized by a maximum density (low or no speed) called the jam density k_j.

In practice, transitions occur in time from one regime to another.

D. First order LWR model

Lighthill, Witham and Richards considered that traffic was an inviscid but compressible fluid (fluid-dynamic model). Densities, speed values and flows were defined as continuous variables in each point in time and space (continuum, macroscopic model). The first order partial differential equation (PDE) from this model is:

$$\frac{\partial k}{\partial t} + \frac{\partial q}{\partial x} = 0 \qquad (5)$$

Crucial to the approach of Lighthill, Witham and Richards was the fundamental hypothesis (see section II.B), i.e. flow is a function of density and speed:

$$\frac{\partial k}{\partial t} + \frac{\partial (k \cdot u)}{\partial x} = 0 \qquad (6)$$

Lighthill and Witham assumed that the fundamental hypothesis holds at all traffic densities, not just for light-density traffic but also for congested traffic conditions. Using the fundamental diagrams (see section II.C) to relate the two dependent variables in the left-hand side of (5) (density k and flow q) to one another, it is possible to solve the partial differential equation, given initial and boundary conditions.

Equation (6) can be made simpler by assuming a constant speed. In this paper a constant speed is assumed, i.e. $u = u_0$:

$$\frac{\partial k}{\partial t} + \frac{\partial (k \cdot u_0)}{\partial x} = 0 \qquad (7)$$

Therefore, (7) becomes

$$\frac{\partial k}{\partial t} + u_0 \cdot \frac{\partial k}{\partial x} = 0 \qquad (8)$$

It is worth mentioning that Lighthill, Witham and Richards noted that because of the continuity assumption, the theory only holds for a large number of vehicles (long crowded roads).

III. ANALYTICAL SOLUTION: METHOD OF CHARACTERISTICS

Equation (8) is a first order partial differential equation (more specific: the first order wave equation with speed u_0).

The method of characteristics [10] can be used to find a solution for the initial boundary value problem. An initial boundary value problem assumes (beside the differential equation) two extra equations (continuous or discontinuous):

- Initial values: the density values at time

$$t = 0 \qquad (k(x,0) = f(x))$$

- Boundary values: the density values at distance
$x = 0 \qquad (k(0,t) = g(t))$

The general form of the solution from the first order partial differential equation (8) with constant speed u_0, density k, initial condition $k(x,0) = f(x)$ and boundary condition $k(0,t) = g(t)$ is:

$$k(x,t) = \begin{cases} f(x - u_0 \cdot t) & x \geq u_0 \cdot t \\ g(t - \dfrac{x}{u_0}) & x \leq u_0 \cdot t \end{cases} \qquad (9)$$

Figure 4: A single finite element

IV. NUMERICAL SOLUTION: GALERKIN FEM

Application of the Galerkin FEM to (8) gives:

$$\int_V \delta w \cdot (\frac{\partial k}{\partial t} + u_0 \cdot \frac{\partial k}{\partial x}) dV = 0 \qquad (10)$$

Galerkin sets the weight function δw equal to the shape function vector $[N]$:

$$\int_V [N] \cdot (\frac{\partial k}{\partial t} + u_0 \cdot \frac{\partial k}{\partial x}) dV = 0 \qquad (11)$$

where V is the element volume.

For a linear homogeneous element dV can be replaced by dx and the integration can be done over x:

$$\int_x [N] \cdot (\frac{\partial k}{\partial t} + u_0 \cdot \frac{\partial k}{\partial x}) dx = 0 \qquad (12)$$

A. Single element matrices

This section gives the derivation of the matrices of a single element with length L and two nodal densities k_1 and k_2. Fig. 4 gives a graphical representation of a single traffic flow element.

The traffic density in the element as a function of the nodal densities, degree of freedom (d.o.f.), is given by

$$k = N_1 \cdot k_1 + N_2 \cdot k_2 = [N_1 \quad N_2] \cdot \begin{Bmatrix} k_1 \\ k_2 \end{Bmatrix} \qquad (13)$$

The differentiation of the density with respect to the time t is given by

$$\frac{\partial k}{\partial t} = N_1 \cdot \frac{\partial k_1}{\partial t} + N_2 \cdot \frac{\partial k_2}{\partial t} = [N_1 \quad N_2] \cdot \begin{Bmatrix} \dfrac{\partial k_1}{\partial t} \\ \dfrac{\partial k_2}{\partial t} \end{Bmatrix} \qquad (14)$$

The differentiation of the density with respect to the position x is given by

$$\frac{\partial k}{\partial x} = \frac{\partial N_1}{\partial x} \cdot k_1 + \frac{\partial N_2}{\partial x} \cdot k_2 = \left[\frac{\partial N_1}{\partial x} \quad \frac{\partial N_2}{\partial x} \right] \cdot \begin{Bmatrix} k_1 \\ k_2 \end{Bmatrix} \qquad (15)$$

The shape functions N_1 and N_2 are

$$[N] = [N_1 \quad N_2] = \left[1 - \frac{x}{L} \quad \frac{x}{L} \right] \qquad (16)$$
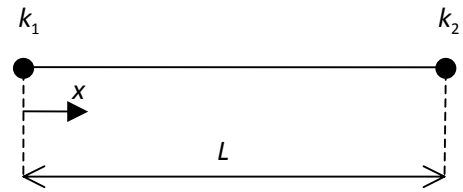
Substituting (13), (14), (15) in (12) gives:

$$\int_x (\begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \cdot \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{Bmatrix} \dfrac{\partial k_1}{\partial t} \\ \dfrac{\partial k_2}{\partial t} \end{Bmatrix}) dx$$

$$+ \int_x (u_0 \cdot \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \cdot \begin{bmatrix} \dfrac{\partial N_1}{\partial x} & \dfrac{\partial N_2}{\partial x} \end{bmatrix} \cdot \begin{Bmatrix} k_1 \\ k_2 \end{Bmatrix}) dx = 0 \tag{17}$$

Simplifying the notation of (17) gives:

$$\int_x ([N]^T \cdot [N] \cdot \left(\frac{\partial k}{\partial t} \right)) dx + \int_x (u_0 \cdot [N]^T \cdot \left[\frac{\partial N}{\partial x} \right] \cdot \{k\}) dx = 0 \tag{18}$$

After derivation and integration of the shape functions, (18) becomes:

$$\frac{L}{6} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \cdot \left\{ \frac{\partial k}{\partial t} \right\} + \frac{u_0}{2} \cdot \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \cdot \{k\} = 0 \tag{19}$$

Because of the time-dependency, there is a need for a time-integration algorithm. Application of the Euler backward time-integration algorithm [9] $\left(\dfrac{\partial k}{\partial t} = \dfrac{k^{t+\Delta t} - k^t}{\Delta t} \right)$ results in:

$$\frac{L}{6 \cdot \Delta t} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \cdot \left\{ k^{t+\Delta t} \right\} - \frac{L}{6 \cdot \Delta t} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \cdot \left\{ k^t \right\}$$

$$+ \frac{u_0}{2} \cdot \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \cdot \left\{ k^{t+\Delta t} \right\} = 0 \tag{20}$$

Re-writing (20) gives:

$$\big[[A] + [B] \big] \cdot \left\{ k^{t+\Delta t} \right\} = [A] \cdot \left\{ k^t \right\} \tag{21}$$

where

$$[A] = \frac{L}{6 \cdot \Delta t} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{22}$$

$$[B] = \frac{u_0}{2} \cdot \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \tag{23}$$

$$\left\{ k^{t+\Delta t} \right\} = \begin{Bmatrix} k_1^{t+\Delta t} \\ k_2^{t+\Delta t} \end{Bmatrix} \tag{24}$$

$$\left\{ k^t \right\} = \begin{Bmatrix} k_1^t \\ k_2^t \end{Bmatrix} \tag{25}$$
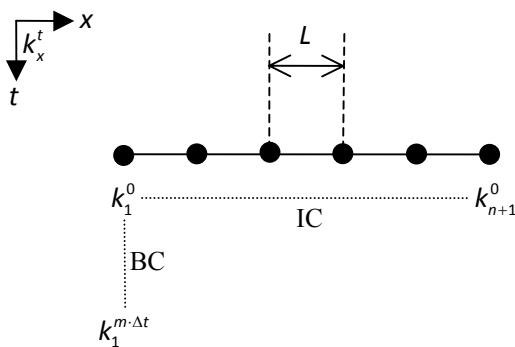
B. General solution for m time steps and n elements

Figure 5: Problem definition for *n* finite elements and *m* time steps

Given a road, which is divided in *n* elements with uniform length L, initial conditions, at $t = 0$ s, k_1^0 to k_{n+1}^0 (Fig. 5) and boundary conditions, at $x = 0$ m, k_1^0 to $k_1^{m \cdot \Delta t}$ (Fig. 5) then, the densities at all times and all positions can be calculated from:

$$\left\{ k^{t+\Delta t} \right\} = \big[[A] + [B] \big]^{-1} \cdot [A] \cdot \{k^t\} \tag{26}$$

where

$$[A] = \frac{L}{6 \cdot \Delta t} \begin{bmatrix} 2 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & 4 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 4 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 2 \end{bmatrix} \tag{27}$$

$$[B] = \frac{v_0}{2} \begin{bmatrix} -1 & 1 & 0 & \cdots & \cdots & 0 \\ -1 & 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 & 1 \\ 0 & \cdots & \cdots & 0 & -1 & 1 \end{bmatrix} \tag{28}$$

V. SIMULATION OF BLOCK DRIVING

By choosing a practical traffic example, block driving, the Galerkin FEM solution method is tested on accuracy, convergence and computational needs.

Block driving is a technique where cars drive in groups by order of the police, in order to prevent jams.

When the traffic density of the road is high, a car of the police drives with a constant speed in the centre of the road. The vehicles drive behind the police car with the same constant speed. Because of this, acceleration and braking (the harmonica effect) are avoided. The chance of accidents reduces and traffic jams become shorter in distance.

In this section block driving is simulated with the Galerkin FEM.

It should be noted that all densities are expressed in vehicles per meter (veh/m), distances in meters (m) and times in seconds (s).
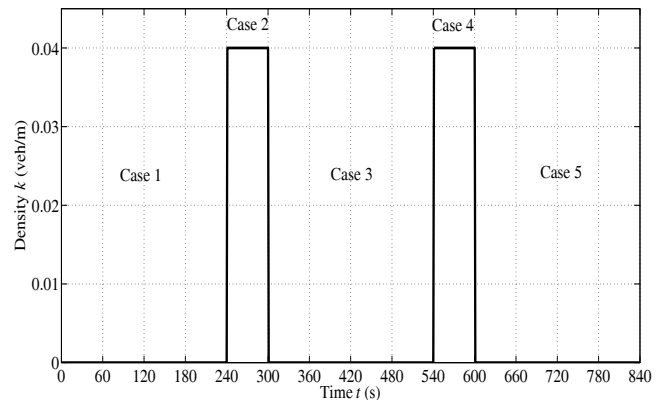
Figure 6: Boundary conditions at the beginning of the road − block driving

A. Details of the simulation

The simulation is described in 5 time intervals (Fig. 6):

- Case 1: no vehicles enters/on the road during 240 seconds.
- Case 2: 60 vehicles enter the road (at 1 vehicle per second driving at 25 m/s) during 60 seconds. This case is responsible for the creation of the first traffic wave.
- Case 3: no vehicles enter the road. The 60 vehicles drive on the road and leave the road whereas the road becomes clear. The time between entering and leaving the road from one car is 200 s (travelling 5000 m with a speed of 25 m/s). This case simulates the evolution of the first traffic wave.
- Case 4: repeating case 2. This case is responsible for the creation of traffic wave 2.
- Case 5: repeating case 3. This case simulates the evolution of traffic wave 2.

B. Mathematical translation

The LWR model with constant speed is given by (8). All simulations are done from 0 s to 840 s and the road length L is 5000 m. The constant speed u_0 is 25 m/s.

Case 1 contains an initial condition. The density at $t = 0$ s is given by:

$$k(x,0) = 0 \qquad 0 \leq x \leq 5000 \qquad (29)$$

Cases 1, 2, 3, 4, and 5 contain boundary conditions. The density at $x = 0$ m (the beginning of the road) is given by:

$$k(0,t) = \begin{cases} 0 & 0 \leq t \leq 240 \\ 0.04 & 240 < t \leq 300 \\ 0 & 300 < t \leq 540 \\ 0.04 & 540 < t \leq 600 \\ 0 & 600 < t \leq 840 \end{cases} \qquad (30)$$

Fig. 6 gives a graphical representation of the boundary conditions.

C. Analytical solution using the method of characteristics

The analytical solution with the method of characteristics (see section III) is given by the following equation:

$$k(x,t) = \begin{cases} 0 & 0 \leq x - u_0 \cdot t \leq 5000 & \rightarrow & f(x - u_0 \cdot t) \\ 0 & 0 \leq t - x/u_0 \leq 240 & \rightarrow & g(t - x/u_0) \\ 0.04 & 240 < t - x/u_0 \leq 300 & \rightarrow & g(t - x/u_0) \\ 0 & 300 < t - x/u_0 \leq 540 & \rightarrow & g(t - x/u_0) \\ 0.04 & 540 < t - x/u_0 \leq 600 & \rightarrow & g(t - x/u_0) \\ 0 & 600 < t - x/u_0 \leq 840 & \rightarrow & g(t - x/u_0) \end{cases} \qquad (31)$$

D. Numerical solution with the Galerkin FEM

The density is calculated for all times (0 to 840 s) and distances (0 to 5000 m) by using (26) and applying the initial and boundary conditions.

The convergence study on the density is done with simulations with different element sizes Δx and time steps Δt. The density versus time at distance 2000 m is fully analysed by using convergence parameters and CPU calculation times.

The evolution of density in distance (at distances 0 m,

2500 m and 5000 m) and time (at times 570 s, 670 s and 770 s) is investigated with the use of a simulation with element size 100 m and time steps of 1 s.

1) Convergence study on density

The analytical solution results in discontinuous traffic block waves. The numerical solution results in continuous rounded traffic waves. The discontinuities become approximated by a continuous function using the Galerkin FEM.

The CPU times of the numerical simulations are presented in Table I.

Table I: CPU times for the numerical simulations of block driving

Δt (s)	Δx (m)	Simulation time (s)
1	1	1563.43
1	10	35.73
1	100	2.76
1	250	2.47
1	1000	1.62
10	100	0.39
30	100	0.35

The accuracy parameters (mean error, standard deviation of the error and total error) of the numerical simulations are presented in Table II. The higher the total error, mean error or standard deviation (std), the less accurate the results. The error results are calculated via the difference between analytical and numerical values.

Table II: Accuracy parameters for the density versus time at distance 2000 m

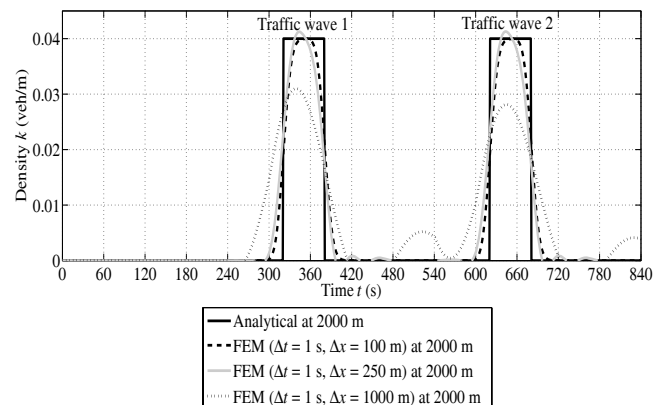| Δt (s)	Δx (m)	Mean error at 2000 m (veh/m)	Std error at 2000 m (veh/m)	Total error at 2000 m (veh/m)
1	1	0.0021	0.0046	1.1452
1	10	0.0021	0.0045	1.1432
1	100	0.0022	0.0046	1.1568
1	250	0.0023	0.0048	1.4650
1	1000	0.0057	0.0073	4.2822
10	100	0.0064	0.0085	4.4135
30	100	0.0072	0.0098	6.2037

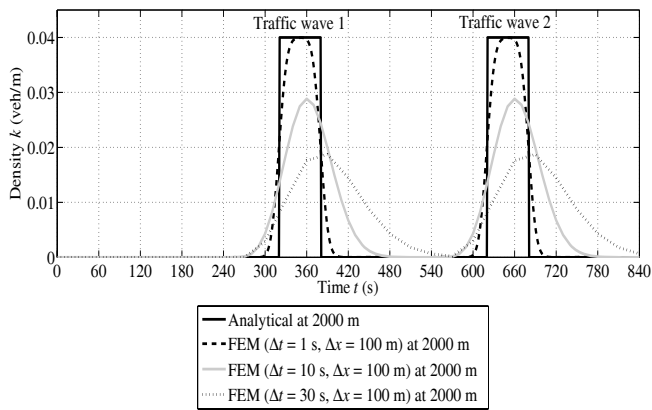Figure 7: Convergence of density as function of Δx – block driving

Figure 8: Convergence of density as function of Δt – block driving

Fig. 7 gives an overview of the density versus time at distance 2000 m for simulations with $\Delta t = 1$ s and $\Delta x = 100$ m, 250 m, and 1000 m.

Simulations with $\Delta t = 1$ s and $\Delta x = 1$ m or 10 m gives good results (both a small mean error of 0.0021 veh/m with a small standard deviation of 0.046 veh/m and 0.045 veh/m), but computationally these simulations are very intensive (high CPU time of 1563.43 s and 35.73 s). This simulations are not presented in Fig. 7 because of the same graphical results as with $\Delta t = 1$ s and $\Delta x = 100$ m. Simulations with $\Delta t = 1$ s and $\Delta x = 100$ m gives good results (a small mean error of 0.0022 veh/m with a small standard deviation of 0.046 veh/m). These simulations are computationally fast (low CPU time of 2.67 s). Simulations with $\Delta t = 1$ s and $\Delta x = 250$ m gives also good results (a small mean error of 0.0023 veh/m with a small standard deviation of 0.048 veh/m) but those are less accurate then simulations with $\Delta x = 100$ m. Simulations with $\Delta t = 1$ s and $\Delta x = 1000$ m gives bad results (a large mean error of 0.0057 veh/m with a large standard deviation of 0.073 veh/m).

The effects of a larger segment length are:
- Broader traffic wave
- Lower maximum densities
- Earlier start of traffic wave

Fig. 8 gives an overview of the density versus time at distance 2000 m for simulations with $\Delta x = 100$ m and $\Delta t = 1$ s, 10 s, and 30 s.

Simulations with $\Delta x = 100$ m and $\Delta t = 0.1$ s gives good results, but computationally these simulations are very intensive (more than 1563.43 s CPU time). This simulation is not presented in Fig. 8 because of the same graphical results as with $\Delta x = 100$ m and $\Delta t = 1$ s. Simulations with $\Delta x = 100$ m and $\Delta t = 1$ s gives good results (a small mean error of 0.0022 veh/m with a small standard deviation of 0.046 veh/m) and they are computationally not intensive (low CPU time of 2.76 s). Simulations with $\Delta x = 100$ m and $\Delta t = 10$ s or 30 s gives bad results (both a large mean error of 0.0064 veh/m and 0.0072 veh/m with a large standard deviation of 0.0085 veh/m and 0.0098 veh/m).

The effects of a larger time step size are:
- Broader traffic wave
- Lower maximum densities
- Later start of traffic wave

Simulations with segment lengths of 100 m and time steps of 1 s give best results complying with computation power and accuracy. This results in the following convergence criteria:

- $\Delta x \leq L/50$
- $\Delta t \leq 1$ s

2) Evolution of density in distance and time

Fig. 9 gives a graphical representation of the evolution of the density in distance at $x = 0$ m, 2500 m, and 5000 m. The movement of traffic waves 1 and 2 is clearly visible in Fig. 9. The analytical and numerical results are the same for $x = 0$ m because of the boundary condition.

Since traffic waves are travelling with a constant speed of 25 m/s, the maximum density at distance 2500 m and time 370 s (point A) becomes at distance 5000 m at time 470 s (point B).

Fig. 10 gives a graphical representation of the evolution of the density in time at $t = 570$ s, 670 s, and 770 s. This is a graphical representation of the movement from traffic wave 2 over the road with length 5000 m.

Since the traffic wave is travelling with a constant speed of 25 m/s, a density at time 570 s and distance 0 m (point A) becomes at time 670 s at distance 2500 m (point B) and at 770 s at distance 5000 m (point C).
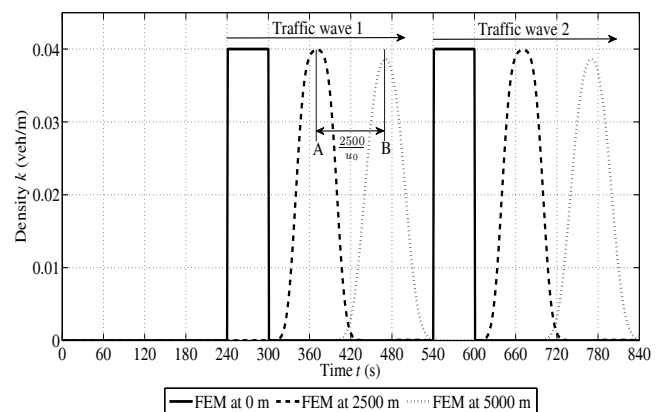

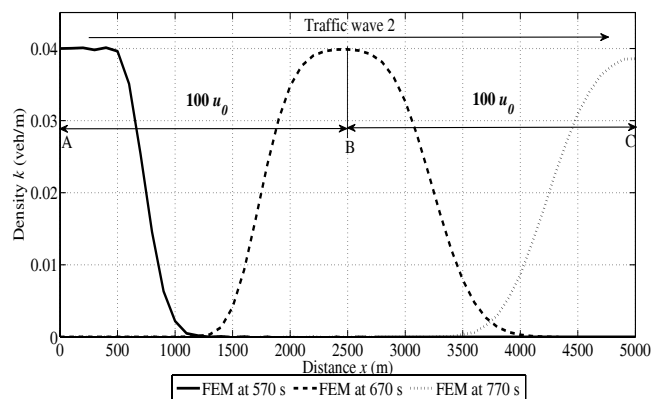Figure 9: Evolution of density in distance – block driving

Figure 10: Evolution of density in time – block driving

VI. CONCLUSIONS AND FUTURE WORK

The Galerkin FEM can be used to solve the first-order macroscopic LWR traffic flow model with constant speed. The density, flow and speed values are calculated in each point on the road, at any time. The results of the Galerkin finite element analysis are compared with that of the analytical method of characteristics.

By the use of an analytical and numerical technique, block driving is simulated. A simulation with road length 5000 m, constant speed of 25 m/s, segment lengths of 100 m and time steps of 1 s results in accurate and fast numerical results. The difficulties with the numerical simulations appear at the discontinuities. This can be prevented by choosing the element size and time steps small enough. Using larger segment lengths and/or time steps can give inaccurate results. Using very small segment lengths and/or time steps can result in intensive simulations.

Future research will concentrate on the application of the Galerkin FEM to the LWR model with non-constant speed. In such a case, the speed is a function of the density. The LWR model can also be extended to include social forces and resistances.

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