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Kinematic Analysis Of Complex Gear Mechanisms

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Abstract—**This paper presents a general kinematic analysis method for complex gear mechanisms. This approach involves the null-space of the adjacency matrix associated with the graph of the mechanism weighted by complex coefficients. It allows to compute the rotational speed ratios of all the links and the frequency of all the contacts in this mechanism (including roll bearings). This approach is applied to various examples including a two degrees of freedom car differential.**

Keywords: Kinematic analysis, gear train, graph theory, car differential

1 Introduction

The research explained in this paper takes its source in the domain of the Health and Usage Monitoring Systems (HUMS) of helicopters. Nowadays, a lot of studies are done on such systems [1]. A very important part of studies to improve the performances of HUMS concerns the vibration analysis of the transmission, and especially of the Main Gear Box (MGB). The aim of these studies is to identify defaults on the MGB using vibration analysis. In fact, each default of contact between the different links of a complex system, as a MGB, can generate an harmonic disturbance at a precise angular frequency in the vibratory signal.

In that domain, a few researches have led to the use of Kalman filters on angularly sampled signals [2]. Such filters can provide a good estimation of the magnitude and phase of an harmonic component in a signal when its frequency is well-known [3]. So, to create the dynamic Kalman model, a very good knowledge of (angular) frequencies of all the contacts in this mechanism (including roll bearings) is required. To determine these frequencies, rotational speed ratios of the various links of the mechanism is required first of all.

There are a lot of kinematic analysis approaches for different types of gear trains. The tabular method is commonly used but can involve a lot of calculation, and cannot give the velocities of elements whose rotation axis are not on the input/output axis [4]. The vector analysis method gives very good results for bevel gears, but is very complex and can lead to human mistakes [5]. The graph theory method can be easily computerized, and can give the velocities of all elements of the gear trains. It can also be adapted for bevel gears [6]. It has been studied by Nelson in [7] so as to find the angular velocities of all links in bevel epicyclic gear trains. It is also limited to gear trains whose input and output axes are co-linear.

In this paper, a new kinematic analysis method, based on the work of Nelson, is introduced. It's objective is to list all the mechanical contacts between all elements in the transmission system (ball-bearing, gears...) and for each of these contact, to find its angular apparition rate, that is the number of times this contact appears for one revolution of the input shaft. Of course, to solve this problem, a general tool to compute all the speed ratios between the links of the transmission is required.

There is a few advantages to this method. The most important of them is that it is possible to analyze very complex mechanisms, as long as its internal composition is known. For example, it is possible to deal with a system whose input and output axes are not co-linear. Systems with several degrees of freedom, as a car differential can also be studied with this method.

The first section presents the kinematic analysis method. In the second section, examples are presented to demonstrate the interest and the generality of this method : a simple epicyclic bevel gear train and a car differential.

2 Kinematic Analysis Method

2.1 Speed ratio matrix

In this section, the kinematic analysis method is introduced. First, it is important to understand that the difference with the kinematic analysis methods already existing, consists in the introduction of complex numbers in the definition of each link of the mechanism.

The first step of that method is to build the table T of mechanism links and joints. For a mechanism with N links this table is a $N \times N$ table representing the kinematics graph of the mechanism. Only mechanisms with turning pairs (revolute joints) or gear pairs are considered here. The element (i, j) of T denotes the interaction of the link i on the link j. The table T is built following

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a few rules:

- Turning-pairs are noted (p) for the elements (i, j) and (j, i).
- For gear pairs, the element (i, j) is equal to $N_i e^{\theta_i \sqrt{-1}}$ and (j, i) to $N_j e^{\theta_j \sqrt{-1}}$, where N_i and N_j are respectively the number of tooth on elements i and j. θ_i (respectively θ_j) is the angle between the axis of rotation and the axis of the gear tooth of element i (respectively j) in contact with element j (respectively i), counted in the positif clockwise and so that $|\theta| < \pi/2$ for an internal gear and $\pi > |\theta| > \pi/2$ for an external gear.
- Otherwise, elements of T stay empty.

Lastly the reference (carrier) link k is also required for each gear pair. The reference link is the link in which the contact point of the gear pair is motionless (the carrier link for an epicycloidal gear). Then it is possible to express the WILLIS formula under the form:

$$T(i, j)(\omega_i - \omega_k) + T(j, i)(\omega_j - \omega_k) = 0 \quad (1)$$

where, $T(i, j)$ and $T(j, i)$ are the elements (i, j) and (j, i) of the matrix T, ω_i and ω_j are respectively the rotational speeds of the links i and j involved in the gear pair, and ω_k is the rotational speed of the reference link k of that gear pair. Note that WILLIS formula is used in a more general context since coefficients of T are complex.

The second step, after the construction of the table T, is to build the adjacency matrix M. This matrix will be used to solve the kinematic analysis. Inspecting the upper triangular part of T, the matrix M is built line per line. For each gear-pair $(i,\ j)$, with reference link k, a new line l is added to matrix M following the rules from equation (1):

- $M(l, i) = T(i, j)$,
- $M(l, j) = T(j, i)$,
- $M(l, k) = -T(i, j) - T(j, i)$,
- 0 elsewhere.

Let us consider $\Omega = [\omega_1,\ \omega_2,\ \cdots \omega_N]^T$ the vector of rotational speeds of the N links of the mechanism. Then Ω is solution of the equation:

$$M\Omega = 0\,, \quad (2)$$

that is:

$$\Omega \in Ker[M] \quad (3)$$

where $Ker(M)$ denotes the null space of matrix M which can be easily computed using linear algebra tools (for example the function `null` in MATLAB). The dimension of this null space represents the number of degrees of freedom (d.o.f.) N_{dof} of the mechanism:

$$N_{dof} = N - rk(M) \quad (4)$$

where $rk(M)$ is the rank of matrix M. In other words $\Omega_0 = Ker(M)$ is the $N \times N_{dof}$ matrix composed of the N_{dof} vectors spanning this null space. In the one d.o.f. case ($N_{dof} = 1$), Ω_0 can be normalized in such a way that $\Omega_0(r) = 1$ where r is the index of the input shaft. This way, $\Omega_0(i)$ corresponds to the speed ratio of the link i w.r.t to link r. In the sequel, Ω_0 is called the **speed ratio vector** (or **matrix** in the multi d.o.f. case). In the general case, the vector of rotational speeds Ω can be parameterized in the following way:

$$\Omega = \Omega_0 \Lambda \quad (5)$$

where Λ is $N_{dof} \times 1$ vector of multiplicative coefficients.

In the multi d.o.f. case, it is possible to add some constraints to be met by Ω in order to reduce the number of degrees of freedom:

$$C\Omega = 0, \quad \Rightarrow \quad \Omega \in Ker \begin{bmatrix} M \\ C \end{bmatrix};$$

For example it is possible to null the speed of link i ($\omega_i = 0$) using :

$$C = \begin{array}{c} \left[\begin{array}{ccccccc} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{array} \right] . \\ \begin{array}{ccccccc} 1 & \cdots & i-1 & i & i+1 & \cdots & N \end{array} \end{array}$$

Lastly, it is important to notice that Ω is a complex vector. That way, the direction of the rotational speed ω_i of link i (in gear pair with link j) can be determined (in the plane containing links i and j axes). That will be illustrated in the first example.

2.2 Contact frequencies

In the context of vibration analysis, it appears that it can be useful to know all the frequencies of contacts in gear-pairs or turning pairs (that is : in ball or roll bearings). A default in a particular contact will produce an harmonic disturbance with a great magnitude at a known frequency.

Default frequencies in gear pairs For a gear-pair $(i,\ j)$ between links i and j with a reference link k, one can distinguish 3 contact default frequencies :

- the gear frequency ω_{ij}^g defined by:
$$\omega_{ij}^g = |T(i,j)||\omega_i - \omega_k| = |T(j,i)||\omega_j - \omega_k|\ ,$$
- the frequency of a default on a single tooth of link i (resp. j):
$$\omega_i^g = |\omega_i - \omega_k| \quad (resp.\ \omega_j^g = |\omega_j - \omega_k|)\ .$$

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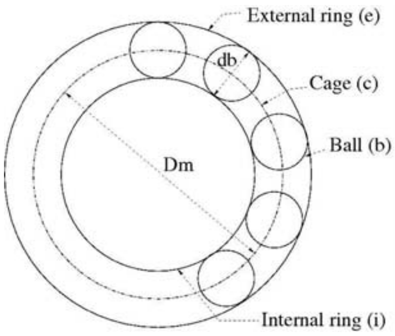

Figure 1: Example of ball bearing

There are $N_i = |T(i,j)|$ (resp. $N_j = |T(j,i)|$) different and independent sources of such an harmonic disturbance because there are N_i (resp. N_j) teeth on link i (resp. j).

Default frequencies in ball (or roll)-bearings

For turning pairs involving ball (or roll)-bearing, contact defaults can appear at several frequencies (even for one element), depending on where the default is located (on the internal or external ring, or on a ball).

The figure (1) represents a ball bearing with :

- D_m : its mean diameter,
- d_b : the ball diameter,
- Z : the number of balls.

In the sequel, indices i, e, c and b refer to the internal ring, the external ring, the cage and the ball, respectively. Under the rolling without slipping assumption, the following formulae allows to compute various default frequencies in a ball bearing [8].

Frequency ω_{di} of apparition of a default on the internal ring

$$\omega_{di} = Z|\omega_i - \omega_c| = \frac{Z}{2}(1 + \frac{d_b}{D_m}|\omega_i - \omega_e|) \tag{6}$$

Frequency ω_{de} of apparition of a default on the external ring

$$\omega_{de} = Z|\omega_e - \omega_c| = \frac{Z}{2}(1 - \frac{d_b}{D_m}|\omega_i - \omega_e|) \tag{7}$$

Frequency ω_{db} of apparition of a default on a ball

$$\omega_{db} = 2|\omega_b - \omega_c| = \frac{D_m}{d_b}(1 - \frac{d_b^2}{D_m^2}|\omega_i - \omega_e|) \tag{8}$$

These formulaes are still true for roll or ball bearings with angular contacts changing d by $d\cos\alpha$ where α is the contact angle (see figure (2)).

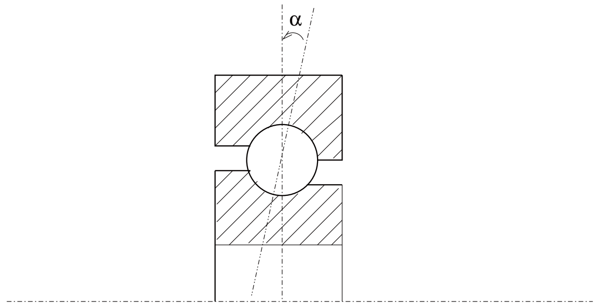

Figure 2: Ball bearing with angular contacts

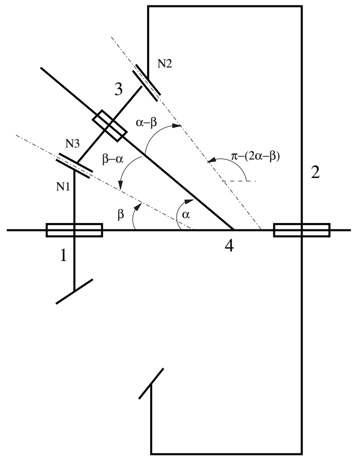

Figure 3: Mechanism with a simple bevel planet [7]

3 Examples

In this section, two examples of mechanisms are analyzed using this method. The first one is a simple bevel planet that was studied in the article of Nelson. The second one is a 2 d.o.f. car differential taking into account roll bearings.

Example 1

The epicyclic gear with a simple bevel planet is shown in figure 3. In this mechanism, the link # 1 is the sun, the link # 2 is the outer ring gear (and the input shaft), link # 3 is the planet and link # 4 is the carrier. The numerical data are: $N_1 = 15$, $N_2 = 25$ and $N_3 = 10$, $\alpha = \pi/3$ and $\beta = \pi/4$.

The table T of links and joints is depicted in Table (1)

Table 1: Table of links and joints (example 1)

	1	2	3	4
1			$N_1 e^{\sqrt{-1}\beta}$	(p)
2			$N_2 e^{\sqrt{-1}(2\alpha-\beta-\pi)}$	(p)
3	$N_3 e^{\sqrt{-1}(\beta-\alpha)}$	$N_3 e^{\sqrt{-1}(\alpha-\beta)}$		(p)
4	(p)	(p)	(p)	

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$$M = \begin{bmatrix} 15e^{\pi/4\sqrt{-1}} & 0 & 10e^{-\pi/12\sqrt{-1}} & -15e^{\pi/4\sqrt{-1}} - 10e^{-\pi/12\sqrt{-1}} \\ 0 & 25e^{-7\pi/12\sqrt{-1}} & 10e^{\pi/12\sqrt{-1}} & -25e^{-7\pi/12\sqrt{-1}} - 10e^{\pi/12\sqrt{-1}} \end{bmatrix} \tag{9}$$

and the adjacency matrix is given by equation (9).

Then, the null space of M reads:

$$Ker(M) = \begin{bmatrix} 0.357e^{2.147\sqrt{-1}} & 0.729e^{-0.070\sqrt{-1}} \\ 0.412e^{-0.686\sqrt{-1}} & 0.444e^{0.633\sqrt{-1}} \\ 0.828e^{0.125\sqrt{-1}} & 0.069e^{-0.106\sqrt{-1}} \\ 0.136e^{-0.382\sqrt{-1}} & 0.517e^{0.284\sqrt{-1}} \end{bmatrix} \tag{10}$$

As it has already been said, the speed ratios are complex in this method, so as to express the different angles of rotational speed vectors. The nominal value of the speed is in fact the modulus of the element of $Ker(M)$.

As it is proposed in the example of Nelson, ω_1 is set to zero. With this constraint (that is: $C = [1\ 0\ 0\ 0]$), the speed ratio vector normalized w.r.t the input shaft ($r = 2$) reads:

$$\Omega_0 = Ker\left(\begin{bmatrix} M \\ C \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 1.36e^{0.639\sqrt{-1}} \\ 0.6250 \end{bmatrix} \tag{11}$$

Remark: one can compute the angular velocity of 3 with respect to 4 ($\omega_{3/4}$) and check that the direction of the relative velocity is given by the angle α. Indeed :

$$\omega_{3/4} = \omega_3 - \omega_4 = 0.9375e^{\pi/4\sqrt{-1}} \tag{12}$$

From (11), $\omega_2 = 1.6\omega_4$. It is the same result as the one of Nelson [7]. Lastly, for $\omega_2 = 10\,Hz$, the gear frequencies are :

$$\omega_{13}^g = \omega_{23}^g = 93.75\,Hz\ . \tag{13}$$

Example 2 In the following example we consider a car differential depicted in Figures 4 and 5.

The table T of links and joints is given in Table (2) with $N_1 = 13$, $N_2 = 65$, $N_3 = 10$, $N_4 = 14$, $\theta_1 = 0$ and $\theta_2 = \pi/4$. Then, the corresponding adjacency matrix is given by equation (14).

Imposing the rotational speed of the car frame (the link # 1) is equal to 0, then the speed ratio matrix Ω_0 can be normalized with respect to the two wheels (links # 3

Figure 4: Picture of the car differential

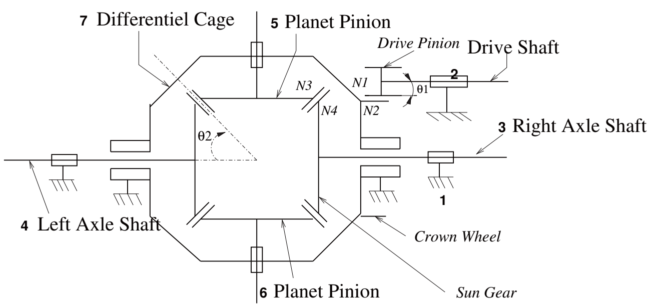

Figure 5: Car Differential

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Table 2: Table of links and joints for the car differential (example 3)

	1	2	3	4	5	6	7
1		(p)	(p)	(p)			(p)
2	(p)						$N_1 e^{\theta_1\sqrt{-1}}$
3	(p)				$N_4 e^{-\theta_2\sqrt{-1}}$	$N_4 e^{\theta_2\sqrt{-1}}$	(p)
4	(p)				$N_4 e^{\theta_2\sqrt{-1}}$	$N_4 e^{-\theta_2\sqrt{-1}}$	(p)
5			$N_3 e^{\theta_2\sqrt{-1}}$	$N_3 e^{-\theta_2\sqrt{-1}}$			(p)
6			$N_3 e^{-\theta_2\sqrt{-1}}$	$N_3 e^{\theta_2\sqrt{-1}}$			(p)
7	(p)	$N_2 e^{-\theta_1\sqrt{-1}}$	(p)	(p)	(p)	(p)	

$$M = \begin{bmatrix} -78 & 13 & 0 & 0 & 0 & 0 & 65 \\ 0 & 0 & 14e^{-\pi/4\sqrt{-1}} & 0 & 10e^{\pi/4\sqrt{-1}} & 0 & -14e^{-\pi/4\sqrt{-1}} - 10e^{\pi/4\sqrt{-1}} \\ 0 & 0 & 14e^{\pi/4\sqrt{-1}} & 0 & 0 & 10e^{-\pi/4\sqrt{-1}} & -14e^{\pi/4\sqrt{-1}} - 10e^{-\pi/4\sqrt{-1}} \\ 0 & 0 & 0 & 14e^{\pi/4\sqrt{-1}} & 10e^{-\pi/4\sqrt{-1}} & 0 & -14e^{\pi/4\sqrt{-1}} - 10e^{-\pi/4\sqrt{-1}} \\ 0 & 0 & 0 & 14e^{-\pi/4\sqrt{-1}} & 0 & 10e^{\pi/4\sqrt{-1}} & -14e^{-\pi/4\sqrt{-1}} - 10e^{\pi/4\sqrt{-1}} \end{bmatrix} \quad (14)$$

and 4):

$$\Omega_0 = \begin{bmatrix} 0 & 0 \\ -2.5 & -2.5 \\ 1 & 0 \\ 0 & 1 \\ 0.86e^{0.95\sqrt{-1}} & 0.86e^{-0.95\sqrt{-1}} \\ 0.86e^{-0.95\sqrt{-1}} & 0.86e^{0.95\sqrt{-1}} \\ 0.5 & 0.5 \end{bmatrix} . \quad (15)$$

The car differential is obviously a 2 d.o.f mechanism, as long as it is made so that the two wheels of the car can spin at different speeds. The two columns of Ω_0 give the speed ratios when one wheel is locked and the other is free. The most common behavior (**driving straight ahead**) corresponds when both wheels spin at the same speed $\omega_3 = \omega_4 = \omega$. Then the rotational speed vector is:

$$\Omega = \Omega_0 \begin{bmatrix} \omega \\ \omega \end{bmatrix} = \begin{bmatrix} 0 & -5 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T \omega .$$

Another well-known behavior appears when there is no transmission $\omega_2 = 0$ and the car is jacked up. Then the two wheels spin in opposite sense at the same speed $\omega_3 = -\omega_4 = \omega$. Indeed:

$$\begin{aligned} \Omega &= \Omega_0 \begin{bmatrix} \omega \\ -\omega \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1.4e^{\pi/2\sqrt{-1}} \\ -1.4e^{-\pi/2\sqrt{-1}} \\ 0 \end{bmatrix} \omega . \end{aligned}$$

Numerical application : let's consider the example of a car turning a right corner at a speed of $30km/h$. It can be shown that the rotational speeds of the two wheels are $\omega_3 = 26\,rd/s$ and $\omega_4 = 30\,rd/s$. Then the rotational speed vector becomes:

$$\Omega = \Omega_0 \begin{bmatrix} 26 \\ 30 \end{bmatrix} = \begin{bmatrix} 0 \\ -140 \\ 26 \\ 30 \\ 28.14e^{-0.10\sqrt{-1}} \\ 28.14e^{0.10\sqrt{-1}} \\ 28 \end{bmatrix} (rd/s) . \quad (16)$$

Now it is possible to take into account the data relative to the 2 roll bearings in the revolute joint between links 1 and 7:

$$D_m = 54mm, d_b = 5mm, Z = 25, \alpha = 15 .$$

Then, the Table (A) in appendix lists all the frequencies of defaults that could be found in this mechanism, their locations, and the number of different sources.

4 Conclusions

The kinematic method introduced in this paper is an improvement of the Nelson's method. Based on the same principle and the same formula (the Willis Formula), this approach solves the model using the null space of the adjacency matrix associated to the kinematic graph of the mechanism. It allows complex mechanisms with several degrees of freedom to be solved.

The other advantage of this method is the introduction of complex coefficients in the adjacency matrix. It is now possible to deal with complex systems with non co-linear input and output axis.

It is also possible to have access to the gear frequencies and the frequencies of all defaults which could appear in

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the various contacts of the mechanism including contacts inside ball (roll)-bearings. Such a kinematic analysis can be very useful in the context of vibration analysis.

Further works will be focused on different directions:

- this approach will be applied to analyze the Main Gear Box (MGB) of an helicopter. This analysis will be used in the KALMAN filter involved in the signal processing of sensors (accelerometers) distributed on the MGB in order to diagnose its health,
- the approach will be also linked to the graph theory (HSU and LAM [6]) in order to develop a procedure to find automatically the reference (carrier) link of gear pairs,
- lastly, the aim of that study was to obtain a list of frequencies at which a default can appear. In that context, there was no interest in Power-flow efficiency or Efficiency analysis. It may be interesting to lead some studies to extend this method to these two domains.

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A Appendix

Table A: List of possible default frequencies in the car

Pair	Frequency (rd/s)	Description	Number of sources
(1,7)	381.3	Default on the inside ring of bearing 1	1
	318.7	Default on the outside ring of bearing 1	1
	310.6	Default on a roll of bearing 1	25
	381.3	Default on the inside ring of bearing 2	1
	318.7	Default on the outside ring of bearing 2	1
	310.6	Default on a roll of bearing 2	25
(2,7)	1820	Gear frequency	
	140	Default on a tooth of link 2	13
	28	Default on a tooth of link 7	65
(3,5), (3,6), (4,5), (4,6)	28	Gear frequency	
	2	Defaults on a tooth of links 3 and 4	14
	2.8	Defaults on a tooth of links 5 and 6	10

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