The Impact of the Magnetic Field on the Transmission of Longitudinal Waves in a Rod

V. I. Erofeyev and A. O. Malkhanov

***Abstract* – This work is devoted to the problem of magnetoelastic waves propagation in rods. The influence of a magnetic field on dispersion, dissipative properties of linear waves as well as impact on parameters of nonlinear waves which propagate in rods has been studied.**

***Index Terms* – magnetic field, waves in rods, magnetoelasticity.**

The effect of magnetoelasticity has been discovered by the Italian physicist E. Villari in 1865. However, magnetoelasticity as a branch of science began to develop at the end of the 1950-ies [Chadwick (1957)]. It arose on the interface of solid mechanics, electrodynamics and acoustics. The first papers were inspired by the problems of geophysics. The goal was to describe the wave dynamics of deep layers of the Earth taking into account its conductivity and interaction with the geomagnetic field.

Since then the study of dynamic processes during the interaction of deformation and electromagnetic fields received close attention. Attention to the problems of magnetoelasticity motivated by the numerous physical, technical and technological applications. Among them are forging and welding of metal constructions by the magneto-pulse method, magnetoacoustic heat of materials, and the problems of defectoscopy [Beliy, Fertik and Klimenko (1997)], [Hefni, Ghaleb and Maugin (1995)], McCarthy (1968)], [Kulikovskii and Sveshnikova (1995)].

Taking into account physical phenomena of the different nature during the modeling of mechanical systems opens new opportunities for the technical and technological development. Effects of magnetoelasticity appear in strong magnetic fields when the induced strains have major influence on wave and dissipative characteristics of elastic media such as rods, planes, and membranes. Anisotropy of properties is typical for boundless magnetoelastic media in a magnetic field. In a medium with finite conductivity, the magnetic field leads to the additional dissipation mechanism. Such features of magnetoelastic systems open new possibilities for practical applications.

As a rule the interaction problem of different physical fields are solved in linear formulation. However, there are a number of publications where non-linear approach was used. [Pospelov (1963)], [Baser and Ericson (1974)], [Donato (1976, 1987)], [Domanski (1993)], [Erofeyev and Kovalev (1997, 1998)] studied nonlinear magnetoelastic waves.

In magnetoelasticity, the influence of a magnetic field on a deformational field is described employing the Lorentz forces

$$\vec{F}_m = \rho_e \vec{E} + \vec{j} \times \vec{B}, \tag{1}$$

which enter equations of motion of an elastic body

$$\rho \frac{\partial^2 \vec{u}}{\partial \tau^2} = (\lambda + \mu) grad\ div\ \vec{u} + \mu \Delta \vec{u} + \vec{F}_{nonlinear} + \vec{F}_m . \tag{2}$$

Here $\vec{E}$ is the intensity of the magnetic field ; $\vec{j}$ is the vector of the electric current density, $\vec{B}$ is the magnetic induction vector; ρ_e is the volumetric density of the electric charges; $\vec{u}$ is displacements vector ; λ, μ are the Lame constants ; ρ is the density of material; τ is the time.

The force $\vec{F}_{nonlinear}$ includes elements which result from the consideration of elastic nonlinearity. If only the quadratic nonlinearity is taken into account, then components of the vector can be represented through the gradients of displacements as follows [7]:

$$\begin{aligned} F_i = &\left(\mu + \frac{A}{4}\right)\left(u_{l,kk}u_{l,i} + u_{i,l}u_{l,kk} + 2u_{l,k}u_{i,lk}\right) + \\ &+\left(\lambda + \mu + \frac{A}{4} + B\right)\left(u_{l,k}u_{l,ik} + u_{i,l}u_{k,lk}\right) + (\lambda + B)\, u_{l,l}u_{i,kk} + \\ &+ (B + 2C)\, u_{l,l}u_{k,ki} + \left(\lambda + \mu + \frac{A}{4} + B\right)\left(u_{l,k}u_{l,ki} + u_{i,l}u_{k,lk}\right) \end{aligned} \tag{3}$$

Here A, B, C are the Landau constants; (index after comma means differentiation with respect to the corresponding coordinate; repeating indices mean summation.

From Maxwell equations, one can obtain equations for electric and magnetic inductions $\left(\vec{D}\right)$ and $\left(\vec{B}\right)$, respectively:

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$$\frac{\partial \vec{D}}{\partial \tau} = rot\, \vec{H} - \vec{j}\,, \tag{4}$$

$$\frac{\partial \vec{B}}{\partial \tau} = rot\left[\frac{\partial \vec{u}}{\partial \tau} \times \vec{B}\right] + \frac{c^2}{4\pi\sigma} \Delta \vec{B}, \tag{5}$$

which along with electromagnetic equations of state

$$\vec{j} = \sigma \vec{E}\,, \vec{D} = \varepsilon_e \vec{E}\,, \vec{B} = \mu_e \vec{H}\,, \tag{6}$$

need to be added to equations (1), (2). Here $\vec{H}$ is intensity of the magnetic field, σ is the conductivity, ε_e is the permittivity and μ_e is the magnetic conductivity, c is the electromagnetic constant.

In magnetoelasticity, are neglected both biasing current and electric field. Due to this, equations of magnetoelasticity can be written as follows:

$$\rho \frac{\partial^2 \vec{u}}{\partial \tau^2} = (\lambda + \mu) grad\ div\ \vec{u} + \mu \Delta \vec{u} +$$

$$+ \vec{F}_{nonlinear} + \frac{1}{4\pi}\left(rot\ \vec{H} \times \vec{H}\right), \tag{7}$$

$$\frac{\partial \vec{H}}{\partial \tau} = rot\left[\frac{\partial \vec{u}}{\partial \tau} \times \vec{H}\right] + \frac{c^2}{4\pi\sigma} \Delta \vec{H}.$$

We consider propagation of the longitudinal waves in a homogeneous nonlinear elastic rod placed in an external magnetic field. Let us suppose that external constant magnetic field with intensity H_0 is transverse to the direction of the waves' propagation (see Fig. 1).

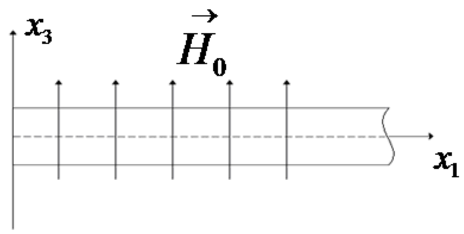

Fig. 1

Generally, magnetic field which results from the interaction between external constant magnetic field and the deformation field can be represented as follows:

$$\vec{H} = \vec{H}_0\, \vec{n} + \vec{h}, \tag{8}$$

where $\vec{h}$ is a small disturbance of the magnetic field, $\vec{n}$ is the normal vector.

For longitudinal elastic waves in the rod and for the magnetic field, we obtain the following expressions:

$$\vec{u} = (u_1, 0, 0), \vec{h} = (h_1, h_2, h_3), \vec{H} = (h_1, h_2, H_0 + h_3). \tag{9}$$

The system of equations of magnetoelasticity, according to the Bishop's model of the rod, can be written as follows:

$$\frac{\partial^2 \vec{u}}{\partial \tau^2} - c_0^2 \left(1 + \frac{6\alpha_1}{E} \frac{\partial \vec{u}}{\partial x_1}\right) \frac{\partial^2 \vec{u}}{\partial x_1^2} -$$

$$- \nu^2 R^2 \frac{\partial^2}{\partial x_1^2}\left(\frac{\partial^2 \vec{u}}{\partial \tau^2} - c_\tau^2 \frac{\partial^2 \vec{u}}{\partial x_1^2}\right) + \frac{1}{4\pi\rho} \vec{H} \frac{\partial \vec{H}}{\partial x_1} = 0, \tag{10}$$

$$\frac{\partial \vec{H}}{\partial \tau} + \vec{H} \frac{\partial^2 \vec{u}}{\partial x_1 \partial \tau} + \frac{\partial \vec{H}}{\partial x_1} \frac{\partial \vec{u}}{\partial \tau} - \frac{c^2}{4\pi\sigma} \frac{\partial^2 \vec{H}}{\partial x_1^2} = 0.$$

Here $c_0 = \sqrt{E/\rho}$ is the velocity of longitudinal waves, $c_\tau = \sqrt{\mu/\rho}$ is the velocity of shear waves.

According to the model under the consideration, only a transverse component of the magnetic field (h_3) is taken into the account. Other items in the system (10), which include components $(h_1),(h_2)$ have smaller order than others and therefore can be neglected. Thus expressions (9) can be presented as follows:

$$\begin{aligned} &\vec{u} = (u_1, 0, 0) \equiv u(x_1, \tau), \vec{h} = (0, 0, h_3) \equiv h(x_1, \tau), \\ &\vec{H} = (0, 0, H_0 + h_3) \equiv H_0 + h(x_1, \tau). \end{aligned} \tag{11}$$

First of all, we consider linear Bernoulli model for longitudinal waves in a rod. It can be obtained if all the items corresponding to the kinetic energy of transverse deformations, potential energy of shear deformations and all nonlinear items in (10) will be neglected. The following system is obtained:

$$\begin{aligned} &\frac{\partial^2 u}{\partial \tau^2} - c_0^2 \frac{\partial^2 u}{\partial x_1^2} + \frac{H_0}{4\pi\rho} \frac{\partial h}{\partial x_1} = 0, \\ &\frac{\partial h}{\partial \tau} + H_0 \frac{\partial^2 u}{\partial x_1 \partial \tau} - \frac{c^2}{4\pi\sigma} \frac{\partial^2 h}{\partial x_1^2} = 0. \end{aligned} \tag{12}$$

We suppose that the magnetic field is stationary i.e. $h(x_1, \tau) \equiv h(x_1)$, then system (12) can be rewritten as follows:

$$\begin{aligned} &\frac{\partial^2 u}{\partial \tau^2} - c_0^2 \frac{\partial^2 u}{\partial x_1^2} + \frac{H_0}{4\pi\rho} \frac{\partial h}{\partial x_1} = 0, \\ &\frac{4\pi\sigma H_0}{c^2} \frac{\partial^2 u}{\partial x_1 \partial \tau} = \frac{\partial^2 h}{\partial x_1^2}. \end{aligned} \tag{13}$$

Thus, the following equation is obtained:

$$\frac{\partial^2 u}{\partial \tau^2} - c_0^2 \frac{\partial^2 u}{\partial x_1^2} + \frac{\sigma H_0^2}{\rho c^2} \frac{\partial u}{\partial \tau} = 0. \tag{14}$$

From equation (14) one can see that the external magnetic field leads to the appearance of the effective viscous dissipation.

For further analysis, let us rewrite system (12) in a non-dimensional form:

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$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + C\frac{\partial h}{\partial x} = 0,$$
$$\frac{\partial h}{\partial t} + \frac{\partial^2 u}{\partial x \partial t} - \frac{1}{\Sigma}\frac{\partial^2 h}{\partial x^2} = 0 \quad (15)$$

Here $C = \frac{c_A^2}{c_0^2}$ is the non-dimensional wave velocity, $\Sigma = \frac{\sigma}{\sigma_0}$ is the non-dimensional conductivity, t is the non-dimensional time, x is the non-dimensional coordinate, $c_A = \sqrt{\frac{H_0^2}{4\pi\rho}}$ is the Alfven wave velocity.

Assuming that

$$u = u_0 e^{i(\omega t - kx)},$$
$$h = h_0 e^{i(\omega t - kx)} \quad (16)$$

we obtain the dispersion equation:

$$\frac{1}{\Sigma}k^4 - \left(\frac{1}{\Sigma}\omega^2 - i\omega(1+C)\right)k^2 - i\omega^3 = 0. \quad (17)$$

If the rod under consideration represents an ideal conductor $(\Sigma = \infty)$, then the following equation is deduced:

$$(1+C)k^2 + \omega^2 = 0. \quad (18)$$

Dependency of the wavenumber (k) on the frequency (ω) of longitudinal waves for the case of ideally conductive rod is shown in Fig. 2:

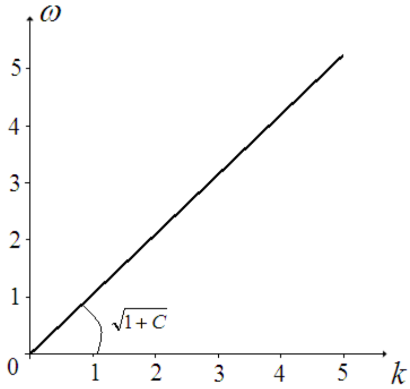

Fig. 2

The intensity of the magnetic field determines the value of the phase velocity: the larger the intensity, the larger the phase velocity. The phase velocity (V_{phase}) is shown in Fig. 3 as a function of the wave frequency (ω):

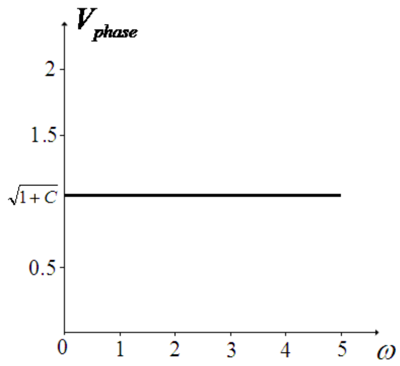

Fig. 3

The finite conductivity of the rod $(\Sigma \neq \infty)$ leads to the appearance of the imaginary part of wavenumber $(\mathrm{Re}\,(k))$ that causes wave decay. Results of numeric solution of equation (17) are shown in Figs. 4, 5:

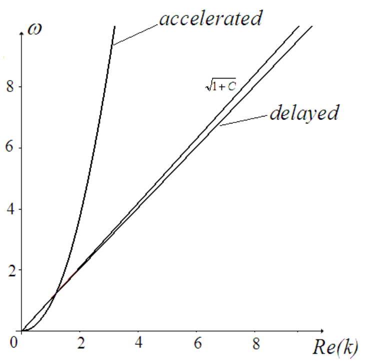

Fig. 4

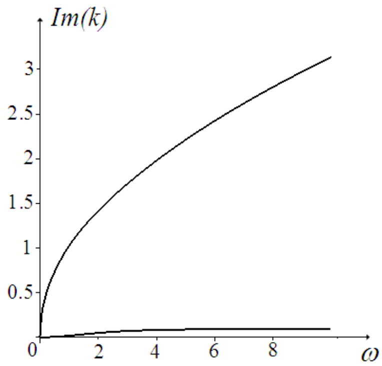

Fig. 5

For the numeric simulations, the following values of non-dimensional velocity and conductivity have been chosen: *C=0.1, Σ=2.* Using Figs. 4, 5, one can conclude that in case of finite conductivity of the rod there exist two waves. One of them is delayed and the other one is accelerated in relation to the wave in the case of ideal conductivity of the rod. Both waves propagate with negligible decay since for both of them the real part $(\mathrm{Re}\,(k))$ of wave number is larger than the imaginary one $(\mathrm{Im}\,(k))$.

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The phase velocity $\left(V_{phase}\right)$ as a function of frequency (ω) is shown in Figs. 6, 7.

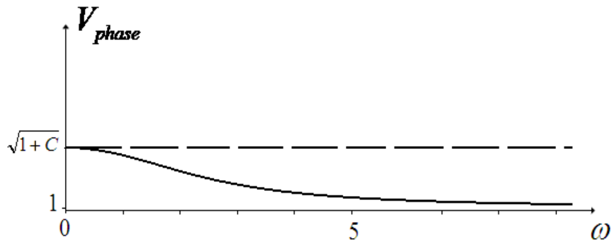

Fig. 6

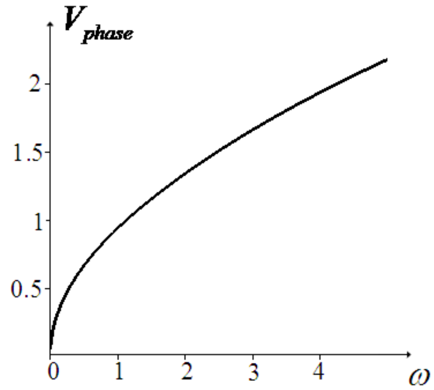

Fig. 7

One can see from Fig. 6 that the presence of the magnetic field leads to the increase in the phase velocity. At the same time it decreases as a function of frequency. The phase velocity of the second wave increases independently of the magnetic field (see Fig. 7).

For the further research, we need to obtain the evolutionary equation. To achieve that, we make the change of the variables and introduce a small parameter. Let us rewrite system (10) in the form

$$\begin{aligned}&\frac{\partial G}{\partial \tau}-c_0^2\left(1+\frac{6\alpha_1}{E}Q\right)\frac{\partial Q}{\partial x_1}-\nu^2R^2\frac{\partial^2}{\partial x_1^2}\left(\frac{\partial G}{\partial \tau}-c_\tau^2\frac{\partial Q}{\partial x_1}\right)+\\&\quad+\frac{1}{4\pi\rho}\left(H_0+h_3\right)\frac{\partial h_3}{\partial x_1}=0,\\&\frac{\partial Q}{\partial \tau}-\frac{\partial G}{\partial x_1}=0,\\&\frac{\partial h_3}{\partial \tau}+\left(H_0+h_3\right)\frac{\partial G}{\partial x_1}+G\frac{\partial h_3}{\partial x_1}-\frac{c^2}{4\pi\sigma}\frac{\partial^2 h_3}{\partial x_1^2}=0.\end{aligned}\tag{19}$$

Here $Q=\frac{\partial u}{\partial x}$ is the deformation, $G=\frac{\partial u}{\partial \tau}$, ν is the Poisson coefficient , $R=\sqrt{J_0/F}$ is the polar radius of gyration, $J_0=\iint\limits_F\left(x_2^2+x_3^2\right)dF$ is the polar moment of inertia, F is the area of cross-section of the rod, $E=\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$ is the modulus of elasticity, $\alpha_1=\frac{E}{2}+\frac{3\lambda}{2}+A+B(1-2\nu)+\frac{C}{3}(1-6\nu)$ is the elastic nonlinearity coefficient.

We introduce non-dimensional variables

$$\begin{aligned}&U=Q, V=\frac{G}{c_0}, W=\frac{h_3}{H_0},\\&\tilde{x}_1=\frac{1}{\nu R}x_1, \tilde{\tau}=\frac{c_0}{\nu R}\tau\end{aligned}\tag{20}$$

as well as a moving reference system:

$$x=\tilde{x}_1-V_p\tilde{\tau}, t=\varepsilon\tilde{\tau}\tag{21}$$

where V_p is characteristic wave velocity, not known in advance, ε is a small parameter.

Substituting (20) and (21) into (19) and omitting items one of the second order and higher, we obtain the following systems of equations:

$$\begin{aligned}&-V_p\frac{\partial V}{\partial x}-\frac{\partial U}{\partial x}+\frac{c_A^2}{c_0^2}\frac{\partial W}{\partial x}=0,\\&-V_p\frac{\partial U}{\partial x}-\frac{\partial V}{\partial x}=0,\\&-V_p\frac{\partial W}{\partial x}+\frac{\partial V}{\partial x}=0.\end{aligned}\tag{22}$$

$$\begin{aligned}&\frac{\partial V}{\partial t}-\frac{6\alpha_1}{E}U\frac{\partial U}{\partial x}+V_p\frac{\partial^3 V}{\partial x^3}+\frac{c_\tau^2}{c_0^2}\frac{\partial^3 U}{\partial x^3}+\frac{c_A^2}{c_0^2}W\frac{\partial W}{\partial x}=0,\\&\frac{\partial W}{\partial t}+W\frac{\partial V}{\partial x}+V\frac{\partial W}{\partial x}-\frac{c^2}{4\pi\sigma c_0\nu R}\frac{\partial^2 W}{\partial x^2}=0.\end{aligned}\tag{23}$$

which represent zero and first dimensionless approximations of the system (19), respectively. Using the 2nd and 3rd equations in (22), we obtain the connection between the functions:

$$U=-W, V=-V_pU,\tag{24}$$

and from the 1st equation we determine the velocity:

$$V_p=\sqrt{1+\frac{c_A^2}{c_0^2}}.\tag{25}$$

Substituting (24) and (25) into (23) and summing the obtained equations, we transform it to the equation of the form

$$\frac{\partial U}{\partial t}-\alpha U\frac{\partial U}{\partial x}+\beta\frac{\partial^3 U}{\partial x^3}-\delta\frac{\partial^2 U}{\partial x^2}=0.\tag{26}$$

Here $\alpha=\dfrac{\frac{c_A^2}{c_0^2}-\frac{6\alpha_1}{E}+2V_p}{1+V_p}$, $\beta=\dfrac{V_p^2-\frac{c_\tau^2}{c_0^2}}{1+V_p}$, $\delta=\dfrac{c^2}{4\pi\cdot\sigma c_0\nu R(1+V_p)}$.

Further, we consider equations derived from the initial evolutionary equation deprived of this or that component. We will consider solutions for the localized waves only. This is due to the localized wave energy focused in a limited segment

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unlike the energy of periodic waves, which causes higher loads in the rod that in its turn may lead to its destruction.

First of all, let us review the initial evolutionary equation (26). It represents Korteweg - de Vries - Burgers equation. This equation has a localized solution in the form of a soliton that can be represented as a blast wave:

$$U = A \cdot \exp(\xi) \operatorname{sech}^2\left(\frac{\xi}{2}\right), \xi = \xi_0(x - 2at),$$
$$\xi_0 = \sqrt{-\frac{a}{3\beta}}, a = -\frac{3\delta^2}{25\beta}, \tag{27}$$
$$A = -\frac{a}{\alpha}, V_k = 2a, \Delta = \frac{2}{\xi_0}.$$

The form of this wave is shown in Fig. 8:

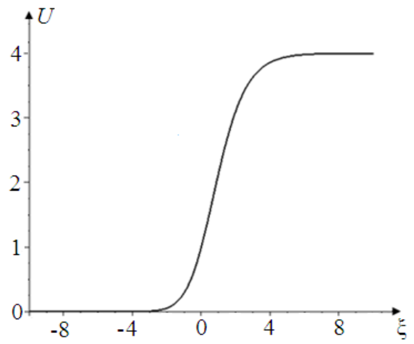

Fig. 8

For condensed media in magnetic fields under 10 tesla, Alfven wave velocity is smaller than a longitudinal wave velocity. That is why, changing of all parameters is shown on $0 \le c_A/c_0 < 1$ interval.

One can see from the figures how the external magnetic field intensity affects the amplitude (A), velocity (V_k) and width (Δ) of the solution. With the increase in magnetic field, both amplitude and velocity decrease and the width increases.

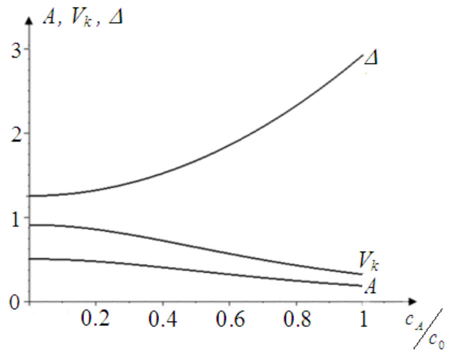

Fig. 9

Let us assume that the rod is an ideal conductor so that the last term in the evolutionary equation with δ coefficient can be neglected. We obtain the Korteweg - de Vries' equation:

$$\frac{\partial U}{\partial t} - \alpha U \frac{\partial U}{\partial x} + \beta \frac{\partial^3 U}{\partial x^3} = 0. \tag{28}$$

It has a localized solution in the form of soliton of negative polarity:

$$U = -\frac{A}{\operatorname{ch}^2\left(\frac{\xi}{\Delta}\right)},$$
$$\xi = x - V_s t, A = \frac{3V_s}{\alpha}, \Delta = \sqrt{\frac{12\beta}{A\alpha}}. \tag{29}$$

The form of the soliton is shown in Fig.10:

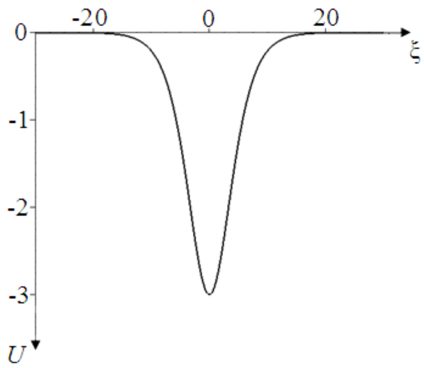

Fig. 10

One can see from Fig. 11 that for a constant velocity (V_s), amplitude (A) and width (Δ) increase, while for a constant amplitude, velocity decreases with increase of the magnetic field.

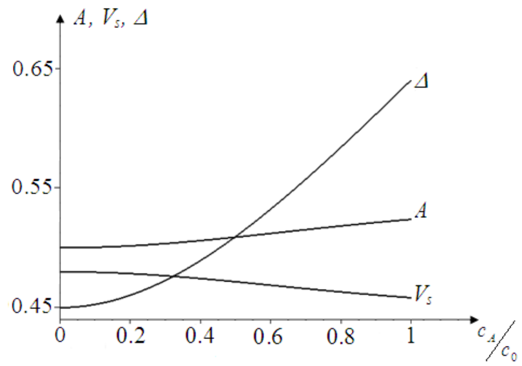

Fig. 11

Finally, let us suppose that the initial system of magnetoelasticity doesn't take into account the kinetic energy of transverse deformations. In this case, from the initial evolutionary equation one will obtain the Burgers' equation:

$$\frac{\partial U}{\partial t} - \alpha U \frac{\partial U}{\partial x} - \delta \frac{\partial^2 U}{\partial x^2} = 0. \tag{30}$$

It has a localized solution in the form of a shock wave:

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$$U = V_b - A \cdot \mathrm{th}\left[\frac{A(x - V_b t)}{2\delta}\right],$$
$$\xi = x - V_b t, A = \frac{U(-\infty) - U(+\infty)}{2}, \tag{31}$$
$$V_b = \frac{U(-\infty) + U(+\infty)}{2}, \Delta = \frac{2\delta}{A}.$$

The form of the shock wave is shown in Fig.12:

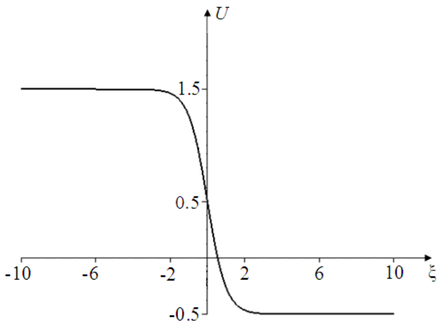

Fig. 12

It has been shown that width (Δ) of the shock wave decreases with the increase of the magnetic field:

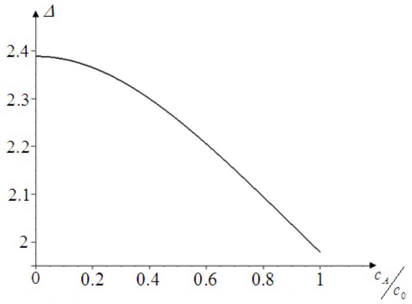

Fig. 13

In this paper, we have obtained the evolutionary equation for the system of magnetoelasticity equations; considered different possible cases of its transformation, and studied localized solutions and their dependencies in a magnetic field. The analysis performed proves that the magnetic field affects parameters of longitudinal deformation waves in a rod.

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