Proceedings of the World Congress on Engineering 2010 Vol III
WCE 2010, June 30 - July 2, 2010, London, U.K.

Properties of Modified p-Cyclic Self-Maps in Metric Spaces

M. De la Sen

Abstract- **This paper investigates p-cyclic self-maps $T: X \to X$ in problems involving perturbations which satisfy a distance constraint in a metric space which mixed non-expansive or contractive properties and potentially expansive properties related to some distance threshold. The above mentioned constraint is feasible in certain real -world problems. Two classes of self-maps are investigated, namely, those which become p-cyclic strict contractions in the absence of perturbations and those which in the same conditions become p-cyclic contractions of Meir-Keeler type**.

Keywords-**strictly contractive maps, non-expansive maps, metric space, fixed points, p-cyclic contractions, perturbations.**

I. INTRODUCTION

Recently, the subsequent set sophisticated related problems are under strong research activity: **1)** In the, so-called, p (≥ 2) -cyclic non-expansive or contractive self-maps map each element of a subset $A_i \in \{A_1, A_2, ..., A_p\}$ of an either metric or Banach space **B** to an element of the next subset A_{i+1} in a strictly ordered chain of p subsets of **B** such that $A_{p+1} = A_1$. If the above subsets do not intersect then fixed points do not exist and their potential relevance in Analysis is played by best proximity points, [1-2]. Best proximity points are also of interest in hyperconvex metric spaces, [3-4]. **2)** The so-called Kannan maps are also being intensively investigated in the last years as well as their relationships with contractive maps. See, for instance, [5-6], [11]. **3)** Although there is an increasing number of theorems about fixed points in Banach or metric spaces, new related recent results have been proven. Some of those novel results are, for instance, the generalization in [7] of Edelstein´s fixed point theorem for metric spaces by proving a new theorem. Also, an iterative algorithm for searching a fixed point in a closed convex subset of a Banach space has been proposed in [8]. On the other hand, an estimation of the size of an attraction ball to a fixed point has been provided in [9] for nonlinear differentiable maps. **4)** Self-maps T in complete (or compact) metric spaces (X, d) are classified in four classes in [12], namely: T is said to be of Leader-type (or Picard operator) if it guarantees the convergence of any iteration through T to the unique fixed point . Also, T is said to be of Unnamed-type if convergence of all the iterations to the unique fixed point is not guaranteed . In the so-called Subrahmanyam -type (or weakly Picard operator), all iterations converge to a fixed point which can be non-unique. Finally, T is said to be of Caristi-type if iterations not necessarily converge to some eventually non-unique fixed points. **5)** Fixed point theory can be also used successfully to find oscillations of solutions of differential or difference equations which can be themselves characterized as fixed points. The formalism is also useful to investigate stability and boundedness of the solutions in time-delay and continuous/ discrete hybrid dynamic systems. See, for instance, [9-10], [13- 15], [19], [25]. On the other hand, the existence of positive solutions of some useful differential equations can be investigated by using the fixed point index . See, for instance, [22-23] and references therein. This manuscript is devoted to investigate the nonexpansive ([2], [7], [24]) and contractive properties of self-maps $T: X \to X$ in a metric space (X, d) which satisfy the constraint:

$$\begin{aligned} -K_{1i}\, d(x,y) + K_{1i}\, d_i + M_{1i}(x,y) &\leq d(Tx, Ty) - d(x,y) \\ &\leq -K_{2i}\, d(x,y) + K_{2i}\, d_i + M_{2i}(x,y) \quad (1.1) \end{aligned}$$

; $\forall (x, y) \in A_i \times A_{i+1}$ for some real constants K_{1i}, K_{2i} and some real functions $M_{ji}: A_i \times A_{i+1} \to \mathbf{R}_{0+}$; $\forall i \in \overline{p} := \{1, 2..., p\}$; j=1, 2 where $\mathbf{R}_{0+} := \{z \in \mathbf{R}_+ : z \geq 0\}$ and $d_i := dist(A_i, A_{i+1})$; $\forall i \in \overline{p}$ are the distances between adjacent subsets A_i and A_{i+1} of X ; $\forall i \in \overline{p}$ .The p-cyclic restricted self -map $T: \bigcup_{i \in \overline{p}} A_i \to \bigcup_{i \in \overline{p}} A_i$ (i.e. the domain and the image of $T: X \to X$ are restricted to $\bigcup_{i \in \overline{p}} A_i$) is subject to the set of constraints $T(A_i) \subseteq A_{i+1}$; $\forall i \in \overline{p}$. The set of fixed points of the map T is denoted through the manuscript by $F(T)$.

II. MAIN RESULTS FOR MODIFIEC CYCLIC CONTRACTIONS

This section is mainly concerned with the derivation of some mathematical results about uniform boundedness of the iteration of distances of pairs of points belonging to adjacent subsets of X in the presence of perturbations given by the functions $M_{ji}: A_i \times A_{i+1} \to \mathbf{R}_{0+}$; $\forall i \in \overline{p} := \{1, 2..., p\}$; j=1, 2. First note that Eq.(1.1) is equivalent to:

$$\begin{aligned} (1-K_{1i})\, d(x,y) + K_{1i}\, d_i + M_{1i}(x,y) &\leq d(Tx, Ty) \\ &\leq (1-K_{2i})\, d(x,y) + K_{2i}\, d_i + M_{2i}(x,y) \qquad (2.1) \end{aligned}$$

Note that if $d_i = M_{2i}(x,y) = 0$, $K_{2i} \in [0,1)$; $\forall (x, y) \in A_i \times A_{i+1}$; $\forall i \in \overline{p}$ then (2.1) implies that

Manuscript received December, 21 2009. This work was supported in part by the Spanish Ministry of Science and Innovation through Grant DPI2009-07197 and also by Basque Government by its support through Grant GIC07143-IT-269-07.

M. De la Sen is with the Institute of Research and Development of Processes, Campus of Leioa , Bizkaia , Aptdo. 644- Bilbao. Spain (e-mail: manuel.delasen@ehu.es).

ISBN: 978-988-18210-8-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$T:\bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is a strict contraction according to Banach contraction principle. If $M_{2i}(x,y)=0$; $\forall i\in\overline{p}$ then $T:\bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ satisfying (2.1) for each $(x,y)\in A_i\times A_{i+1}$; $\forall i\in\overline{p}$ is a po-cyclic contraction [2, 7]. Through this section, it is proven that the contraction principle does not hold, in general. However, $T:\bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is not expansive for sufficiently large distances between $(x,y)\in A_i\times A_{i+1}; \forall i\in\overline{p}$. In this context and in view of (1.1), it is possible to speak about the restricted self-map $T:\bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ being locally non expansive, contractive and expansive as follows. In particular, note that $d(Tx,Ty)\le d(x,y)$; i.e. $T:X|A_i\to A_{i+1}$ is locally non-expansive for a given pair $(x,y)\in A_i\times A_{i+1}$ if $d(x,y)\ge\frac{K_{2i}d_i+M_{2i}(x,y)}{K_{2i}}$.

Also, $d(Tx,Ty)<d(x,y)$; i.e. $T:X|A_i\to A_{i+1}$ is locally contractive for a given pair $(x,y)\in A_i\times A_{i+1}$ if $d(x,y)>\frac{K_{2i}d_i+M_{2i}(x,y)}{K_{2i}}$. Finally, for any given $(x,y)\in A_i\times A_{i+1}$:

$d(x,y)<\frac{K_{1i}d_i+M_{1i_i}(x,y)}{K_{1i}}$

$\Rightarrow d(Tx,Ty)>d(x,y)$; $\forall x,y\in X$

so that $T:X|A_i\to A_{i+1}$ is locally expansive for such a given pair. Since $T:X|A_i\to A_{i+1}$ cannot be simultaneously locally expansive and locally non-expansive for any given pair $(x,y)\in A_i\times A_{i+1}$ the following inequality is not feasible for any $(x,y)\in A_i\times A_{i+1}$:

$$\frac{K_{1i}d_i+M_{1i_i}(x,y)}{K_{1i}}>d(x,y)\ge\frac{K_{2i}d_i+M_{2i}(x,y)}{K_{2i}} \qquad (2.2)$$

The following result is concerned with sets of necessary constraints for (1.1) to hold.

Propositions 2.1. The following propositions follow directly:

(i) If the constants K_{ji} have to satisfy the sets of necessary conditions:

$$M_{1i}(x,y)/M_{2i}(x,y)\le K_{1i}/K_{2i}\ ;\ i\in\overline{p}\ ;\quad j=1,\ 2;\ \forall\,(x,y)\in A_i\times A_{i+1} \qquad (2.3)$$

then the unfeasible condition (2.2) never holds in $A_i\times A_{i+1}$. If

$$M_{1i}(x,y)/M_{2i}(x,y)>K_{1i}/K_{2i} \text{ for some } (x,y)\in A_i\times A_{i+1},\ i\in\overline{p},\ \text{j=1, 2} \qquad (2.4)$$

then

$$d(x,y)\notin\left[\frac{K_{2i}d_i+M_{2i}(x,y)}{K_{2i}},\frac{K_{1i}d_i+M_{1i_i}(x,y)}{K_{1i}}\right)$$

(ii) $(K_{1i}-K_{2i})d(x,y)+(K_{2i}-K_{1i})d_i$

$$+(M_{2i}(x,y)-M_{1i}(x,y))\ge 0 \qquad (2.5)$$

$\forall(x,y)\in A_i\times A_{i+1}$, $\forall i\in\overline{p}$

(iii) If $d_i=0$ (i.e. $A_i\cap A_{i+1}\ne\varnothing$) and $M_i(x,y)=0$; $\forall(x,y)\in A_i\times A_{i+1}$ for some $i\in\overline{p}$ then $1\ge K_{1i}\ge K_{2i}\ge 0$ or $1\ge K_{2i}\ge 0$ and $K_{1i}\ge 1$. In the second case, the first inequality of (2.1) holds trivially everywhere in $A_i\times A_{i+1}$.

Proof: *(i)* Since (1.3) is unfeasible for any pair $(x,y)\in A_i\times A_{i+1}$; $\forall i\in\overline{p}$ then

$$\frac{K_{2i}d_i+M_{2i}(x,y)}{K_{2i}}\ge\frac{K_{1i}d_i+M_{1i}(x,y)}{K_{1i}}$$

which is equivalent to (1.4), guarantees that (2.2) does not hold in $A_i\times A_{i+1}$. Also, (2.4) together with the companion constraint for the distance guarantees that (2.4) does not hold.

(ii)-(iii) follow directly from (2.1) which requires the necessary condition

$$(1-K_{1i})d(x,y)+K_{1i}d_i+M_{1i}(x,y)$$
$$\le(1-K_{2i})d(x,y)+K_{2i}d_i+M_{2i}(x,y)$$

$\forall(x,y)\in A_i\times A_{i+1}$, $\forall i\in\overline{p}$ □

The following result proves uniform boundedness of the distance iterates independently of the iteration index but dependent, in general, of the initial points. The limit superiors of the iterations are uniformly bounded independent of the iteration index and also independent of the initial points.

Theorem 2.2. The following properties hold:

(i) Assume that $M_{2i}(x,y)\le\alpha_{2i}\,d(Tx,Ty)+\gamma_{2i}$ with $\gamma_{2i}\in\mathbf{R}_{0+}$, $0\le\alpha_{2i}<K_{2i}<1$; $\forall(x,y)\in A_i\times A_{i+1}$, $\forall i\in\overline{p}$. Then , $d\left(T^{jp}x,T^{jp}y\right)\le L(x,y)<\infty$; $\forall(x,y)\in A_i\times A_{i+1}$, $\forall i\in\overline{p}$ where $L(x,y)$ being a bound dependent on the pair (x, y) ; $\forall(x,y)\in A_i\times A_{i+1}$; $\forall i\in\overline{p}$ which is uniform for all $j\in\mathbf{Z}_+$ provided that $d(x,y)$ is bounded. Furthermore,

$$\limsup_{j\to\infty}\ d\left(T^{jp}x,T^{jp}y\right)\le\beta_2:=\frac{\sum_{i=1}^{p}\left(\prod_{k=i+1}^{p}\left[\frac{1-K_{2k}}{1-\alpha_{2k}}\right]\frac{K_{2i}d_i+\gamma_{2i}}{1-\alpha_{2i}}\right)}{1-\prod_{i=1}^{p}\left[\frac{1-K_{1i}}{1-\alpha_{1i}}\right]} \qquad (2.6)$$

is uniformly bounded $\forall x,y\in\bigcup_{i\in\overline{p}}A_i$ .Also, there is an upper-bound $\theta_{2j}(x,y)$ of $\sup_{\ell\ge j} d\left(T^{\ell p}x,T^{\ell p}y\right)$; $\forall x,y\in\bigcup_{i\in\overline{p}}A_i$ which is sufficiently close to β_2 for sufficiently large $j\in\mathbf{Z}_+$ in the sense that , for any prescribed arbitrarily small $\varepsilon\in\mathbf{R}_+$, $\left|\theta_{2j}(x,y)-\beta_2\right|\le\varepsilon$; $\forall j\ge N=N(\varepsilon)\in\mathbf{Z}+$ for some finite $N\in\mathbf{Z}+$.

ISBN: 978-988-18210-8-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

(ii) Assume that Property (i) holds and, in addition, $M_{1i}(x,y) \ge \alpha_{1i}\, d(Tx,Ty) + \gamma_{1i}$ with $\alpha_{1i}(\le \alpha_{2i}) \in \mathbf{R}_{0+}$ and $\gamma_{1i}(\le \gamma_{2i}) \in \mathbf{R}_{0+}$ and $0 \le \alpha_{1i} < K_{1i} \le K_{2i}\, min\left(1, \frac{1-\alpha_{1i}}{1-\alpha_{2i}}\right) < 1$; $\forall (x,y) \in A_i \times A_{i+1}$, $\forall i \in \overline{p}$.

Then, $d\left(T^{jp}x, T^{jp}y\right) \in [\beta_1 - \varepsilon_0, \beta_2 + \varepsilon_0]$, $\forall j(\ge max(N, N_0)) \in \mathbf{Z}_+$ and some finite $N_0 = N_0(\varepsilon_0)$ where $\beta_1 := \dfrac{\sum_{i=1}^{p}\left(\prod_{k=i+1}^{p}\left[\frac{1-K_{1k}}{1-\alpha_{1k}}\right]\frac{K_{1i}\, d_i + +\gamma_{1i}}{1-\alpha_{1i}}\right)}{1 - \prod_{i=1}^{p}\left[\frac{1-K_{1i}}{1-\alpha_{1i}}\right]}$.

Furthermore, there is a lower-bound $\theta_{1j}(x,y)$ of $\inf_{\ell \ge j} d\left(T^{\ell p}x, T^{\ell p}y\right)$; $\forall (x,y) \in A_i \times A_{i+1}$; $\forall i \in \overline{p}$ which is sufficiently close to β_1 for sufficiently large $j \in \mathbf{Z}_+$ in the sense that, for any given arbitrarily small $\varepsilon_0 \in \mathbf{R}_+$, $\left|\theta_{1j}(x,y) - \beta_1\right| \le \varepsilon_0$; $\forall j \ge N_0 = N_0(\varepsilon_0) \in \mathbf{Z}+$ for some finite $N_0 \in \mathbf{Z}+$. Also, $d\left(T^{jp}x, T^{jp}y\right) \in [\beta_1 - \varepsilon_0,\ \beta_2 + \varepsilon]$; $\forall j(\ge max(N, N_0)) \in \mathbf{Z}_+$.

(iii) Consider the restricted map $T : X \mid \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ and assume that Property (i) holds and, furthermore, $\gamma_{2i} = 0$; $\forall i \in \overline{p}$ and $\bigcap_{i\in\overline{p}} A_i \ne \varnothing$. Then , $\exists z \in F\left(T \mid \bigcup_{i\in\overline{p}} A_i\right) \subseteq \bigcap_{i\in\overline{p}} A_i$ and the p-cyclic restricted self-map $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is strictly contractive. If, furthermore, (X,d) is complete then the fixed point is unique.

Proof: *(i)* The following chain of inequalities follows by direct inspection of (2.1) provided that $M_{2i}(x,y) \le \alpha_{2i}\, d(Tx,Ty) + \gamma_{2i}$ and $0 \le \alpha_{2i} < K_{2i} < 1$; $\forall (x,y) \in A_i \times A_{i+1}$, $\forall i \in \overline{p}$:

$$d(Tx,Ty) \le (1-K_{2i})d(x,y) + K_{2i}\, d_i + M_{2i}(x,y)$$
$$\le (1-K_{2i})d(x,y) + K_{2i}\, d_i + \alpha_{2i}\, d(Tx,Ty) + \gamma_{2i}$$
$$\Rightarrow d(Tx,Ty) \le \frac{1-K_{2i}}{1-\alpha_{2i}} d(x,y) + \frac{K_{2i}\, d_i + \gamma_{2i}}{1-\alpha_{2i}}$$
$$\Rightarrow d\left(T^p x, T^p y\right) \le \left(\prod_{i=1}^{p}\left[\frac{1-K_{2i}}{1-\alpha_{2i}}\right]\right) d(x,y)$$
$$+ \sum_{i=1}^{p}\left(\prod_{j=i+1}^{p}\left[\frac{1-K_{2j}}{1-\alpha_{2j}}\right]\frac{K_{2i}\, d_i + \gamma_{2i}}{1-\alpha_{2i}}\right) < \infty$$
; $\forall j \in \mathbf{Z}_+$
$$\Rightarrow \limsup_{j\to\infty} d\left(T^{jp}x, T^{jp}y\right) \le \beta_2$$
$$\Rightarrow d\left(T^{jp}x, T^{jp}y\right) \in [0, \beta_2 + \varepsilon] \qquad (2.7)$$

; $\forall (x,y) \in A_i \times A_{i+1}$, $\forall i \in \overline{p}$ for any given $\varepsilon \in \mathbf{R}_+$ and some finite sufficiently large $N = N(\varepsilon) \in \mathbf{Z}_+$ since $\prod_{i=1}^{p}\left[\frac{1-K_{2i}}{1-\alpha_{2i}}\right] < 1$. Property (i) has been proven.

(ii) If Property (i) holds and, furthermore, $M_{1i}(x,y) \ge \alpha_{1i}\, d(Tx,Ty) + \gamma_{1i}$, $\gamma_{1i} \le \gamma_{2i}$ and $0 \le \alpha_{1i} \le \alpha_{2i}$; $\forall (x,y) \in A_i \times A_{i+1}$, $\forall i \in \overline{p}$. Then one gets in a similar way $\liminf_{j\to\infty} d\left(T^{jp}x, T^{jp}y\right) \ge \beta_1$. Thus, for any given $\varepsilon_0 \in \mathbf{R}_+$, $\exists N_0 = N_0(\varepsilon_0) \in \mathbf{Z}_+$ being sufficiently large so that $d\left(T^{jp}x, T^{jp}y\right) \in [\beta_1 - \varepsilon_0,\ \beta_2 + \varepsilon]$; $\forall j(\ge max(N, N_0)) \in \mathbf{Z}_+$. Property (ii) follows.

(iii) Note that $\lim_{j\to\infty} d\left(T^{jp}x, T^{jp}y\right) = 0$; $\forall (x,y) \in A_i \times A_{i+1}$ for any $i \in \overline{p}$ if $\gamma_{2i} = 0$; $\forall i \in \overline{p}$, and $\bigcap_{i\in\overline{p}} A_i \ne \varnothing$ $(\Leftrightarrow d_i = 0; \forall i \in \overline{p})$ as a result so that $T \mid \bigcup_{i\in\overline{p}} A_i := \left(T : X \mid \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i\right)$ has a fixed point . This is obvious since $T(A_i) = A_{i+1}$. Thus , $z \in A_j \cap F\left(T \mid \bigcup_{i\in\overline{p}} A_i\right)$ for some $j \in \overline{p}$ since $z \in \bigcup_{i\in\overline{p}} A_i$. Thus, $Tz = z \in A_j \cap A_{j+1}$ and proceeding recursively:

$$T^p z = z \in \bigcap_{i\in\overline{p}} A_i = \lim_{j\to\infty} T^j x \quad ; \quad \forall x \in \bigcup_{i\in\overline{p}} A_i$$

$F\left(T \mid \bigcup_{i\in\overline{p}} A_i\right) \subset \bigcap_{i\in\overline{p}} A_i$ is proven as follows. Since $T(A_i) \subseteq A_{i+1}; \forall i \in \overline{p}$, then $T^k z = z = \lim_{j\to\infty} T^j x \in F\left(T \mid \bigcup_{i\in\overline{p}} A_i\right) \ne \varnothing \Rightarrow z \in T^i A_\ell \subseteq A_{\ell+i}$ for some $\ell \in \overline{p}$, $\forall x \in \bigcup_{i\in\overline{p}} A_i$, $\forall i \in \overline{p}$. Then, $z \in \bigcap_{i\in\overline{p}} A_i$ which is unique from Banach contraction principle if (X,d) is complete. □

The conditions $0 \le \alpha_{2i} < K_{2i} < 1$ and

$$0 \le \alpha_{1i} < K_{1i} \le K_{2i}\, min\left(1, \frac{1-\alpha_{1i}}{1-\alpha_{2i}}\right) < 1$$

of Theorem 2.2 are now weakened by replacing them by weaker ones related to the whole p-cycle of the restricted map $T \mid \bigcup_{i\in\overline{p}} A_i$ as follows:

ISBN: 978-988-18210-8-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

Corollary 2.3. The following properties hold:
(i) Assume that $M_{2i}(x,y) \le \alpha_{2i}\, d(Tx,Ty) + \gamma_{2i}$ with $0 \le K_{2i} \le 1$ and $\gamma_{2i} \in \mathbf{R}_{0+}$; $\forall (x,y) \in A_i \times A_{i+1}$, $\forall i \in \overline{p}$ and $\rho_2 := \prod_{i=1}^{p} \left[\frac{1-K_{2i}}{1-\alpha_{2i}} \right] < 1$. Then, Theorem 2.2 (i) holds.
(ii) Assume that Property (i) holds and, in addition, $M_{1i}(x,y) \ge \alpha_{1i}\, d(Tx,Ty) + \gamma_{1i}$ with, $0 \le K_{1i} \le K_{2i} \le 1$ $\alpha_{1i} (\le \alpha_{2i}) \in \mathbf{R}_{0+}$ and $\gamma_{1i} (\le \gamma_{2i}) \in \mathbf{R}_{0+}$ and $\rho_1 := \prod_{i=1}^{p} \left[\frac{1-K_{1i}}{1-\alpha_{1i}} \right] < 1$; $\forall (x,y) \in A_i \times A_{i+1}$, $\forall i \in \overline{p}$. Then, Theorem 2.2 (ii) holds.

(iii) Consider the restricted map $T: X \mid \bigcup_{i \in \overline{p}} A_i \to \bigcup_{i \in \overline{p}} A_i$ and assume that Property (i) holds and, furthermore, $\gamma_{2i} = 0$; $\forall i \in \overline{p}$ and $\bigcap_{i \in \overline{p}} A_i \ne \varnothing$. Then, $\exists z \in F\left(T \mid \bigcup_{i \in \overline{p}} A_i \right) \subseteq \bigcap_{i \in \overline{p}} A_i$ and $T: \bigcup_{i \in \overline{p}} A_i \to \bigcup_{i \in \overline{p}} A_i$ is strictly contractive.

Proof: It follows directly from Theorem 2.2. □

Note that an important fact related to the applicability of Corollary 2.3 is that the conditions $0 \le K_{ji} \le 1; \forall i \in \overline{p};\ j = 1,2$ may be achieved in a compatible fashion with $max(\rho_1, \rho_2) < 1$ with only K_{1i} , K_{2j} one such constraints being strictly less than unity for some $i, j \in \overline{p}$. □

The known previous result that the sets A_i; $i \in \overline{p}$ have identical pair-wide distances if non-expansive p-cyclic self-maps on $\bigcup_{i \in \overline{p}} A_i$ exist , [2] , adopts the following characterization.

Proposition 2.4. If there exists a non-expansive p-cyclic self-map $T: \bigcup_{i \in \overline{p}} A_i \to \bigcup_{i \in \overline{p}} A_i$ then $d = d_i$; $\forall i \in \overline{p}$.

Proof: Take $(x,y) \in A_i \times A_{i+1}$ such that $d_i = d(x,y)$ for any $i \in \overline{p}$ and assume that $T: \bigcup_{i \in \overline{p}} A_i \to \bigcup_{i \in \overline{p}} A_i$ is non-expansive. Then ,

$$d_i = d(x,y) \ge max\left(d_i, d(Tx,Ty)\right)$$
$$\ge \max_{j \in \overline{p}} \left(\max_{1 \le i \le j} (d_i), d\left(T^j x, T^j y\right) \right) \ge \max_{j \in \overline{p}} \left(d_j\right) \qquad (2.8)$$

Since $i \in \overline{p}$ is arbitrary $d_i \ge \max_{j \in \overline{p}} \left(d_j\right) \Leftrightarrow d_i = d; \forall i \in \overline{p}$. □

Proposition 2.4 applies in particular if $T: \bigcup_{i \in \overline{p}} A_i \to \bigcup_{i \in \overline{p}} A_i$ is strictly contractive as, for instance, to Theorem 2.2 (iii) and Corollary 2.3 (iii). Note

A relevant result in the context of this paper is related to the fact that under weak conditions the self-map $T: \bigcup_{i \in \overline{p}} A_i \to \bigcup_{i \in \overline{p}} A_i$ cannot be expansive or asymptotically expansive. That means that sufficiently large distances lead to local contractions in the sense that such distances decrease through the iterative process. As a result, if can only be locally expansive as it is proven in the subsequent result:

Theorem 2.5. Assume that:
$min\left(\frac{K_{i2} - \alpha_{i2}}{1-\alpha_{i2}}, \frac{K_{i2} d_i + \gamma_{i2}}{1-\alpha_{i2}} \right) \ge 0$; $\forall i \in \overline{p}$ so that there is at least one $j \in \overline{p}$ such that $min\left(\frac{K_{j2} - \alpha_{j2}}{1-\alpha_{j2}}, \frac{K_{j2} d_i + \gamma_{j2}}{1-\alpha_{j2}} \right) \ge 0$ .Then, $T: \bigcup_{i \in \overline{p}} A_i \to \bigcup_{i \in \overline{p}} A_i$ is neither expansive nor asymptotically expansive.

Proof: Eq. (1.1) may be rewritten as follows:

$$d(Tx,Ty) - d(x,y) \le -\frac{K_{i2} - \alpha_{i2}}{1-\alpha_{i2}} d(x,y) + \frac{K_{i2} d_i + \gamma_{i2}}{1-\alpha_{i2}} \qquad (2.9)$$

; $(x,y) \in A_i \times A_{i+1}$, $\forall i \in \overline{p}$ which implies:

$$d\left(T^p x, T^p y\right) - d(x,y) = \sum_{i=1}^{p} \left(\frac{K_{i2} d_i + \gamma_{i2}}{1-\alpha_{i2}} - \frac{K_{i2} - \alpha_{i2}}{1-\alpha_{i2}} d\left(T^{i-1} x, T^{i-1} y\right) \right) \le 0 \qquad (2.10)$$

provided that

$$\sum_{i=1}^{p} \frac{K_{i2} - \alpha_{i2}}{1-\alpha_{i2}} d\left(T^{i-1} x, T^{i-1} y\right) \ge \sum_{i=1}^{p} \frac{K_{i2} d_i + \gamma_{i2}}{1-\alpha_{i2}}$$

and the given parametrical conditions hold. Now, assume that $T: \bigcup_{i \in \overline{p}} A_i \to \bigcup_{i \in \overline{p}} A_i$ is either expansive or asymptotically expansive. Thus, for any $(x,y) \in A_i \times A_{i+1}$ and any $i \in \overline{p}$, there exists a finite sufficiently large positive integer $j = j(x,y,i) \in \overline{p}$ such that

$$\sum_{i=1}^{p} \frac{K_{i2} - \alpha_{i2}}{1-\alpha_{i2}} d\left(T^{j+i-1} x, T^{j+i-1} y\right) \ge \sum_{i=1}^{p} \frac{K_{i2} d_i + \gamma_{i2}}{1-\alpha_{i2}}$$

which implies that $d\left(T^{p+j} x, T^{p+j} y\right) \le d\left(T^{p+j-1} x, T^{p+j-1} y\right)$ from (2.10). As a result , for each $(x,y) \in A_i \times A_{i+1}$ and each $i \in \overline{p}$, there is a infinite sequence of positive integers $S(x,y)$ such that

ISBN: 978-988-18210-8-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$$d\left(T^{j_{k+1}}x, T^{j_{k+1}}y\right) \le d\left(T^{j_k}x, T^{j_k}y\right) \quad ; \quad \forall j_k = j_k(x,y) \in S(x,y) \quad (2.11)$$

if $T: \bigcup_{i\in\bar{p}} A_i \to \bigcup_{i\in\bar{p}} A_i$ is either expansive or asymptotically expansive what is a contradiction which proves the result. □

Note that Theorem 2.5 is applicable even if K_{i2} ; $i\in\bar{p}$ are real constants non necessarily in $[0,1)$. Note also that Theorem 2.5 does not guarantee that $T: \bigcup_{i\in\bar{p}} A_i \to \bigcup_{i\in\bar{p}} A_i$ so that it is not guaranteed that all distances between the subsets A_i ; $\forall i\in\bar{p}$ arte identical. The following complementary result follows which also ensures that $T: \bigcup_{i\in\bar{p}} A_i \to \bigcup_{i\in\bar{p}} A_i$ is neither globally expansive or asymptotically expansive although it is not ensured to be non-expansive for any points in $\bigcup_{i\in\bar{p}} A_i$:

Proposition 2. 6. Assume that the constraint for $M_{2i}(x,y)$ of Corollary 2.3 (i) is generalized as

$$M_{2i}\left(T^j x, T^j y\right) \le \alpha_{2i} \max_{\ell_1 \le i \le \ell_2} d\left(T^i x, T^i y\right) + \gamma_{2i} \quad (2.12)$$

with $\gamma_{2i} \in \mathbf{R}_{0+}$ and $\alpha_{2i} \in [K_{2i}, 1)$; $\forall i\in\bar{p}$, $0 \le \alpha_{2i} < K_{2i} < 1$; for some $\ell_k = \ell_k(j) \in \mathbf{Z}_{0+}$; k=1, 2 subject to $0 \le \ell_1 \le \ell_2 \le j+1$, $\forall (x,y) \in A_i \times A_{i+1}$, $\forall i\in\bar{p}$. Then, $d\left(T^j x, T^j y\right)$ is iteration- uniformly bounded (in the sense that its upper-bound is independent of the integer $j\in\mathbf{Z}_+$) with uniform bound being dependent on the (bounded) distance of the pair (x, y); $\forall (x,y) \in A_i \times A_{i+1}$; $\forall j \in \mathbf{Z}_{0+}$, $\forall i\in\bar{p}$ so that $T: \bigcup_{i\in\bar{p}} A_i \to \bigcup_{i\in\bar{p}} A_i$ can be locally expansive although it cannot be either expansive or asymptotically expansive. If, furthermore, $\alpha_{2i} \in [K_{2i} + \mu, 1)$; $\forall i\in\bar{p}$ and some $\mu \in \boldsymbol{R}_+$ then

$$\limsup_{k\to\infty} d\left(T^k x, T^k y\right) \le \frac{1}{1-\nu} \max_{i\in\bar{p}} \frac{K_{2i} d_i + \gamma_{2i}}{1-\alpha_{2i}} < \infty$$

for some $\boldsymbol{R} \ni \nu \in (0,1)$ (2.13)

Proof : From (2.12),

$$d\left(T^{j+1}x, T^{j+1}y\right) \le (1-K_{2i}) d\left(T^j x, T^j y\right) + K_{2i} d_i + M_{2i}\left(T^j x, T^j y\right) \le (1-K_{2i}) d\left(T^j x, T^j y\right) + \alpha_{2i} \max_{0\le i\le j+1} d\left(T^i x, T^i y\right) + K_{2i} d_i + \gamma_{2i} \quad (2.14)$$

Define $j = j(k)$

$$:= \left\{ \max \ell \in \boldsymbol{Z}_{0+} : (\ell \le k) \wedge \left(d\left(T^{k+1}x, T^{k+1}y\right) = \max_{0\le j\le k} d\left(T^{j+1}x, T^{j+1}y\right) \right) \right\}$$

for any $k \in \mathbf{Z}_{0+}$ so that , one gets from (2.14)

$$\max_{0\le j\le k} d\left(T^{j+1}x, T^{j+1}y\right) \le (1-K_{2i}) d\left(T^k x, T^k y\right) + \alpha_{2i} \max_{0\le j\le k} d\left(T^{j+1}x, T^{j+1}y\right) + K_{2i} d_i + \gamma_{2i}$$

$$\Rightarrow$$

$$d\left(T^j x, T^j y\right) \le d\left(T^{k+1}x, T^{k+1}y\right) = \max_{0\le j\le k} d\left(T^{j+1}x, T^{j+1}y\right) \le L(x,y) := d\left(T^k x, T^k y\right) + \frac{K_{2\ell} d_\ell + \gamma_{2\ell}}{1-\alpha_{2\ell}} < \infty \quad (2.15)$$

provided that $\alpha_{2i} \in [K_{2i}, 1)$, $\forall (x,y) \in A_i \times A_{i+1}$; $\forall j (\le k+1) \in \mathbf{Z}_{0+}$, $\forall i\in\bar{p}$ and $\ell\in\bar{p}$ being defined as $\ell = k + i - Int\,Part\left(\frac{k+i-1}{p}\right)p$ .Then, $d\left(T^k x, T^k y\right) < \infty$ with uniform bound $\forall k \in \mathbf{Z}_{0+}$ depending on each given $(x,y) \in A_i \times A_{i+1}$; If, $\forall k \in \mathbf{Z}_+$. furthermore, $\alpha_{2i} \in [K_{2i} + \mu, 1)$; $\forall i\in\bar{p}$ then $\nu_i := \frac{1-K_{2i}}{1-\alpha_{2i}} < 1$; $\forall i\in\bar{p}$ so that (2.13) follows from (2.15) with $\nu := \max_{i\in\bar{p}} \nu_i$. □

Proposition 2.6 implies has the following consequent result.

Proposition 2.7. Under the conditions of Proposition 2.6, the following set of relations for the distances between the subsets A_i, $i\in\bar{p}$ of X hold :

$$d_i := dist(A_i, A_{i+1}) \le d_j + \frac{K_{2\ell} d_\ell}{1-\alpha_{2\ell}} \; ; \; \forall i,j,\ell \in \bar{p} \quad (2.16)$$

Proof : Eq. (2.15) implies directly:

$$d_{j+i} \le d\left(T^j x, T^j y\right) \le d\left(T^k x, T^k y\right) + \frac{K_{2\ell} d_\ell}{1-\alpha_{2\ell}} \le d_{k+i} + \frac{K_{2\ell} d_\ell}{1-\alpha_{2\ell}} \quad (2.17)$$

; $\forall (x,y) \in A_i \times A_{i+1}$, $\forall j (\le k+1) \in \mathbf{Z}_{0+}$, $\forall k \in \mathbf{Z}_{0+}$ any $i\in\bar{p}$ (since x and y are any points in $A_i \times A_{i+1}$; $\forall i\in\bar{p}$) and some $\bar{p} \ni \ell = \ell(i,k) = k + i - Int\,Part\left(\frac{k+i-1}{p}\right)p$. Since the integers i and k are arbitrary in $\bar{p} \times \mathbf{Z}_+$, ℓ takes any value in $\bar{p}$ depending on the initial points x and y. Therefore, Eq. (2.16) implies (2.17). □

Note that Proposition 2.7 is applied directly to the case that all distances between adjacent subsets are identical. It is of interest the investigation of the properties of the self-map $T: \bigcup_{i\in\bar{p}} A_i \to \bigcup_{i\in\bar{p}} A_i$ for distances between non-adjacent subsets. Assume the following cases:

a) $x, y \in A_i$ for some $i\in\bar{p}$ so that $Tx, Ty \in A_{i+1}$ subject to:

$$(1-K_{1i}) d(x,y) + M_{1i}(x,y) \le d(Tx, Ty) \le (1-K_{2i}) d(x,y) + M_{2i}(x,y) \quad (2.18)$$

b) $(x,y) \in A_i \times A_j$ for some $i,j\in\bar{p}$ so that $(Tx, Ty) \in A_{i+1} \times A_{j+1}$ with $i \ne j \ne i+1$ then one gets for

ISBN: 978-988-18210-8-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

any set of points satisfying $\left\{ x_j \in A_j : x_i = x, x_j = y ; \ell \in \bar{j} \setminus \overline{i+1} \right\}$ the following constraint by using the triangle inequality for distances together with the upper-bounding constraint in (2.1):

$$d(Tx, Ty) \le \sum_{\ell=i}^{j-1} d\left(Tx_\ell, Tx_{\ell+1}\right) \le \sum_{\ell=i}^{j-1} \left[\left(1 - K_{2\ell}\right) d\left(x_\ell, x_{\ell+1}\right) \right.$$
$$\left. + K_{2\ell} d_\ell + M_{2\ell}\left(x_\ell, x_{\ell+1}\right) \right]$$
$$\le p \max_{i \le \ell \le j-1} \left[\left(1 - K_{2\ell}\right) d\left(x_\ell, x_{\ell+1}\right) + K_{2\ell} d_\ell + M_{2\ell}\left(x_\ell, x_{\ell+1}\right) \right] \tag{2.19}$$

$$\le p \max_{i \le \ell \le j-1} \left(1 - K_{2\ell}\right) d\left(x_\ell, x_{\ell+1}\right)$$
$$+ \max_{i \le \ell \le j-1} \left\lfloor K_{2\ell} d_\ell + M_{2\ell}\left(x_\ell, x_{\ell+1}\right) \right\rfloor \tag{2.20}$$

It follows from (2.20) that Theorem 2.2 [(i), (iii)] (i.e. uniform boundedness distance of iterations for any two initial points in any , in general, distinct non-adjacent, subsets A_i of X under obtained via $T : \bigcup_{i \in \bar{p}} A_i \to \bigcup_{i \in \bar{p}} A_i$) still holds irrespective of $i, j \in \bar{p}$ if $p \max_{i \in \bar{p}} \left(\frac{1 - K_{2i}}{1 - \alpha_{2i}} \right) < 1$, Corollary 2.3 (i) holds if $p \max_{i \in \bar{p}} \left(\frac{1 - K_{2i}}{1 - \alpha_{2i}} \right) < 1$ irrespective of $i, j \in \bar{p}$. Also, Proposition 2.6 holds under the replacement $\nu := \max_{i \in \bar{p}} \nu_i \to p\nu < 1$ irrespective of $i, j \in \bar{p}$. Note that Theorem 2.5 guaranteeing that $T : \bigcup_{i \in \bar{p}} A_i \to \bigcup_{i \in \bar{p}} A_i$ is non-expansive and non-asymptotically expansive also holds under a close and similar proof to the given one. If the initial points are within the same subset A_i of X, the above results are still valid under weaker conditions to the light of (2.18) by zeroing the distances between subsets. Expressions for lower- bounds can be also derived in a similar way using $p \min_{i \le \ell \le j-1} \left(1 - K_{2\ell}\right) d\left(x_\ell, x_{\ell+1}\right)$.

III.MAIN RESULTS ON MODIFIED MEIR-KEELER CONTRACTIONS

p- cyclic Meir- Keeler contractions have been discussed in a number of papers (see, for instance, [1-3], [7]). In the case that $\bigcap_{i \in \bar{p}} A_i = \varnothing$ then $T : \bigcup_{i \in \bar{p}} A_i \to \bigcup_{i \in \bar{p}} A_i$ has no fixed points and their role is played in this case by the so-called best proximity points, [1-4], [16-17]. This section is concerned with the extension of p- cyclic Meir – Keeler contractions to the case of constraints close to (1.1). Roughly speaking $T : \bigcup_{i \in \bar{p}} A_i \to \bigcup_{i \in \bar{p}} A_i$ is not expansive for large distances of the initial points in a similar context as that investigated in Section 2 to the light of Banach contraction principle. A p- cyclic self-map $T : \bigcup_{i \in \bar{p}} A_i \to \bigcup_{i \in \bar{p}} A_i$ is a p-cyclic Meir–Keeler contraction, [2], if for every $\varepsilon \in \mathbf{R}_+$, $\exists \delta = \delta(\varepsilon) \in \mathbf{R}_+$ if

$$d(x, y) < d_i + \delta + \varepsilon \Rightarrow d(Tx, Ty) < d_i + \varepsilon \tag{3.1}$$

or, equivalently, $d(Tx, Ty) < \min\left(d_i + \varepsilon, d(x, y) - \delta\right)$ if $d(x, y) \ge \delta$ .Consider now according to (1.1) , or (2.1), that $M_{2i}\left(T^j x, T^j y\right) \le \alpha_{2i} \max_{\ell_1 \le i \le \ell_2} d\left(T^i x, T^i y\right) + \gamma_{2i}$ as assumed in (2.12) with $\ell_k = \ell_k(j); k = 1,2$; $\forall (x, y) \in A_i \times A_{i+1}$; $\forall i \in \bar{p}$, $\forall j \in \mathbf{Z}_{0+}$ which includes the particular case $M_{2i}(x, y) \le \alpha_{2i}\, d(Tx, Ty) + \gamma_{2i}$, studied in Section 2, for the modified Banach contraction principle. Thus, $T : \bigcup_{i \in \bar{p}} A_i \to \bigcup_{i \in \bar{p}} A_i$ is a modified p-cyclic Meir- Keeler contraction if:

$$d(x, y) - d_i - M_{2i}(x, y) < \delta + \varepsilon$$
$$\Rightarrow d(Tx, Ty) - d_i - M_{2i}(x, y) < \varepsilon \tag{3.2}$$

; $\forall (x, y) \in A_i \times A_{i+1}$; $\forall i \in \bar{p}$. Also, $T : \bigcup_{i \in \bar{p}} A_i \to \bigcup_{i \in \bar{p}} A_i$ is a worst –case modified p-cyclic Meir-Keeler contraction if for any $\varepsilon \in \mathbf{R}_+$; $\exists \delta_{ji} = \delta_{ji}(\varepsilon)$:

$$d\left(T^j x, T^j y\right) - d_i - \alpha_{2i} \max_{\ell_1 \le i \le \ell_2} d\left(T^i x, T^i y\right) - \gamma_{2i} < \delta_{ji} + \varepsilon$$

$$\Rightarrow d\left(T^{j+1} x, T^{j+1} y\right) - d_i - \alpha_{2i} \max_{\ell_1 \le i \le \ell_2} d\left(T^i x, T^i y\right) - \gamma_{2i} < \varepsilon \tag{3.3}$$

Theorem 3.1. Assume that $T : \bigcup_{i \in \bar{p}} A_i \to \bigcup_{i \in \bar{p}} A_i$ is a worst -case modified p- cyclic Meir-Keeler contraction. Then, $d\left(T^j x, T^j y\right)$ is bounded for any $(x, y) \in A_i \times A_{i+1}$; $\forall i \in \bar{p}$ provided that $d(x, y)$ is bounded.

Proof : One gets from (3.3)

$$d\left(T^{j+1} x, T^{j+1} y\right)$$
$$< \min\left(\varepsilon + d + \alpha_2 \max_{\ell_1 \le i \le \ell_2} d\left(T^i x, T^i y\right) + \gamma_2, d\left(T^j x, T^j y\right) - \delta \right)$$

$$= \min\left(\varepsilon + d + \alpha_2\, d\left(T^{i_j} x, T^{i_j} y\right) + \gamma_2, d\left(T^j x, T^j y\right) - \delta \right) \tag{3.4}$$

$\forall (x, y) \in A_i \times A_{i+1}$, for some integer $i_j \in \left[\ell_1(j), \ell_2(j)\right]$, some $\ell_k = \ell_k(j); k = 1,2$; $\forall i \in \bar{p}$, $\forall j \in \mathbf{Z}_{0+}$ where $d = \max_{i \in \bar{p}} d_i$, $\gamma_2 = \max_{i \in \bar{p}} \gamma_{2i}$, $\alpha_2 = \max_{i \in \bar{p}} \alpha_{2i}$ and $\delta = \max_{i \in \bar{p}, \ell \in \mathbf{Z}_+} \delta_{\ell i}$ for some sequence of positive real constants $\left\{\delta_{\ell i}\right\}_{i=0}^{j}$; $\forall j \in \mathbf{Z}_{0+}$, $\forall i \in \bar{p}$. Proceeding recursively with (3.4), one gets:

$$d\left(T^{j+1} x, T^{j+1} y\right)$$
$$< \min\left(\varepsilon + d + \gamma_2 + \alpha_2 \max_{\ell_1 \le i \le \ell_2} d\left(T^i x, T^i y\right), d\left(T^j x, T^j y\right) - \delta \right)$$
$$= \min\left(\varepsilon + d + \gamma_2 + \alpha_2\, d\left(T^{i_j} x, T^{i_j} y\right), d\left(T^j x, T^j y\right) - \delta \right) \tag{3.5}$$

ISBN: 978-988-18210-8-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

provided that $\min_{0\le\ell\le j}\left(d\left(T^{\ell}x,T^{\ell}y\right)\ge\delta_{\ell i}\right)$. Assume that it exists $(x,y)\in A_i\times A_{i+1}$ such that the real sequence $\left\{d\left(T^{j+1}x,T^{j+1}y\right)\right\}_{j\in\mathbf{Z}_{0+}}$ is unbounded so that some subsequence of it diverges as $\ell\to+\infty$. Thus, for some arbitrarily large $\Delta\in\mathbf{R}_{0+}$, $\exists k=k(n_0,\Delta)(\ge n_0)\in\mathbf{Z}_+$ and some sufficiently large $n_0\in\mathbf{Z}_+$ such that from (3.5), one gets.

$$\Delta<\min\left(\left(\varepsilon+d+\gamma_2\right)\left(1+\sum_{k=0}^{j+1}\alpha_2^{k}\right)+\alpha_2^{j+2}\max_{0\le i\le j+1}d\left(T^ix,T^iy\right),\ \min_{0\le i\le j}\left(d\left(T^ix,T^iy\right)-\delta\right)\right)$$

where $d=\max_{i\in\overline{p}}d_i$, $\gamma_2=\max_{i\in\overline{p}}\gamma_{2i}$, $\alpha_2=\max_{i\in\overline{p}}\alpha_{2i}$ and $\delta=\max_{i\in\overline{p},\ell\in\mathbf{Z}_+}\delta_{\ell i}$ what implies that $\min_{0\le\ell\le j}d\left(T^{\ell}x,T^{\ell}y\right)$ is unbounded leading to a contradiction. □

Some further properties of worst-case modified p-cyclic Meir – Keeler contractions are now investigated. Define the sequence $\left\{f_j\right\}_0^{\infty}$ of real functions $f_j:\left(\bigcup_{i\in\overline{p}}A_i\right)\times\left(\bigcup_{i\in\overline{p}}A_i\right)\times\mathbf{R}_{0+}\to\mathbf{R}$; $\forall j\in\mathbf{Z}_{0+}$ from (3.3) for a given sequence of nonnegative real numbers $\left\{m_j\right\}_0^{\infty}$ as follows:

$$f_j\left(x,y,m_j\right):=-\alpha_2\max_{j-\ell_2\le i\le j-\ell_1}d\left(T^ix,T^iy\right)-d-\gamma_2-m_j \qquad (3.6)$$

$$f_{j+1}\left(x,y,m_{j+1}\right)\equiv g_j\left(x,y,m_{j+1}\right):=d\left(T^{j+1}x,T^{j+1}y\right)-\alpha_2\max_{j+1-\ell_2\le i\le j+1-\ell_1}d\left(T^ix,T^iy\right)-d-\gamma_2-m_{j+1} \qquad (3.7)$$

where $d\left(T^jx,T^jy\right)\ge\alpha_2\max_{j-\ell_2\le i\le j-\ell_1}d\left(T^ix,T^iy\right)+d+\gamma_2+m_j+M$ for some $M\in\mathbf{R}_{0+}$; $\forall j\in\mathbf{Z}_{0+}$ and $d=\max_{i\in\overline{p}}d_i$, $\gamma_2=\max_{i\in\overline{p}}\gamma_{2i}$ and $\alpha_2=\max_{i\in\overline{p}}\alpha_{2i}$, $\forall(x,y)\in\left(\bigcup_{i\in\overline{p}}A_i\right)\times\left(\bigcup_{i\in\overline{p}}A_i\right);\forall i\in\overline{p}$. It is assumed for simplicity that $\ell_{1,2}\in\mathbf{Z}_{0+}$ are constant with $\ell_2\ge\ell_1\ge0$ The following result is an ad-hoc extension of [2], Lemma 2.2, [20-21], which is useful for studying worst-case modified p-cyclic Meir-Keeler contractions:

Lemma 3.2. Assume that $d\left(T^jx,T^jy\right)\ge\alpha_2\max_{j-\ell_2\le i\le j-\ell_1}d\left(T^ix,T^iy\right)+d+\gamma_2+m_j+M$; $\forall(x,y)\in Y(\ne\varnothing)\subset\left(\bigcup_{i\in\overline{p}}A_i\right)\times\left(\bigcup_{i\in\overline{p}}A_i\right)$; $\forall j\in\mathbf{Z}_{0+}$. Then , the following properties are equivalent:

(1) For each $M\in\mathbf{R}_+\Rightarrow\exists\delta=\delta(M)\in\mathbf{R}_+$ such that $M\le f_j\left(x,y,m_j\right)<M+\delta\Rightarrow0\le f_{j+1}\left(x,y,m_{j+1}\right)<M$ for $(x,y)\in Y$ and some $m_j,m_{j+1}\in\mathbf{R}_{0+}$.

(2) $\exists$ a nondecreasing continuous L-function $\phi:\mathbf{R}_{0+}\to\mathbf{R}_{0+}$ (*i.e.* $\phi(0)=0$ and $\exists\delta\in\mathbf{R}_+$ fulfilling $0<\phi(t)\le s$; $\forall t\in[s,s+\delta]$, [2],[20-21]) such that for $m_j,m_{j+1}\in\mathbf{R}_{0+}$ fulfilling the above property (1):

$f_j\left(x,y,m_j\right)>0\Rightarrow f_{j+1}\left(x,y,m_{j+1}\right)$, for $(x,y)\in Y$, the second inequality following sine ϕ is an L-function, and

$$f_j\left(x,y,m_j\right)=0\Rightarrow f_{j+1}\left(x,y,m_{j+1}\right)=0<\phi\left(f_j\left(x,y,m_j\right)\right)\le f_j\left(x,y,m_j\right)$$ □

The following concerns the boundedness of $\left(d\left(T^jx,T^jy\right)-\alpha_2\max_{j-\ell_2\le i\le j-\ell_1}d\left(T^ix,T^iy\right)\right)$.

Lemma 3.3. Assume that $T:\bigcup_{i\in\overline{p}}A_i\to\bigcup_{i\in\overline{p}}A_i$ is a worst –case modified p- cyclic Meir-Keeler contraction which satisfies $d\left(T^jx,T^jy\right)\ge\alpha_2\max_{j-\ell_2\le i\le j-\ell_1}d\left(T^ix,T^iy\right)+d+\gamma_2+m_j+M$; $\forall(x,y)\in Y(\ne\varnothing)\subset\left(\bigcup_{i\in\overline{p}}A_i\right)\times\left(\bigcup_{i\in\overline{p}}A_i\right)$; $\forall j\in\mathbf{Z}_{0+}$. Assume also that Lemma 3.2 holds with $\left\{m_j\right\}_0^{\infty}$ being an uniformly bounded nonnegative real sequence ; $\forall j\in\mathbf{Z}_{0+}$. Then, the real sequence

$$S:=\left\{s_k:=d\left(T^jx,T^jy\right)-\alpha_2\max_{j-\ell_2\le i\le j-\ell_1}d\left(T^ix,T^iy\right)\right\}_0^{\infty} \qquad (3.8)$$

is uniformly bounded from below and from above .

Proof : It is directly bounded from below since . It is now proven that it is bounded from above. $d\left(T^jx,T^jy\right)-\alpha_2\max_{j-\ell_2\le i\le j-\ell_1}d\left(T^ix,T^iy\right)\ge d+\gamma_2+m_j+M$

First note that $s_k\in\mathbf{R}_{0+}$. The proof follows by contradiction. Assume that S is unbounded to that there is a strictly monotone increasing subsequence $S':=\left\{s_{j_k}\right\}_{j_k\in\mathbf{Z}'}$, with $\mathbf{Z}'\subset\mathbf{Z}$ being countable, such that $s_{j_k}\to\infty$ as $j_k\to\infty$. Then, $s_{j_k}\ge\theta$ for any prefixed arbitrary $\theta\in\mathbf{R}_+$ and all $\ell(\ge N)\in S'$ for some $N=N(\theta)\in S'$. From Lemma 3.2(i), for each $M\in\mathbf{R}_+\Rightarrow\exists\delta=\delta(M)\in\mathbf{R}_+$ such that

$$\delta_j+M>f_j\left(x,y,m_j\right):=d\left(T^jx,T^jy\right)-\alpha_2\max_{j-\ell_2\le i\le j-\ell_1}d\left(T^ix,T^iy\right)-d-\gamma_2-m_j\ge\max\left(M,\theta-d-\gamma_2-m_j\right) \qquad (3.9)$$

$$M>f_{j+1}\left(x,y,m_{j+1}\right)>\theta-d-\gamma_2-m_{j+1}\Rightarrow\theta<M+d+\gamma_2+m_{j+1}\le M+d+\gamma_2+\max_{0\le j<\infty}m_j \qquad (3.10)$$

so that θ cannot be fixed arbitrarily what leads to a contradiction. □

The above result together with Lemma 3.2 lead to the self-map $T:\bigcup_{i\in\overline{p}}A_i\to\bigcup_{i\in\overline{p}}A_i$ to be neither expansive nor asymptotically expansive.

Proposition 3.4. Assume that $T:\bigcup_{i\in\overline{p}}A_i\to\bigcup_{i\in\overline{p}}A_i$ is a worst –case modified p- cyclic Meir-Keeler contraction whichsatisfies

$$d\left(T^j x,T^j y\right)\ge\alpha_2\max_{j-\ell_2\le i\le j-\ell_1}d\left(T^i x,T^i y\right)+d+\gamma_2+m_j+M$$

; $\forall(x,y)\in Y(\ne\varnothing)\subset\left(\bigcup_{i\in\overline{p}}A_i\right)\times\left(\bigcup_{i\in\overline{p}}A_i\right)$; $\forall j\in\mathbf{Z}_{0+}$. Assume also that Lemma 3.2 holds with $\{m_j\}_0^\infty$ being a nonnegative nonotone decreasing real sequence. Then, $T:\bigcup_{i\in\overline{p}}A_i\to\bigcup_{i\in\overline{p}}A_i$ is neither expansive nor asymptotically expansive.

Proof. Note that

$$\left(d\left(T^{j+1}x,T^{j+1}y\right)-\alpha_2\max_{j+1-\ell_2\le i\le j+1-\ell_1}d\left(T^i x,T^i y\right)\right)$$
$$-\left(d\left(T^j x,T^j y\right)-\alpha_2\max_{j-\ell_2\le i\le j-\ell_1}d\left(T^i x,T^i y\right)\right)$$
$$<m_{j+1}-m_j \tag{3.11}$$

From Lemma 3.3, the real sequence S consisting of elements

$$s_k=d\left(T^j x,T^j y\right)-\alpha_2\max_{j-\ell_2\le i\le j-\ell_1}d\left(T^i x,T^i y\right)$$

is uniformly bounded for $k\in\mathbf{Z}_{0+}$:

$$\infty>2C>d\left(T^j x,T^j y\right)-d\left(T^{j+1}x,T^{j+1}y\right)$$
$$+\alpha_2\left(\max_{j+1-\ell_2\le i\le j+1-\ell_1}d\left(T^i x,T^i y\right)-\max_{j-\ell_2\le i\le j-\ell_1}d\left(T^i x,T^i y\right)\right)$$
$$=\left(d\left(T^j x,T^j y\right)-\alpha_2\max_{j-\ell_2\le i\le j-\ell_1}d\left(T^i x,T^i y\right)\right)$$
$$-\left(d\left(T^{j+1}x,T^{j+1}y\right)-\alpha_2\max_{j+1-\ell_2\le i\le j+1-\ell_1}d\left(T^i x,T^i y\right)\right)$$
$$>m_j-m_{j+1}\ge 0 \tag{3.12}$$

The following situations can occur for the sequences below:

1) $d\left(T^j x,T^j y\right)-d\left(T^{j+1}x,T^{j+1}y\right)$,

$$\left(\max_{j+1-\ell_2\le i\le j+1-\ell_1}d\left(T^i x,T^i y\right)-\max_{j-\ell_2\le i\le j-\ell_1}d\left(T^i x,T^i y\right)\right) \tag{3.13}$$

if $\alpha_2\ne 0$ are both bounded real sequences. Then , $T:\bigcup_{i\in\overline{p}}A_i\to\bigcup_{i\in\overline{p}}A_i$ is neither expansive nor asymptotically expansive.

2) $d\left(T^j x,T^j y\right)-d\left(T^{j+1}x,T^{j+1}y\right)$

$$\alpha_2\left(\max_{j+1-\ell_2\le i\le j+1-\ell_1}d\left(T^i x,T^i y\right)-\max_{j-\ell_2\le i\le j-\ell_1}d\left(T^i x,T^i y\right)\right) \tag{3.14}$$

are unbounded sequences with the first one having a sequence diverging to $+\infty$ and the second one having a sequence diverging at the same rate to $-\infty$ or vice-versa. In the first case, $d\left(T^j x,T^j y\right)>d\left(T^{j+1}x,T^{j+1}y\right)$ for $j\in\mathbf{Z}_\alpha\subset\mathbf{Z}_{0+}$ and $\mathbf{Z}_\alpha$ being countable of infinite cardinal. This implies that $T:\bigcup_{i\in\overline{p}}A_i\to\bigcup_{i\in\overline{p}}A_i$ is neither expansive nor asymptotically expansive. In the contrary case, there is a diverging sequence fulfilling $\max_{j+1-\ell_2\le i\le j+1-\ell_1}d\left(T^i x,T^i y\right)>\max_{j-\ell_2\le i\le j-\ell_1}d\left(T^i x,T^i y\right)$ so that $d\left(T^{j_{i+1}+i}x,T^{j_{i+1}+i}y\right)>d\left(T^{j_i+i}x,T^{j_i+i}y\right)$ for some integers $j_{i+1}=j_{i+1}(i+1)>j_i=j_i(i)$ with $j_i\in\left[i-\ell_2,i-\ell_1\right]$. Again, $T:\bigcup_{i\in\overline{p}}A_i\to\bigcup_{i\in\overline{p}}A_i$ is neither expansive nor asymptotically expansive. □

Remark 3.5. Note that if $\ell_1=\ell_1(k)=0$, then Lemmas 3.3-3.4 imply that $d\left(T^j x,T^j y\right)\ge\dfrac{d+\gamma_2+m_j+M}{1-\alpha_2}$ while $T:\bigcup_{i\in\overline{p}}A_i\to\bigcup_{i\in\overline{p}}A_i$ is neither expansive nor asymptotically expansive. □

Some elementary results concerning the comparative values of interest of distances through the worst-case modified p-cyclic self-map $T:\bigcup_{i\in\overline{p}}A_i\to\bigcup_{i\in\overline{p}}A_i$ are provided in the next two Propositions. Some conditions for the above self-map being non expansive, non asymptotically expansive or expansive are discussed in the next two results.

Proposition 3.6. Assume that $\exists M\in\mathbf{R}_+$, $\exists\delta_j=\delta_j(M)\in\mathbf{R}_+$ and $\exists\{m_j\}_0^\infty$ being a nonnegative monotone nonincreasing real sequence such that

$$M\le f_j\left(x,y,m_j\right)=d\left(T^j x,T^j y\right)+M$$
$$-\alpha_2\max_{j-\ell_2\le i\le j-\ell_1}d\left(T^i x,T^i y\right)-d-\gamma_2-m_j<\delta_j$$
$$\Rightarrow\left[f_{j+1}\left(x,y,m_{j+1}\right)\equiv g_j\left(x,y,m_{j+1}\right)<M\right.$$
$$\left.\Leftrightarrow d\left(T^{j+1}x,T^{j+1}y\right)\right]$$
$$<\alpha_2\max_{j+1-\ell_2\le i\le j+1-\ell_1}d\left(T^i x,T^i y\right)+d+\gamma_2+m_{j+1}+M \tag{3.15}$$

, equivalently $f_j\left(x,y,m_j\right)\ge M$, for

$$(x,y)\in Y\subset\left(\bigcup_{i\in\overline{p}}A_i\right)\times\left(\bigcup_{i\in\overline{p}}A_i\right).$$

Then, the following inequalities hold:

$$d\left(T^{j+1}x,T^{j+1}y\right)\le d\left(T^j x,T^j y\right);$$
$$\max_{j+1-\ell_2\le i\le j+1-\ell_1}d\left(T^i x,T^i y\right)\le\max_{j-\ell_2\le i\le j-\ell_1}d\left(T^i x,T^i y\right)$$
$$\max_{j-\ell_2\le i\le j-\ell_1}d\left(T^i x,T^i y\right)$$
$$\le\max\left(d\left(T^{j-\ell_2}x,T^{j-\ell_2}y\right),\max_{j+1-\ell_2\le i\le j+1-\ell_1}d\left(T^i x,T^i y\right)\right)$$

ISBN: 978-988-18210-8-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

(3.16)

or, equivalently one has according to Lemma 3.2:

$$f_j(x,y,m_j)>0 \Rightarrow \left(f_{j+1}(x,y,m_{j+1})<\phi\left(f_j(x,y,m_j)\right)\le f_j(x,y,m_j)\right)$$
$$\wedge\, f_j(x,y,m_j)=0 \Rightarrow f_{j+1}(x,y,m_{j+1})=0\Big] \quad (3.17)$$

Two cases can occur, namely:

a) Case 1:

$$\max_{j+1-\ell_2\le i\le j+1-\ell_1} d\left(T^i x, T^i y\right) \le \max_{j-\ell_2\le i\le j-\ell_1} d\left(T^i x, T^i y\right) \quad (3.18)$$

$$\Rightarrow \max\left(d\left(T^{j-\ell_2}x, T^{j-\ell_2}y\right), d\left(T^{j+1-\ell_1}x, T^{j+1-\ell_1}y\right)\right) \le \max_{j-\ell_2\le i\le j-\ell_1} d\left(T^i x, T^i y\right) \quad (3.19)$$

Then, it follows that

$$d\left(T^{j+1}x, T^{j+1}y\right) < d\left(T^j x, T^j y\right) + m_{j+1} - m_j \le d\left(T^j x, T^j y\right) \quad (3.20)$$

so that $T: \bigcup_{i\in\bar{p}} A_i \to \bigcup_{i\in\bar{p}} A_i$ is not either expansive or asymptotically expansive.

b) Case 2:

$$d\left(T^{j+1-\ell_1}x, T^{j+1-\ell_1}y\right) = \max_{j+1-\ell_2\le i\le j+1-\ell_1} d\left(T^i x, T^i y\right) > \max_{j-\ell_2\le i\le j-\ell_1} d\left(T^i x, T^i y\right)$$

$$\Rightarrow d\left(T^{j-\ell_1}x, T^{j-\ell_1}y\right) < d\left(T^{j+1-\ell_1}x, T^{j+1-\ell_1}y\right); \quad \forall j\in \mathbf{Z}_{0+} \quad (3.21)$$

then $\left\{ d\left(T^{j+1-\ell_1}x, T^{j+1-\ell_1}y\right)\right\}_0^\infty$ is unbounded so that it has a subsequence which diverges . With no loss in generality, assume that $\left\{ d\left(T^{j+1-\ell_1}x, T^{j+1-\ell_1}y\right)\right\}_0^\infty$ diverges so that, one gets from the worst-case modified p-cyclic contraction properties in Lemma 3.2:

$$f_{j+1}(x,y,m_{j+1}) = d\left(T^{j+1}x, T^{j+1}y\right) - \alpha_2 \max_{j+1-\ell_2\le i\le j+1-\ell_1} d\left(T^i x, T^i y\right) - d - \gamma_2 - m_{j+1}$$
$$= d\left(T^{j+1}x, T^{j+1}y\right) - \alpha_2\, d\left(T^{j+1-\ell_1}x, T^{j+1-\ell_1}y\right) - d - \gamma_2 - m_{j+1} \le M$$
$$\Leftrightarrow d\left(T^{j+1}x, T^{j+1}y\right) - \max_{j+1-\ell_2\le i\le j+1-\ell_1} d\left(T^i x, T^i y\right)$$
$$< -(1-\alpha_2)\, d\left(T^{j+1-\ell_1}x, T^{j+1-\ell_1}y\right) + d + \gamma_2 + m_{j+1} + M < 0 \quad (3.22)$$

provided that $0\le \alpha_2 < 1$ and $d\left(T^{j+1-\ell_1}x, T^{j+1-\ell_1}y\right) > \dfrac{d+\gamma_2+m_{j+1}+M}{1-\alpha_2}$. As a result, if $\left\{ d\left(T^{j+1-\ell_1}x, T^{j+1-\ell_1}y\right)\right\}_0^\infty$ is unbounded then $\limsup_{j\to\infty} d\left(T^j x, T^j y\right) < \infty$; $\forall (x,y)\in Y \subset \left(\bigcup_{i\in\bar{p}} A_i\right)\times\left(\bigcup_{i\in\bar{p}} A_i\right)$ what leads to a contradiction. As a result, $T: \bigcup_{i\in\bar{p}} A_i \to \bigcup_{i\in\bar{p}} A_i$ cannot be either expansive or asymptotically expansive. □

Proposition 3.7. The following properties hold:

(i) If $d\left(T^j x, T^j y\right) \ge \alpha_2 \max_{j-\ell_2\le i\le j-\ell_1} d\left(T^i x, T^i y\right) + d + \gamma_2 + m_j + M$; $\forall j\in \mathbf{Z}_{0+}$ then

$$d\left(T^{j+1}x, T^{j+1}y\right) > \max_{j-\ell_2\le i\le j-\ell_1} d\left(T^i x, T^i y\right)$$

if and only if

$$d\left(T^j x, T^j y\right) \ge \alpha_2 \max\left(\max_{j-\ell_2\le i\le j-\ell_1} d\left(T^i x, T^i y\right), d\left(T^{j+1}x, T^{j+1}y\right) - \rho_{j+1}\right) + d + \gamma_2 + m_j + M$$

for some positive real sequence $\left\{\rho_k\right\}_0^\infty$. Also, $d\left(T^{j+1}x, T^{j+1}y\right) \ge \max_{j-\ell_2\le i\le j-\ell_1} d\left(T^i x, T^i y\right)$ if and only if the constraint holds with the above sequence being identically zero.

(ii) If $d\left(T^j x, T^j y\right) \ge \max_{j-\ell_2\le i\le j-\ell_1} d\left(T^i x, T^i y\right) + d + \gamma_2 + m_j + M$, $\forall j\in \mathbf{Z}_{0+}$ (i.e. $\alpha_2\ge 1$) then $T: \bigcup_{i\in\bar{p}} A_i \to \bigcup_{i\in\bar{p}} A_i$ is expansive.

Proof: It follows directly from the following relations:

$$d\left(T^j x, T^j y\right) \ge \alpha_2 \max_{j-\ell_2\le i\le j-\ell_1} d\left(T^i x, T^i y\right) + d + \gamma_2 + m_j + M$$
$$= \alpha_2\left(d\left(T^{j+1}x, T^{j+1}y\right) - \rho_{j+1}\right) + d + \gamma_2 + m_j + M$$
$$\Leftrightarrow d\left(T^j x, T^j y\right) - \alpha_2\, d\left(T^{j+1}x, T^{j+1}y\right) \ge \alpha_2\, \rho_{j+1} + d + \gamma_2 + m_j + M$$

If $\alpha_2 \in [0,1)$, one gets proceeding recursively:

$$d\left(T^j x, T^j y\right) \ge \alpha_2 \max_{j-\ell_2\le i\le j-\ell_1} d\left(T^i x, T^i y\right) + d + \gamma_2 + m_j + M$$
$$\ge \alpha_2^k \max_{j-\ell_2-k+1\le i\le j-\ell_1} d\left(T^i x, T^i y\right) + \left(\sum_{\ell=0}^{k-1}\alpha_2^\ell\right)\left(d+\gamma_2+m_j+M\right)$$
$$> \left(\sum_{\ell=0}^{k-1}\alpha_2^\ell\right)\left(d+\gamma_2+m_j+M\right) = \frac{1-\alpha_2^k}{1-\alpha_2}\left(d+\gamma_2+m_j+M\right)$$

for any fixed $k\in\mathbf{Z}_+$, and

$$\liminf_{j\to\infty} d\left(T^j x, T^j y\right) \ge \frac{d+\gamma_2+m_j+M}{1-\alpha_2} ;$$

$\forall (x, y\ne x)\in A_i\times A_{i+1}$; $\forall i\in\bar{p}$ even in the event that a fixed point $z\in\bigcap_{i\in\bar{p}} A_i$ exists if the sets A_i intersect provided that either $x\ne z$ or $y\ne z$ since it exists $k\in\mathbf{Z}_+$ such tha $\forall i\in\bar{p}$ t $T^k x\ne T^k y \Rightarrow \max_{0\le i<\infty} d\left(T^i x, T^i y\right)\ne 0$. On the other hand, if $\alpha_2\ge 1$ then from the above relationships:

$$\lim_{j\to\infty} d\left(T^j x, T^j y\right) = \limsup_{j\to\infty} d\left(T^j x, T^j y\right) = \liminf_{j\to\infty} d\left(T^j x, T^j y\right) = +\infty$$

so that $T: \bigcup_{i\in\bar{p}} A_i \to \bigcup_{i\in\bar{p}} A_i$ is expansive. □

ISBN: 978-988-18210-8-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

Remark 3.8, [2]. An important result for p-cyclic contractive self- maps $T:\bigcup_{i\in\bar{p}}A_i\rightarrow\bigcup_{i\in\bar{p}}A_i$, namely, $\alpha_{2i}=\gamma_{2i}=M(x,y)=0$; $\forall i\in\bar{p}$, $\forall x,y\in\bigcup_{i\in\bar{p}}A_i$ is that since they are non-expansive they have the same distance between any pair of subsets ; i.e. $d_i=d$; $\forall i\in\bar{p}$ so that a fixed point exists in $\bigcap_{i\in\bar{p}}A_i$ provided that such an intersection is non-empty. If the intersection of the subsets is empty and those subsets are closed and convex then

$$\lim_{j\rightarrow\infty} d\left(T^{jp+i}x,T^{jp+i}y\right)$$
$$=\lim_{j\rightarrow\infty} d\left(T^{jp+i}z_i,T^{jp+i}z_{i+1}\right)=d(z_i,Tz_i)$$

; $\forall(x,y)\in A_i\times A_{i+1}$ where $z_i\in A_i$, $z_{i+1}\in A_{i+1}$; $\forall i\in\bar{p}$ are best proximity points. □

IV. EXAMPLES

Example 4.1. In order to discuss the feasibility of (1.1) for p=1 ; i.e. the self-map $T:X\rightarrow X$ is not p-cyclic under the constraint $M_{2i}(x,y)=M_2(x,y)=M\in\boldsymbol{R}_{0+};\forall x,y\in X$, note the following:

1) If $M=0$ and $K\in(0,1]$ then (2.1) is the usual contractive constraint of Banach contraction principle and $T:X\rightarrow X$ is strictly contractive . If $K=M=0$ then $T:X\rightarrow X$ is non-expansive. If $M=0$, $K=1$ and the inequality in (2.1) is strict for $x,y(\neq x)\in X$ then $T:X\rightarrow X$ is weakly contractive.

2) If K=1 then $d(Tx,Ty)\leq M$; $\forall x,y\in X$. Since T is a self-map on X, the validity of the constraint (2.1) is limited to the set family $\hat{A}_T:=\{A_i\subset X:(diam(A_i)\leq M\wedge T(A_i)\subset A_j;\ some\ A_j\in\hat{A}_T)\}$ of bounded subsets of X. In this case, $d\left(T^jx,T^jy\right)\leq M$; $\forall j\in\boldsymbol{Z}_+$ provided that $x,y\in A_\alpha\in\hat{A}_T$ and T maps X to some member A_i of $\hat{A}_T$ for each given $x,\ y\in X$. In other words, the image of T is restricted so that $T:X\rightarrow X\,|\,A_i$ (for some $A_i\in\hat{A}_T$ which depends, in general, on x and y) so that $d(Tx,Ty)\leq M$ in order to (2.1) to be feasible, i.e. Tx,Ty are in some set of the family $\hat{A}_T$ if the pair x, y in X is such that $d(x,y)>M$. Note that $T:X\rightarrow X$ is not necessarily a retraction from X to some element of $\hat{A}_T$ since $T(A_i)\subseteq A_j$ for $A_i,A_j(\neq A_i)\in\hat{A}_T$. Note that $T:X\rightarrow X\,|\,A_i$ can possess a fixed point if K=1 and (2.1) holds.

3) If $K>1$ then

$$d(x,y)\geq M/(K-1)\Rightarrow 0\leq d(Tx,Ty)\leq d(x,y)-\frac{M}{K-1}<d(x,y)$$

if $x,y(\neq x)\in X$

As a result, if $x,y\in X$ exist such that $d(x,y)\in\left(0,\frac{M}{K-1}\right)$ then the constraint (2.1) is impossible for any self-map T on X since it would imply $d(Tx,Ty)<0$. But for large enough distances satisfying $d(x,y)\in\left[\frac{M}{K-1},\infty\right)$ ($d(x,y)\in\left(\frac{M}{K-1},\infty\right)$, the self-map $T:X\rightarrow X$ is locally non-expansive (locally contractive). This can be expected to the light of some results provided in [18]. Fixed points can exist only in trivial cases as, for instance , $X:=\left\{x:d(x,y)\geq\frac{M}{K-1};\forall y\in X\right\}$ is a set of isolated points with a minimum pair-wise distance threshold so that $T:X\rightarrow X$ is such that $T(y)=x\in X$; $\forall y\in X$.

4) The case of interest discussed through this paper for (2.1) is when $M>0$ and $K\in[0,1)$. It is shown that the self-map $T:X\rightarrow X$ exhibits contractive properties for sufficiently large distances which exceed a minimum real threshold while it might possibly be expansive for distances under such a threshold. A related motivating example follows.

Example 4.2: Note that the second inequality in (2.1) with $p=1,K_{2i}\,d_i+M_{2i}(x,y)\equiv K_2\,d+M_2(x,y)=M$, $d=0$;$\forall x,y\in X$ is equivalent to :

$$d(Tx,Ty)\leq(1-K)d(x,y)+M \tag{4.1}$$

; $\forall x,y\in X$, for some $M>0$. Eq. (4.1) is relevant, for instance, in the following important problem. Let a linear time-invariant n-th order dynamic system be:

$$\dot{x}(t)=Ax(t)+\eta_x(t) \tag{4.2}$$

with $A\in\boldsymbol{R}^{n\times n}$ being a stability matrix whose fundamental matrix satisfies $\left\|e^{At}\right\|\leq K_0e^{-\alpha_0t}$; $\forall t\geq 0$ for some positive real constants K_0 (being norm- dependent) and α_0 and $\eta:[0,\infty)\times X\rightarrow\boldsymbol{R}^n$ being an unknown uniformly bounded perturbation of essential supremum bound satisfying $\underset{\infty>t\geq 0}{ess\ sup}\ \|\eta_x(t)\|\leq M_0<\infty$; $\forall x\in X$. The unique solution of (4.3) for $x(0)=x_0$ is:

$$x(t)=e^{At}x_0+\int_0^t e^{A(t-\tau)}\eta_x(\tau)d\tau \tag{4.3}$$

Direct calculation with (2.4) for the norm –induced distance $d(x,y):=\|x-y\|$; $\forall x,y\in X$ yields:

$$d(x(t),y(t))=\|x(t)-y(t)\|\leq K_0e^{-\alpha_0t}\|x_0-y_0\|$$
$$+\frac{K_0}{\alpha_0}\sup_{0\leq\tau<\infty}\left\|\eta_x(\tau)-\eta_y(\tau)\right\|$$
$$\leq(1-K)d(x_0,y_0)+M \tag{4.4}$$

; $\forall t\geq h_0:=\frac{1}{\alpha_0}ln\frac{K_0}{1-K}$ with $\infty>M\geq\frac{2K_0M_0}{\alpha_0}$, $K:=1-K_0e^{-\alpha_0h_0}\in(0,1)$. Now, let $X\subset\boldsymbol{R}^n$ the state space of (4.1), generated by (4.4) , subject to $x_0\in X$ and (X,d) is a complete metric space. Define the state transformation $T_h\,x(kh)=x\left[(k+1)h\right]$ on X which

ISBN: 978-988-18210-8-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

generates the sequence of states $\left\{x(kh)\right\}_{k=0}^{\infty}$ being in X if $x_0 \in X$ with h being any real constant which satisfies $h \ge h_0$. Then, the self-map $T_h : X \to X$ satisfies (4.1).

Note that the system (4.3) is always globally Lyapunov stable for any bounded initial conditions in view of (4.5). If the perturbation is identically zero then the origin is globally asymptotically Lyapunov stable since A is a stability matrix. This follows also from (4.5) since the self-map T_h on X is a contraction which has zero as its unique fixed and equilibrium point so that $x(kh+\tau) = e^{A\tau} x(kh) \to 0$ as $k \to \infty$; $\forall \tau \in [0, h)$; $\forall h \ge h_0$. Thus, $x(t) \to 0$ as $t \to \infty$. However, in the presence of the perturbation, the origin is not globally asymptotically stable (although the system is globally stable) and it exhibits ultimate boundedness since for sufficiently large distances $d(T_h x(kh), T_h y(kh)) \ge \frac{M}{K}$ (respectively, $d(x(kh), y(kh)) > \frac{M}{K}$), the self-map is non-expansive (respectively, contractive). Then, $0 \le d(T_h x(kh), T_h y(kh)) \le d(x(kh), y(kh))$ respectively, $d(T_h x(kh), T_h y(kh)) < d(x(kh), y(kh))$). But such properties are not guaranteed if $d(x(kh), y(kh)) < \frac{M}{K}$ which can lead to $T_h : X \to X$ being expansive. □

Examples 4.1-4.2 emphasize the fact that some real-world problems exist where certain self-maps T from X to X are neither contractive nor expansive everywhere in X while such a map is guaranteed to be contractive for sufficiently large distances between any two points in X exceeding a known real threshold. For small distances, the self-map could be potentially expansive, or, as in the dynamic system of Example 4.1, unclassified as expansive, non-expansive or contractive since those potential behaviors have a local character. Note that in the self-map X on X is point-wise nonexpansive, contractive or potentially expansive for each given pair in X accordingly to the distance between them.

Example 4.3: Let the metric space be $(\boldsymbol{R}, d)$ endowed with Euclidean distance. Consider the self- map $T: X \to X$ defining the discrete scalar dynamic controlled system:

$$x_{k+1} = a_k x_k + b_k u_k ;\ \forall k \in \boldsymbol{Z}_{0+},\ x(0) = x_0 \quad (4.5)$$

subject to $b_k \ge \underline{b} > 0$ under the control law

$$u_k = \begin{cases} -K_k x_k \,sgn\, x_k & if\ x_k \ne 0 \\ 0 & if\ x_k = 0 \end{cases} \quad (4.6)$$

$$K_k = K_k(x_k) = \frac{\lambda_k + a_k}{b_k \ sign\, x_k} \quad (4.7)$$

is a sequence from $(\boldsymbol{R} \setminus \{0\}) \times \boldsymbol{Z}_{0+} \to \boldsymbol{R} \times \boldsymbol{Z}_{0+}$ where λ_k is a discrete real sequence on $[0, 1)$. Consider the self-map $T: \boldsymbol{R} \to \boldsymbol{R}$ defining the discrete closed-loop dynamic system obtained by combining (4.5)-(4.7).

$$x_{k+1} = Tx_k \equiv a_k x_k + b_k u_k = (a_k - b_k K_k sign\ x_k) x_k = -\lambda_k x_k ;\ \forall k \in \boldsymbol{Z}_{0+},\ x(0) = x_0 \quad (4.8)$$

which is also valid for $x_k = 0$. Define subsets $A_1 = \boldsymbol{R}_{0+}$ and $A_2 = \boldsymbol{R}_{0-}$ of $\boldsymbol{R}$ and note that the self-map is a 2-cyclic self-map from $\boldsymbol{R} = A_1 \cup A_2$ to itself with $A_1 \cap A_2 = \{0\}$ fulfilling $T A_{1,2} = A_{2,1}$. Note that $0 \in F(T)$. Since $\lambda \in [0, 1)$, $x_k \to 0 \in F(T)$ for any $x_0 \in \boldsymbol{R}$ so that $T: A_1 \cup A_2 \to A_1 \cup A_2$ is a 2-cyclic strict contraction.

Example 4.4: Example 4.3 is re-examined under a class of parametrical perturbations. Let the system be modified as follows:

$$x_{k+1} = (a_k + a_{0k}) x_k + b_k u_k ;\ \forall k \in \boldsymbol{Z}_{0+},\ x(0) = x_0 \quad (4.9)$$

subject to $b_k \ge \underline{b} > 0$ under the parametrical perturbation constraint $a_{0k} \in [0, a_0]$. Now, redefine $A_1 \equiv \boldsymbol{R}_{0\varepsilon} := \{z \in \boldsymbol{R} : z \ge \varepsilon > 0\}$ and $A_2 \equiv \boldsymbol{R}_{-0\varepsilon} := \{z \in \boldsymbol{R} : z \le -\varepsilon < 0\}$ for some given $\varepsilon \in \boldsymbol{R}_+$ as two disjoint sets of $\boldsymbol{R}$. The control law is:

$$u_k = K_k x_k = -\frac{1}{b_k}(1 + a_k + a_0 - \delta_k \,sign\, x_k) x_k \quad (4.10)$$

A direct calculation shows that

$$x_k \in A_1 \Rightarrow \left[x_{k+1} = -x_k - (a_0 - a_{0k} + \delta_k) x_k \le -(1 + \delta_k)\varepsilon\right]$$
$$\Rightarrow x_{k+1} \in A_2$$
$$x_k \in A_2 \Rightarrow \left[x_{k+1} = (1 + a_0 - a_{0k} + \delta_k)\left|x_k\right| \ge (1 + \delta_k)\varepsilon\right]$$
$$\Rightarrow x_{k+1} \in A_1 \quad (4.11)$$

where $\{\delta_k\}_0^\infty$ is any arbitrary nonnegative real sequence. One gets from (4.11) that

$$1 + \frac{x_{k+1}}{x_k} = -(a_0 - a_{0k} + \delta_k);\ \forall k \in \boldsymbol{Z}_{0+} \quad (4.12)$$

If $\{\delta_k\}_0^\infty$ is identically zero and $a_{0k} = a_0 ; \forall k \in \boldsymbol{Z}_{0+}$ then $a_0 - a_{0k} + \delta_k \equiv 0$

$$-x_{k+1} = x_k = (-1)^k sign\ x_0$$

so that $x_{k+1} \in A_{i+1}$ if $x_k \in A_i$ provided that $|x_0| \ge \varepsilon$. If $a_0 - a_{0k} + \delta_k \to 0$ as $k \to \infty$ then $sign\, x_{k+1} = -sign\, x_k$ and $x_{k+1} \to -x_k \to L$ as $k \to \infty$ for some real constant $L \in (-\infty, -\varepsilon] \cup [\varepsilon, \infty)$ provided that $|x_0| \ge \varepsilon$.If the control law (4.10) is replaced with:

$$u_k = K_k x_k = -\left(\frac{1 - \alpha^k}{b_k}\right)(1 + a_k + a_{0k}) x_k \quad \text{for some}$$
$$\alpha \in [0, 1),\ \forall k \in \boldsymbol{Z}_{0+} \quad (4.13)$$

Then, $sign\, x_{k+1} = -sign\, x_k$, $x_{k+1} \to -x_k \to \pm\varepsilon$ as $k \to \infty$ if $|x_0| \ge \varepsilon$.

Example 4.5: A variant of Example 4.4 is re-examined under additive perturbations. Let the system be modified as follows:

$$x_{k+1} = a_k x_k + b_k u_k + g_k ;\ \forall k \in \boldsymbol{Z}_{0+},\ x(0) = x_0 \quad (4.14)$$

The control law is computed so that $x_{k+1} = -\lambda_k x_k$ provided that the real additive perturbation sequence $\{g_k\}_0^\infty$ is identically zero. This yields:

ISBN: 978-988-18210-8-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$$u_k=-b_k^{-1}(\lambda_k+a_k)x_k \Rightarrow x_{k+1}=-\lambda_k x_k+g_k \quad (4.15)$$

$$x_k \ge \varepsilon \Rightarrow \left[x_{k+1}=-\lambda_k x_k+g_k \le -\varepsilon\right] \Leftrightarrow g_k \le \lambda_k x_k-\varepsilon \quad (4.16)$$

and the last implication is guaranteed if $g_k \le (\underline{\lambda}-1)\varepsilon \le (\lambda_k-1)\varepsilon$. Also,

$$x_k \le -\varepsilon \Rightarrow \left[x_{k+1}=-\lambda_k x_k+g_k \ge \varepsilon\right] \Leftrightarrow g_k \ge \lambda_k x_k+\varepsilon \quad (4.17)$$

and the last implication is guaranteed if $g_k \ge (1-\underline{\lambda})\varepsilon \ge (1-\lambda_k)\varepsilon$. Then,

$$\left[x_k \in A_1 \wedge g_k \le (\underline{\lambda}-1)\varepsilon\right] \Rightarrow x_{k+1} \in A_2 \, ,$$
$$\left[x_k \in A_2 \wedge g_k \ge (1-\underline{\lambda})\varepsilon\right] \Rightarrow x_{k+1} \in A_1 \quad (4.18)$$

provided that $1 \le \underline{\lambda} \le \min_{0\le j<\infty} \lambda_j \Rightarrow g_k/\varepsilon \in \left[1-\underline{\lambda}\ ,\underline{\lambda}-1\right]$; $\forall k \in \mathbf{Z}_{0+}$. Note that if $\underline{\lambda}=1$ then $\{g_k\}_0^\infty$ is identically zero and no disturbance is admitted. The 2-cyclic self-map $T: A_1 \cup A_2 \to A_1 \cup A_2$ fulfils $TA_{1,2}=A_{2,1}$ and is expansive for the class of perturbations fulfilling $g_k/\varepsilon \in \left[1-\underline{\lambda}\ ,\underline{\lambda}-1\right]$ if $\underline{\lambda}>1$ and nonexpansive if $\underline{\lambda}=1$ with $g_k \equiv 0$. If $g_k/\varepsilon \in \left[1-\lambda_k\ ,\lambda_k-1\right]$; with $\lambda_k \ge 1$, $\forall k \in \mathbf{Z}_{0+}$ then T can be iteration-dependent locally expansive or non-expansive depending on λ_k being unity or larger than unity.

Example 4.6: Example 4.5 is reformulated by modifying the controls to better overcome the perturbations as follows. Assume that the perturbation sequence is unknown but upper- bounding and lower-bounding sequences of it $\{g_{k1}\}_0^\infty$, $\{g_{k2}\}_0^\infty$ are known and the controls are corrected with the sequence $\{\omega_k\}_0^\infty$ by using the above knowledge so that:

$$u_k=-b_k^{-1}\left[(\lambda_k+a_k)x_k+\omega_k\right] \ ; \ \forall k \in \mathbf{Z}_{0+} \quad (4.19)$$

in order to achieve that that $T: A_1 \cup A_2 \to A_1 \cup A_2$ be a 2-cyclic self-map so that:

$$x_k \in A_1 \Rightarrow x_{k+1}(\in A_2)=-\lambda_k x_k+g_k-\omega_k=\varepsilon+\nu_{k+1} \ge \varepsilon$$
$$x_k \in A_2 \Rightarrow x_{k+1}(\in A_1)=-\lambda_k x_k+g_k-\omega_k=-(\varepsilon+\nu_{k+1}) \le -\varepsilon \quad (4.20)$$

with $\{\nu_k\}_0^\infty$ being some real nonnegative sequence. Combining these two constraints with (4.19) and replacing the obtained results in (4.14) yields for the elements of $\{\nu_k\}_0^\infty$:

$$\nu_{k+1}=\lambda_k x_k-g_k+\omega_k-\varepsilon \ge 0$$
$$\Rightarrow \omega_k := g_{k2}-\lambda_k x_k+\sigma_k+\varepsilon \ge g_k-\lambda_k x_k+\varepsilon \quad (4.21)$$

, $\forall k \in \mathbf{Z}_{0+}$ if $x_k \in A_2$, and

$$\nu_{k+1}=-\lambda_k x_k+g_k-\omega_k-\varepsilon \ge 0$$
$$\Rightarrow \omega_k := g_{k1}-\lambda_k x_k-\sigma_k-\varepsilon \le g_k-\lambda_k x_k-\varepsilon \quad (4.22)$$

; $\forall k \in \mathbf{Z}_{0+}$ if $x_k \in A_1$ by using an arbitrary bounded nonnegative real sequence $\{\sigma_k\}_0^\infty$ with $\sigma_k \le \overline{\sigma}$; $\forall k \in \mathbf{Z}_{0+}$. Thus, the controller is given by (4.19) together with the companion equation:

$$\omega_k=\begin{cases} g_{k2}-\lambda_k x_k+\sigma_k+\varepsilon & \text{if} \quad x_k \in A_2 \\ g_{k1}-\lambda_k x_k-\sigma_k-\varepsilon & \text{if} \quad x_k \in A_1 \end{cases}$$
$$\lambda_k \in [0,1) \, ; \sigma_k \ge 0 \, ; \forall k \in \mathbf{Z}_{0+} \quad (4.23)$$

The controlled system through (4.19) becomes

$$x_{k+1}=-\lambda_k x_k+g_k-\omega_k \, ; \ \forall k \in \mathbf{Z}_{0+} \quad (4.24)$$

subject to (4.23). Combining (4.22) - (4.24) yields:

$$x_k \in A_1 \Rightarrow \nu_{k+1}=g_k-g_{k1}+\sigma_k$$
$$x_k \in A_2 \Rightarrow \nu_{k+1}=g_{k2}-g_k+\sigma_k \quad (4.25)$$
$$x_k \in A_1 \Rightarrow x_{k+1}=\nu_{k+1}+\varepsilon=g_k-g_{k1}+\sigma_k+\varepsilon$$
$$x_k \in A_2 \Rightarrow x_{k+1}=-(\nu_{k+1}+\varepsilon)=g_k-g_{k2}-(\sigma_k+\varepsilon) \quad (4.26)$$

Define $\overline{\sigma} := \max_{0\le k<\infty} \sigma_k$. Then, one gets proceeding inductively from (4.26) provided that $\overline{g}_k := \max\left(\left|\overline{g}_{k1}\right|,\left|\overline{g}_{k2}\right|\right) \le \vartheta \max_{0\le j\le k}\left|x_j\right|$ for some positive real constant $\vartheta<1$

$$\left|x_{k+1}\right| \le \overline{g}_k+\overline{\sigma}+\varepsilon \le \vartheta \max_{0\le j\le k}\left|x_j\right|+\overline{\sigma}+\varepsilon \Rightarrow$$
$$\max_{0\le j\le k}\left|x_j\right| \le \vartheta \max_{0\le j\le k}\left|x_j\right|+\overline{\sigma}+\varepsilon+\left|x_0\right|$$
$$\Rightarrow \max_{0\le j\le k}\left|x_j\right| \le \frac{\overline{\sigma}+\varepsilon+\left|x_0\right|}{1-\vartheta}<\infty \quad (4.27)$$

since $\max_{0\le j\le k}\left|x_j\right| \le \max_{0\le j\le k+1}\left|x_j\right|$. It has been proven the following:

Proposition 4.7. Assume that the system of Example 4.6 is subject to a controller (4.20), (4.23). Then , the self-map $T: A_1 \cup A_2 \to A_1 \cup A_2$ which defines the trajectory solution is a 2-cyclic self-map. Furthermore, assume that the perturbation satisfies $\left|g_k\right| \le \vartheta \max_{0\le j\le k}\left|x_j\right| < \max_{0\le j\le k}\left|x_j\right|$.

Then, the controlled system (4.14), (4.20), (4.23) is globally Lyapunov stable so that the 2-cyclic self-map $T: A_1 \cup A_2 \to A_1 \cup A_2$ is neither expansive nor asymptotically expansive. □

The conditions for a modified 2-cyclic contraction under perturbations are obtained by taking three consecutive points of the controlled system trajectory solution as follows:

$$x_k(\ge\varepsilon) \in A_2 \, ; \ y_k=x_{k+1}(\le -\varepsilon) \in A_1 \, ;$$
$$z_k(\ge\varepsilon) := x_{k+2}=y_{k+1}=Ty_k=T^2 x_k \in A_2 \quad (4.28)$$

Proposition 4.8. Assume that Assume that Proposition 4.7 holds. Assume also that

$$M(x_k, x_{k+1}) \le \alpha\, d(x_{k+1}, x_{k+2})+\gamma_k=\alpha\left(\left|x_{k+1}\right|+\left|x_{k+2}\right|\right)+\gamma_k$$

; $\forall k \in \mathbf{Z}_{0+}$ for some uniformly bounded nonnegative real sequence $\{\gamma_k\}_0^\infty$. Then, $T: A_1 \cup A_2 \to A_1 \cup A_2$ a 2-cyclic strict contraction if $\alpha < \max_{0\le k<\infty}(\lambda_k \lambda_{k+1}) \in (0,1)$ and $K_0 := \dfrac{\max_{0\le k<\infty}(\lambda_k \lambda_{k+1})-\alpha}{1-\alpha} < \dfrac{\alpha+\gamma}{\alpha+\varepsilon} \in [0,1)$. If $\varepsilon=0$ then $F(T)=A_1 \cap A_2=\{0\}$.

ISBN: 978-988-18210-8-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

Proof: Direct calculations yield:

$$d\left(x_{k+2},x_{k+1}\right)=d\left(Tx_k\ ,T^2x_k\right)=\left|x_{k+2}\right|+\left|x_{k+1}\right|$$
$$\le\left(1-K\right)d\left(x_{k+1}\ ,x_k\right)+2K\varepsilon+\alpha\left(\left|x_k\right|+\left|x_{k+1}\right|\right)+\gamma_k$$
$$=\left(1-K\right)\left(\left|x_{k+1}\right|+\left|x_k\right|\right)+2K\varepsilon+\alpha\left(\left|x_{k+2}\right|+\left|x_{k+1}\right|\right)+\gamma_k \tag{4.29}$$

$\forall k\in\boldsymbol{Z}_{0+}$ provided that

$$\left|x_{k+2}\right|\le\frac{1-K}{1-\alpha}\left|x_k\right|+2K\varepsilon+\left(\gamma_k-\frac{K-\alpha}{1-\alpha}\left|x_{k+1}\right|\right)=$$
$$\left(1-K_0\right)\left|x_k\right|+2K_0\varepsilon+\left(\bar{\gamma}+\alpha\left(1-K_0\right)-\frac{K-\alpha}{1-\alpha}\left|x_{k+1}\right|\right)$$
$$;\forall k\in\boldsymbol{Z}_{0+} \tag{4.30}$$

where $K_0:=\frac{K-\alpha}{1-\alpha}=1-\frac{1-K}{1-\alpha}\in\left[0,1\right)$ if $\alpha<K:=\max\limits_{0\le k<\infty}\left(\lambda_k\lambda_{k+1}\right)\in\left(0,1\right)$ and $\bar{\gamma}:=\max\limits_{0\le k<\infty}\gamma_k$. Since $\left|x_k\right|\ge\varepsilon\,;\forall k\in\boldsymbol{Z}_{0+}$ then (4.9) implies that

$$\left|x_{k+2}\right|\le\left(1-K_0\right)\left|x_k\right|+2K_0\varepsilon\ \ ;\ \forall k\in\boldsymbol{Z}_{0+} \tag{4.31}$$

so that $T:A_1\cup A_2\to A_1\cup A_2$ a 2-cyclic strict contraction if $K_0<\frac{\alpha+\gamma}{\alpha+\varepsilon}\in\left[0,1\right)$. □

Note that if the perturbation is subject to $M\left(x_k,x_{k+1}\right)\le\alpha\,d\left(x_k,x_{k+1}\right)+\gamma_k=\alpha\left(\left|x_k\right|+\left|x_k\right|\right)+\gamma_k$; $\forall k\in\boldsymbol{Z}_{0+}$ then Proposition 4.8 can be formulated " mutatis-mutandis" under alternative sufficiency-type conditions.

Remark 4.9: Note that if Proposition 4.8 holds then from (4.30) since $\left|x_k\right|\ge\varepsilon\,;\ \forall k\in\boldsymbol{Z}_{0+}$:

$$\left|x_{k+2}\right|-\left|x_k\right|\le-K_0\left|x_k\right|+2K_0\varepsilon\Rightarrow\left|x_{k+2}\right|\le\left|x_k\right|$$

; $\forall k\in\boldsymbol{Z}_{0+}$ and for any $\delta\in\boldsymbol{R}_+$

$$\left|x_{k+2}\right|<\rho_0:=2\varepsilon<\left|x_k\right|<\rho_0+\delta\ ;\ \forall k\in\boldsymbol{Z}_{0+} \tag{4.32}$$

which is equivalent to

$$d\left(x_{k+2},x_{k+1}\right)=\left|x_{k+2}\right|+\left|x_{k+1}\right|<\rho\left(k\right):=\rho_0+\left|x_{k+1}\right|$$
$$<d\left(x_{k+1},x_k\right)=\left|x_k\right|+\left|x_{k+1}\right|<\rho\left(k\right)+\delta\ ;\forall k\in\boldsymbol{Z}_{0+} \tag{4.33}$$

so that $T:A_1\cup A_2\to A_1\cup A_2$ is also a worst-case modified 2-cyclic Meir-Keeler contraction.

ACKNOWLEDGMENTS

The author thanks the Spanish Ministry of Education by its support of this work through Grant DPI2009-07197. He is also grateful to the Basque Government by its support SAIOTEK S-PE07UN04.

REFERENCES

[1] A. Meir and E. Keeler, "A theorem on contraction mappings", *Journal of Mathematical Analysis and Applications*, Vol. 28, No. 2 , pp. 326-329, 1969.

[2] S. Karpagan and S. Agrawal, "Best proximity point theorems for p-cyclic Meir- Keeler contractions", *Fixed Point Theory and Applications*, Vol. 2009, Article ID 197308, 9 pages, doi:10.1155/FPTA/2009/197308.

[3] Chi-Ming Chen and Tong-Huei-Chang, "Fixed Point theorems for a weaker Meir- Keeler type ψ - contraction in metric spaces", *Fixed Point Theory and Applications* , Vol. 2009, Article ID 129124, 8 pages , 2009. doi: 10.1155/2009/129124.

[4] A.A. Eldred and P. Veeramani, "Existence and convergence of best proximity points", *Fixed Point Theory*, Vol. 323, No. 2, pp. 1001-1006, 2006.

[5] M. Kikkawa and T. Suzuki, "Some similarity between contractions and Kannan mappings", *Fixed Point Theory and Applications*, Vol. 2008, Article ID 649749, 8 pages, doi:10.1155/FPTA/2008/649749.

[6] Y. Enjouji, N. Nakanishi and T. Suzuki, "A generalization of Kannan´s fixed point theorem", *Fixed m Point Theory and Applications*, Vol. 2009, Article ID 192872, 10 pages, doi:10.1155/FPTA/2009192872.

[7] M. De la Sen, "Some combined relations between contractive mappings, Kannan mappings, reasonable expansive mappings and T- stability", *Fixed Point Theory and Applications*, Vol. 2009, Article ID 815637, 25 pages, doi:10.1155/FPTA/2009/815637.

[8] T. Dominguez Benavides and S. Phothi, "The fixed point property under renorming in some classes of Banach spaces", *Nonlinear Analysis*, doi: 10.1016/j.na.2009.08.024.

[9] Dž . Burgic′ , S. Kalabušic′ , and M.R.S. Kulenovic′ , "Global attractivity results for mixed –monotone mappings in partially ordered complete metric spaces", *Fixed Point Theory and Applications,* vol. 2009, Article ID 762478, 17 pages, doi: 10.1155/2009/762478.

[10] J. J. Nieto, R.I. Pouso and R. Rodriguez- Lopez, "Fixed point theorems in ordered abstract spaces", *Proc. Amer. Math. Soc*., Vol. 135, pp. 2505-2517, 2007.

[11] M. O. Olatinwo, "Some common fixed point theorems for self-mappings satisfying two contractive conditions of integral type in a uniform space", *Cent. Eur. J. Math.* , Vol. 6, No. 2 , pp. 335-341, 2008.

[12] T. Suzuki, "A new type of fixed point theorem in metric spaces", *Nonlinear Analysis*, Vol. 71, pp. 5313-5317, 2009.

[13] V. Azhmyakov, " Convexity of the set of fixed points generated by some control systems", *Journal of Applied Mathematics,* Vol. 2009, Article ID 291849, 14 pages, doi: 10.1155/2009/291849.

[14] M. De la Sen, "Total stability properties based on fixed point theory for a class of hybrid dynamic systems", *Fixed Point Theory and Applications,* vol. 2009, Article ID 826438, 19 pages, doi: 10.1155/2009/826438.

[15] M. De la Sen, "About robust stability of dynamic systems with time delays through fixed point theory", *Fixed Point Theory and Applications,* vol. 2008, Article ID 480187, 20 pages, doi: 10.1155/2008/480187.

[16] A. Amini-Harandi, A.P. Farajzadeh, D. O. Reagan and R.P. Agarwal, " Coincidence point best approximation and best proximity theorems for condensing set-valued maps in hyperconvex metric spaces", *Fixed Point Theory and Applications*, Vol. 2009, Article ID 543154 , doi:10.1155/FPTA/2009/543154.

[17] A. Amini-Harandi, A.P. Farajzadeh, D. O. Reagan and R.P. Agarwal, "Best proximity pairs for upper semicontinuous set-valued maps in hyperconvex metric spaces ", *Fixed Point Theory and Applications*, Vol. 2009, Article ID 648985, doi:10.1155/FPTA/2009/648985.

[18] E. Catinas, "Estimating the radius of an attraction ball ", *Applied Mathematics Letters*, Vol. 22, No. 5, pp. 712-714 , 2009.

[19] M.R. Pournaki and A. Razani, "On the existence of periodic solutions for a class of generalized forced Lienard equations", *Applied Mathematics Letters*, Vol. 20, No. 3, pp. 248-254, 2009.

[20] T. C. Lim, "On characterizations of Meir- Keeler contractive maps", *Nonlinear Analysis: Theory, Methods and Applications* ,Vol. 46, No. 1, pp. 113-120, 2001.

[21] T. Suzuki, "On characterizations of Meir- Keeler contractions and L-functions*", Bulletin of the Kyushu Institute of Technology*, No. .53, No. 1, pp. 1-13, 2006.

[22] B.Q. Yan, D. O ´Reagan and R.P. Agarwal, " Positive solutions for second-order singular boundary value problems with derivative dependence on infinite intervals", *Acta Applicandae Mathematicae*", Vol. 103, No. 1, pp. 19-57, 2008.

[23] X.K. Zhao and W.G. Ge, " Multiple positive solutions for time scale boundary value problems on infinite intervals", *Acta Applicandae Mathematicae*", Vol. 106, No. 2, pp. 265-273, 2009.

[24] X.L. Qin, Y.J. Cho, S. M. Kang and H.Y. Zhou, "Convergence of a hybrid projection algorithm in Banach spaces", *Acta Applicandae Mathematicae*", Vol. 108, No. 2, pp. 299-313, 2009.

[25] Y.F. Jia, J.H. Wu, and H. Nie, "The coexistence states of a predator-prey model with nonmonotonic functional response and diffusion", *Acta Applicandae Mathematicae*", Vol. 108, No. 2, pp. 413-428, 2009. 187, 20 pages, doi: 10.1155/2008/480187.