Presentation of an Efficient Parallelepiped Finite Element Based on the Strain Approach "SBP8C"

D. HAMADI[1] and T. MAALEM[2]

Abstract— **A new parallelepiped finite element, simple and effective baptized SBP8C (Strain Based Parallelepiped 8-nodes Condensed) is presented in this paper. It is formulated by the use of the static condensation, contributing to enrich the existing finite elements library. This element can be used for the analysis of three dimensional problems and also for the thin and thick plates bending. Tests on standard problems have been examined. This element has a better performance compared to the one based on the displacement model.**

Index *Terms*— Parallelepiped Element, Three-dimensional Elasticity, Strain Approach, Plate Bending, Static condensation.

I. INTRODUCTION

Numerous studies, theoretical and numerical were dedicated to the bending plate. Numerically, the calculation of the thick plate with 3D finite elements has been examined by several authors, references [1] and [2] used these elements by maintaining 3D constants, let us quote for example the brick with twenty nodes, B20 and bricks without intermediate nodes following thickness. According to these authors, 3D elements give good results in this last case, but do not approach known solutions for the thin plates [3]. The major inconvenience in the use of these elements of superior order is the high cost because of the large number of points of numeric integration necessary for the exact evaluation of the element stiffness matrix.

The objective of this paper, is to present a new parallelepiped finite element, simple and effective baptized **SBP8C** (Strain Based Parallelepiped 8-nodes condensed), contributing to enrich the existing finite elements library. This last one is formulated, by the use of the static condensation, not only for the study of the 3D problems but also and especially for the thin and thick plates bending.

II. DESCRIPTION OF THE SBP8C ELEMENT [4]

Figure 1 shows the geometry of the element **SBP8C** and the correspondent kinematic variables. Each node (i) is attributed the three degrees of freedom (d.o.f) Ui, Vi and Wi.

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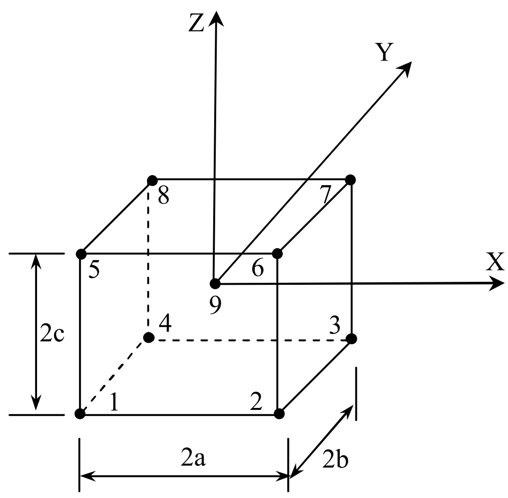

Fig.1. Geometry of the element "SBP8C"

III. ANALYTICAL FORMULATION OF THE SBP8C ELEMENT

A. Displacement field

For a linear theory where the strains are small, there are six strain components occurring in completely 3D analysis.

$$\varepsilon_{xx} = U,_x \qquad \gamma_{xy} = U,_y + V,_x \qquad (1\ a,b)$$

$$\varepsilon_{yy} = V,_y \qquad \gamma_{yz} = V,_z + W,_y \qquad (1\ c,d)$$

$$\varepsilon_{zz} = W,_z \qquad \gamma_{xz} = W,_x + U,_z \qquad (1\ e,f)$$

U, V and W: are the displacements in the three directions X, Y and Z respectively.

Equations (2) represent the condition of the rigid body modes (RBM). We have:

$$\varepsilon_{ii} = 0 \qquad (2a)$$

$$\gamma_{ij} = 0 \qquad (2b)$$

By integrating equations (2), we obtain a particular solution:

$$U_R = a_1 + a_4\,y + a_6\,z \qquad (3a)$$

$$V_R = a_2 - a_4\,x - a_5\,z \qquad (3b)$$

$$W_R = a_3 + a_5\,y - a_6\,x \qquad (3c)$$

Equations (3) represent the displacement fields corresponding to the rigid body modes (RBM).

The present element is an eight parallelepiped node in addition to the central node, with three degrees of freedom by node (Fig.1). Therefore, the field of displacement has to

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contain twenty-seven independent constants. Six of them (a_1, a_2 ... a_6) are already used to represent the RBM, so the remaining twenty-one (a_7, a_8 ... a_{27}) represent in a rough way strains in the element, while verifying the six equations of compatibility. The strain field is:

$$\varepsilon_{xx} = a_7 + a_8\, y + a_9\, z + a_{10}\, yz + a_{25}\, x \qquad (4a)$$

$$\varepsilon_{yy} = a_{11} + a_{12}\, x + a_{13}\, z + a_{14}\, xz + a_{26}\, y \qquad (4b)$$

$$\varepsilon_{zz} = a_{15} + a_{16}\, x + a_{17}\, y + a_{18}\, xy + a_{27}\, z \qquad (4c)$$

$$\gamma_{yz} = -a_{10}\, x^2 - a_{19} + a_{20}\, x + a_{22}\, x \qquad (4d)$$

$$\gamma_{xz} = -a_{14}\, y^2 + a_{21} + a_{22}\, y + a_{24}\, y \qquad (4e)$$

$$\gamma_{xy} = -a_{18}\, z^2 + a_{20}\, z + a_{23} + a_{24}\, z \qquad (4f)$$

Substituting equations (2) and (4) into (1) and solving the resulting differential equations gives:

$$\begin{aligned} U = {} & a_1 + a_4\, y + a_6\, z + a_7\, x + a_8\, xy + a_9\, xz + a_{10}\, xyz \\ & - 0.5\, a_{12}\, y^2 - 0.5\, a_{14}\, y^2 z - 0.5\, a_{16}\, z^2 - 0.5\, a_{18}\, yz^2 \\ & + 0.5\, a_{21}\, z + 0.5\, a_{23}\, y + a_{24}\, yz + 0.5\, a_{25}\, x^2 \end{aligned} \qquad (5a)$$

$$\begin{aligned} V = {} & a_2 - a_4\, x - a_5\, z - 0.5\, a_8\, x^2 - 0.5\, a_{10}\, x^2 z + a_{11}\, y \\ & + a_{12}\, xy + a_{13}\, yz + a_{14}\, xyz - 0.5\, a_{17}\, z^2 - 0.5\, a_{18}\, xz \\ & + 0.5\, a_{19}\, z + a_{20}\, xz + 0.5\, a_{23}\, x + 0.5\, a_{26}\, y^2 \end{aligned} \qquad (5b)$$

$$\begin{aligned} W = {} & a_3 + a_5\, y - a_6\, x - 0.5\, a_9\, x^2 - 0.5\, a_{10}\, x^2 y \\ & - 0.5\, a_{13}\, y^2 - 0.5\, a_{14}\, xy^2 + a_{15}\, z + a_{16}\, xz \\ & + a_{17}\, yz + a_{18}\, xyz + 0.5\, a_{19}\, y + 0.5\, a_{21}\, x \\ & + a_{22}\, xy + 0.5\, a_{27}\, z^2 \end{aligned} \qquad (5c)$$

It should be noticed here, that the final displacement functions contain quadratic terms so allowing the change of curvature. The parallelepiped element having the displacement fields given by equations (5) is referred to as **SBP8C**. The classic element based on the displacement model will be referred to as DBB8.

B. Evaluation of the element stiffness matrix $[K_e]$

The evaluation of the element stiffness matrix is summarized with the evaluation of the following expression:

$$\left[K_e\right] = \left[A^{-1}\right]^T \left[K_0\right]\left[A^{-1}\right] \qquad (6)$$

Where

$$\left[K_0\right] = \iiint_v \left[Q\right]^{\mathrm{T}} \left[\mathrm{D}\right]\left[Q\right] dx\, dy dz \qquad (7)$$

With

$$\left[\underset{6x27}{\mathrm{Q}}\right] = \begin{bmatrix} 0&0&0&0&0&0&1&y&z&yz&0&0&0&0&0&0&0&0&0&0&0&0&0&0&x&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&x&z&xz&0&0&0&0&0&0&0&0&0&0&0&y&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&x&y&xy&0&0&0&0&0&0&0&0&z \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-z^2&0&z&0&0&1&z&0&0&0 \\ 0&0&0&0&0&0&0&0&0&-x^2&0&0&0&0&0&0&0&0&1&x&0&x&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&-y^2&0&0&0&0&0&0&1&y&0&y&0&0&0 \end{bmatrix} \qquad (8)$$

Since [A] and its inverse can be evaluated numerically, the evaluation of the integral (7) becomes the key of the problem. While the shape of the element is regular, numerical integration is reduced to an analytical integration [4].

C. Mechanical characteristics of the material

The matrix (9) is a modified form of the material matrix properties by introducing the plane stress constants and a corrective coefficient of transverse shearing (TS) noted K [5].

$$[\mathrm{D}] = \begin{bmatrix} \mathrm{D1} & \mathrm{D2} & 0 & 0 & 0 & 0 \\ \mathrm{D2} & \mathrm{D3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{D4} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{D5} & 0 & 0 \\ 0 & 0 & 0 & 0 & K\,\mathrm{D6} & 0 \\ 0 & 0 & 0 & 0 & 0 & K\,\mathrm{D7} \end{bmatrix} \qquad (9)$$

Where:

$$D1 = D3 = \frac{E}{\left(1-\nu^2\right)} \;; \qquad D2 = \frac{\nu E}{\left(1-\nu^2\right)} \;;$$

$$D4 = \frac{E\left(1-\nu\right)}{\left(1-2\nu\right)\left(1+\nu\right)} \;; \qquad D5 = \frac{E}{2\left(1+\nu\right)} \;;$$

$$D6 = D7 = K\,\frac{E}{2\left(1+\nu\right)}$$

E is the modulus of elasticity; ν is the Poisson's ratio
K= $\pi^2/12$ in Uflyand-Hencky-Mindlin's theory
K= 5/ 6 in Reissner's theory

IV. NUMERICAL EXAMPLES

The performance of the present element **SBP8C** is estimated through a series of standard tests to show the interest of the strain model. The peculiarity of these examples lies generally, on one hand, in their geometrical simplicities, and on the other hand, in their very varied behaviour toward the phenomenon of locking in transverse shearing (TS). These two aspects make these examples an ideal tool for the validation of new models of finite elements.

A. Plate patch tests

In plate problems, the importance of the patch tests is paramount [1]. A number of popular numerical problems mainly extracted from the proposed standard set of problems by White and Abel [6]. All reference solutions are taken from the same paper unless stated otherwise.

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1) Constant bending moment patch test for plates

The response of single element cantilever to a constant bending moment applied as shown in Fig.2(c) is considered. Vertical deflections at the tip of the plate are calculated. It is seen in Table I that the **SBP8C** shows the same tip deflection as theory and gives more accurate results.

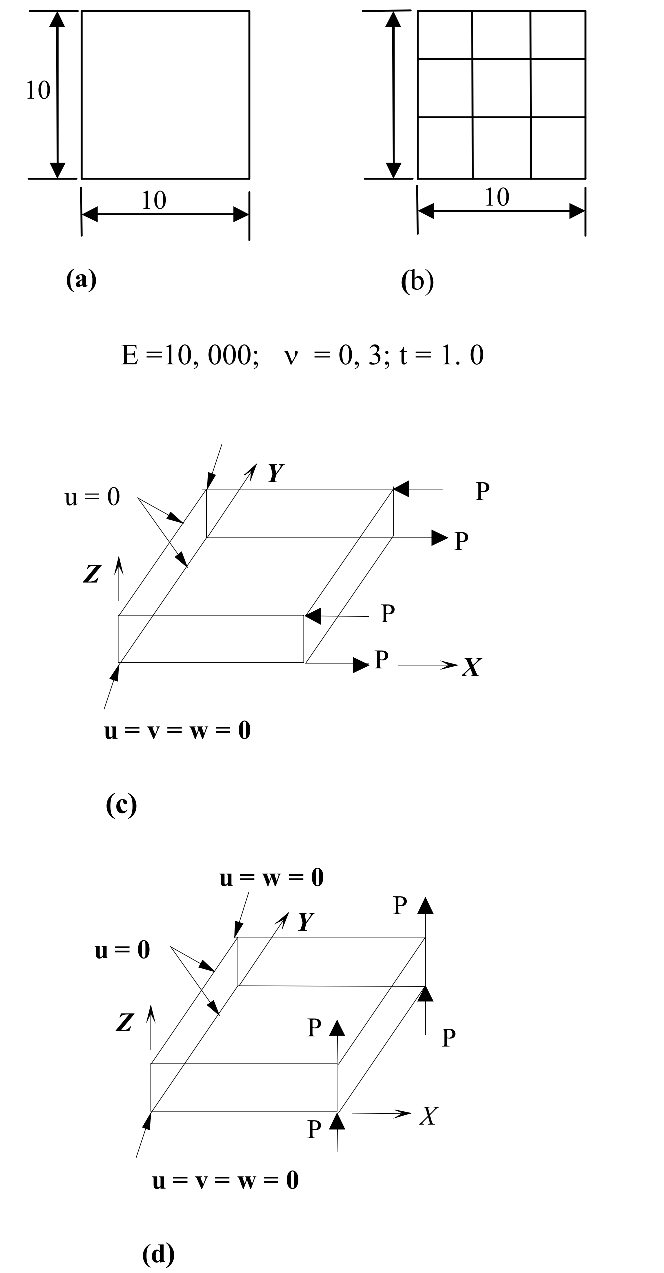

Fig.2. Plate patch tests (P = 1.0); Mesh :(a) regular 1x1; (b) regular 3x3.
(c) Constant bending moment test; (d) Out-of-plane shear load test

TABLE I: CONSTANT BENDING MOMENT PATCH TEST FOR PLATES

	Tip deflection W (x 10^{-1})			
Mesh	Theory	PN30 [7]	ANSYS	**SBP8C**
1 x 1	0.12	0.1092	0.1092	0.12
3 x 3	0.12	0.1106	0.1092	0.12

2) Out-of-plane patch test for plates

We use the same meshes as in previous section. The boundary conditions and end shear loading used are shown in Fig.2 (d). The solutions obtained are shown in Table II. It is seen for the SBP8C that the results are satisfactory and convergence to the analytical solution is obtained as the number of elements used is increased.

TABLE II: OUT-OF-PLANE PATCH TEST FOR PLATES

	Tip deflection W (x 10^{-1})			
Mesh	Theory	PN30 [7]	ANSYS	**SBP8C**
1 x 1	0.16	0.132	0.121	0.1268
3 x 3	0.16	0.151	0.147	0.1459

B. Simply Supported Square Plate

The test of the simply supported square plate is examined with either a uniform loading (q = 1) or with a concentrated load (P = 1) at the centre (Fig.3). The quarter of the plate is divided into a mesh of N x N elements. The convergence tests are carried out on two different L/h ratios of 10 and 100 for thick and thin plates respectively. The results for the central deflection are given in Table III and Table IV. The effect of L/h ratio on the deflection at the centre WC for a plate is studied. The results presented in Table V are given for the 12x12 meshes in terms of WC/Wk where Wk is the reference Kirchhoff solution [1] for thin plates.

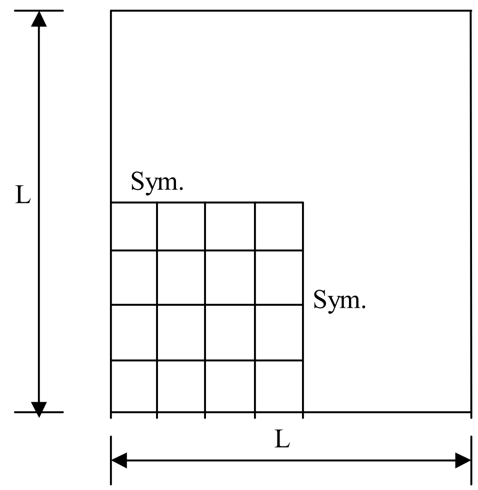

Fig.3: Simply supported square plate
(L = 10, h = 1. or 0.1, E =10.92, ν = 0.25)

The numerical tests show that:

- The strain based element **SBP8C** has quite rapid rate of convergence to reference solutions for both thick and thin plates.

- The **SBP8C** element is free from any shear locking since it converge to the Kirchhoff solution for thin plates, contrarily for the corresponding displacement based element DBB8

- SBH8 and **SBP8C** have similar behaviour, and they have the advantages to be valid for both thin and thick plates.

- The influence of the transverse shear for the strain based elements is much more important for plates with concentrated load than for those with uniform load.

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Table III: Central deflection of a simply supported plate with a uniform load

Mesh	$\frac{wD}{qL^4}$ x100					
	L/h=10			L/h=100		
	SBP8C	SBH8 [8]	DBB8	**SBP8C**	SBH8 [8]	DBB8
2 x 2	0.3812	0.326	0.2283	0.0349	0.0523	0.0045
4 x 4	0.4218	0.4048	0.351	0.2563	0.3081	0.0171
8 x 8	0.4229	0.4145	0.3982	0.3856	0.3883	0.0582
12 x 12	0.4270	0.4249	0.4171	0.4033	0.4029	0.0786
Exact solution [9]	0.427			0.406		

TABLE IV: CENTRAL DEFLECTION OF A SIMPLY SUPPORTED PLATE WITH A CONCENTRATED LOAD

Mesh	$\frac{wD}{PL^2}$ x100					
	L/h=10			L/h=100		
	SBP8C	SBH8 [8]	DBB8	**SBP8C**	SBH8 [8]	DBB8
2 x 2	1.1745	0.9907	0.7269	0.113	0.1452	0.0134
4 x 4	1.321	1.243	1.097	0.789	0.8387	0.0481
8 x 8	1.363	1.333	1.289	1.108	1.115	0.1636
12 x 12	1.372	1.364	1.344	1.152	1.145	0.2269
Kirchhoff solution [9]				1.16		
Ref. [2]	1.346					

TABLE V: INFLUENCE OF L/H ON THE CENTRAL DEFLECTION FOR SIMPLY SUPPORTED PLATES

L/h	W_c/W_{ref}					
	Uniform load			Concentrated load		
	SBP8C	SBH8 [8]	DBB8	**SBP8C**	SBH8[8]	DBB8
5	1.2067	1.2024	1.2016	1.739	1.7317	1.7338
10	1.0522	1.0466	1.0273	1.1866	1.1759	1.1586
20	1.0143	1.0074	0.9206	1.0456	1.0363	0.9473
40	1.0019	0.9975	0.7027	1.0086	1.0008	0.6987
50	1.000	0.996	0.6000	1.0038	0.9959	0.5919
100	0.9931	0.9924	0.1936	0.9895	0.9871	0.1956
W_{ref}	$0.406x10^{-2}qL^4/D$			$1.16x10^{-2}PL^2/D$		

V. CONCLUSION

The present element (**SBP8C**) passes the plate patch tests. Numerical results obtained using this element tends to agree well with those from other investigations and theoretical results for both thin and thick plates. The robustness of the present element was demonstrated. The plate bending can be very well simulated with a simple parallelepiped element (**SBP8C**) based on the strain approach.

The performance of this element has been demonstrated in plate bending, and the advantages of using the strain approach are again confirmed.

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