A $Geo_m/G/1/n$ Queueing System with $LIFO$ Discipline, Service Interruptions and Resumption, and Restrictions on the Total Volume of Demands

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Abstract— **Consideration is given to a discrete-time queueing system with inverse discipline, service interruption and resumption, second-order geometrical demand arrival, arbitrary (discrete) distribution of demand length and finite storage. Each demand entering the queue has random volume besides its length. The total volume of the demands in the queue is limited by a certain number. Formulae for the stationary probabilities of states and the stationary waiting time distribution in the queuing system are obtained.**

Keywords: ***queueing system, discrete time, finite buffer, the demand length and volume***

1 Introduction

In many service facilities, in particular in modern informationand computer complexes, along with the restriction on the total volume of demands the restrictions on other parameters are assigned as well. The most common situation occurs when each demand has some random volume (for instance, each program requires a certain memory size for its performance) and the newly arriving demands can be accepted for serving only in case when their volume does not surpass resources available in the server.

Despite doubtless importance of researching queueing systems (QS) with the restriction on the total volume of demands, there are very few works devoted to development of analytical methods of calculation of such systems ([1], [2], [3]). This happens because, for a correct construction of a Markovian process, one should take into account the volumes of all the demands in the system that in a sense makes QS with restriction on the total volume of demands related to multilinear QS (studying of the latter system also meets similar difficulty).

However, if we consider the inversion service order ($LIFO$) in the systems with the restriction on the total volume of demands, then, as was shown in [4], [5], [6], it would be possible to receive suitable-to-use algorithms for the calculation of stationary characteristics. In these works three variants of $LIFO$ discipline were considered: disciplines with service interruption and service resumption, with service interruption and repeat again service, and without service interruption. Besides, in [7] the results of calculations performed on the basis of the obtained formulae were presented.

It should be noted that all the quoted papers examined continuous-time QS while one of the distinctive features of modern information-telecommunication systems is the universal implementation of digital technologies which generates a need for studying discrete-time QS. Studying discrete-time queueing systems in many cases involves additional difficulties related to the possibility of simultaneous occurrence of several events.

In the present article consideration is given to QS $Geo_m/G/1/n$ with $LIFO$ discipline functioning in discrete time and restriction on the total volume of demands, which is similar to QS $M_l/G/1/n$ functioning in the continuous time and which was described in [6]. The relations are obtained allowing the calculation of the basic stationary characteristics of this system.

2 System description

Let us consider a unilinear discrete-time QS $Geo_m/G/1/n$ with n, $0 \le n < \infty$, waiting spaces, and with second kind Geometrical input flow, i.e. a demand flow with a_m, $0 \le m \le n+1$, the probability of a demand arrival during a time slot depends on the number m of demands residing in the QS just before the slot begins.

Each demand arriving in the system along with its length

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has random positive volume. Joint distribution of the demand length and volume is defined by the probability $B_k(x)$ of the event that the demand length (the number of service time slots) is equal to $k,\ k \geq 0$, and its volume does not exceed x. We shall assume the fulfillment of natural condition that the length of the demand cannot be equal to zero i.e. $B_0(x) = 0$ for any x. For the sake of simplicity we shall suppose that for each $k,\ k \geq 1$, there exists the derivative $b_k(x) = B'_k(x)$ (the density function of the volume).

The total volume of demands in the queueing system is limited by a (nonrandom) number $Y,\ 0 < Y < \infty$. If the aggregate volume of arriving demand and the demands in the system exceeds Y, then the arriving demand will be lost. The arriving demand will be necessarily lost also if at the moment of arrival it finds in the queue n other demands (and in this moment the demand serving is not finished at the server). For definiteness we shall assume that if at the moment of a new demand arrival the served demand leaves the queuing system, then its volume will be disregarded while defining the aggregate volume.

The inversion service order with service interruption and resumption is realized in the system whereby a demand accepted by the queuing system is directed to the server and drives out the demand being served to the beginning of the queue. In particular, if at the moment of new demand arrival the served demand leaves the system, then upon availability of free volume a new demand is directed to the server. The service of an interrupted demand will be resumed.

3 Stationary probabilities of conditions

Denote the stationary probability of absence of demands in the queuing system through p_0. Let $p_{k,i}(y_1,\ldots,y_i)$, $i = \overline{1,n+1},\quad k \geq 0$, be the stationary density of probabilities (with arguments $y_1,\ldots,y_i$) of the event that there are i demands in the system, and the (served) length and the volume of the demand at the server are equal to k and y_1 respectively, and the volumes of other demands situated in the system are equal (in order of the queue) to $y_2,\ldots,y_i$.

As the total amount of demands situated in the system ranges between 0 and Y, then $p_{k,i}(y_1,\ldots,y_i) = 0$ at $(y_1,\ldots,y_i) \notin D_i$, where D_i is a simplex limited by hyperplanes $y_k = 0,\ \ k = \overline{1,i}$, and $y_1 + \ldots + y_i = Y$. Therefore, without paying special attention to it we shall assume henceforth $(y_1,\ldots,y_i) \in D_i,\ \ i = \overline{1,n+1}$.

Let also:

$b(y) = \sum\limits_{k=0}^{\infty} b_k(y)$ be the density function of the demand volume distribution;

$B(y) = \int\limits_0^y b(u)\,du = \int\limits_0^y \sum\limits_{k=0}^{\infty} b_k(u)\,du = \sum\limits_{k=0}^{\infty} B_k(u)$ be the demand volume distribution function;

$b(k|y) = b_k(y)/b(y),\ k \geq 1$, be the conditional probability that demand length is equal to k given that its volume is equal to y;

$B(k|y) = \sum\limits_{i=k}^{\infty} b(i|y),\ k \geq 1$, be the conditional probability that the demand length is no less than k given that its volume is equal to y;

$\beta(s|y) = \sum\limits_{k=1}^{\infty} s^k b(k|y)$ be the generating function of the demand length given that its volume is equal to y;

$m_y = \sum\limits_{k=1}^{\infty} kb(k|y) = \sum\limits_{k=1}^{\infty} B(k|y)$ be the conditional mathematical expectation of the demand length given that its volume is equal to y.

Since for the QS under discussion the necessary and sufficient condition of the stationary mode existence has a complicated form, we will bring forward the following simple sufficient condition: $m_y \leq C < \infty$.

By using the method of states elimination (see. [8], p. 22), we obtain the system of equations

$$p_{k,i}(y_1,\ldots,y_i) = \frac{B(k+1|y_1)}{B(k|y_1)}\,p_{k-1,i}(y_1,\ldots,y_i),$$

$$i = \overline{1,n+1},\quad k \geq 1, \tag{1}$$

with the initial condition

$$p_{0,1}(y_1)$$

$$= a_0 b(y_1) p_0 + a_1 b(y_1) \int\limits_0^Y \sum_{k=1}^{\infty} \frac{b(k|y)}{B(k|y)}\,p_{k-1,1}(y)\,dy, \tag{2}$$

$$p_{0,i}(y_1,\ldots,y_i)$$

$$= a_{i-1} b(y_1) \sum_{k=1}^{\infty} \frac{B(k+1|y_2)}{B(k|y_2)}\,p_{k-1,i-1}(y_2,\ldots,y_i)$$

$$+\ a_i b(y_1) \int\limits_0^{Y-y_2-\ldots-y_i} \sum_{k=1}^{\infty} \frac{b(k|y)}{B(k|y)}\,p_{k-1,i}(y,y_2,\ldots,y_i)\,dy,$$

$$i = \overline{2,n+1}. \tag{3}$$

From equations (1) we deduce:

$$p_{k,i}(y_1,\ldots,y_i) = B(k+1|y_1)\,p_{0,i}(y_1,\ldots,y_i),$$

$$i = \overline{1,n+1},\quad k \geq 1. \tag{4}$$

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By plugging into the initial conditions (2) and (3) their values from formula (4) instead of $p_{k,i}(y_1,\dots,y_i)$, $i=\overline{1,n+1}$, $k\ge 1$, we obtain:

$$p_{0,1}(y_1)=a_0 b(y_1)p_0+a_1 b(y_1)\int_0^Y p_{0,1}(y)\,dy, \quad (5)$$

$$p_{0,i}(y_1,\dots,y_i)=a_{i-1}b(y_1)(m_{y_2}-1)p_{0,i-1}(y_2,\dots,y_i) + a_i b(y_1)\int_0^{Y-y_2-\dots-y_i} p_{0,i}(y,y_2,\dots,y_i)\,dy,$$
$$i=\overline{2,n+1}. \quad (6)$$

Solving equations (5) and (6), we obtain:

$$p_{0,1}(y_1)=\frac{a_0 b(y_1)}{1-a_1B(Y)}p_0, \quad (7)$$

$$p_{0,i}(y_1,\dots,y_i) =\frac{a_{i-1}b(y_1)(m_{y_2}-1)}{1-a_iB(Y-y_2-\dots-y_i)}p_{0,i-1}(y_2,\dots,y_i),$$
$$i=\overline{2,n+1}. \quad (8)$$

Therefore, formulae (7), (8) and (4) allow to calculate recurrently by i from $i=1$ up to $n+1$ the stationary probability densities $p_{k,i}(y_1,\dots,y_i)$, $i=\overline{1,n+1}$, $k\ge 0$, accurate to the stationary probability p_0; implementation of normalization condition is required for defining p_0:

$$p_0+\sum_{i=1}^{n+1}\int\cdots\int_{y_1+\dots+y_i\le Y}\sum_{k=0}^{\infty}p_{k,i}(y_1,\dots,y_i)\,dy_1\cdots dy_i$$
$$=p_0+\sum_{i=1}^{n+1}\int\cdots\int_{y_1+\dots+y_i\le Y} m_{y_1}p_{0,i}(y_1,\dots,y_i)\,dy_1\cdots dy_i=1. \quad (9)$$

As a rule for practical calculations it is sufficient to know only the volume of the demand at the server and the aggregate volume of all demands situated in the queue. Then denoting by

$$p_{k,i}(y,z)=\int\cdots\int_{y_3+\dots+y_i\le z} p_{k,i}(y,z-y_3-\dots-y_i)\,dy_3\cdots dy_i,$$
$$i=\overline{3,n+1},\quad k\ge 0,$$

the stationary density of the probability that there are i demands in the system where the (served) length and volume of the demand at the server are equal to k and y respectively while the total volume of the demands situated in the system is equal to z, we use the same reasoning as before and obtain

$$p_{k,i}(y,z)=B(k+1|y)p_{0,i}(y,z),\quad i=\overline{3,n+1},\quad k\ge 1,$$

where

$$p_{0,i}(y,z)=\frac{a_{i-1}b(y)}{1-a_iB(Y-z)}\int_0^z (m_u-1)p_{0,i-1}(u,z-u)\,du,$$
$$i=\overline{3,n+1}.$$

The recurrent procedure of defining $p_{k,i}(y,z)$ (in view of formula (7) and also formulae (8) at $i=2$ and (4) at $i=1,2$) remains the same. The normalization condition (9) takes the form:

$$p_0+\int_0^Y m_y p_{0,1}(y)\,dy+\sum_{i=2}^{n+1}\iint_{y+z\le Y} m_y p_{0,i}(y,z)\,dy\,dz=1.$$

Let us also put down the expressions for some stationary characteristics related to the stationary state probabilities. The stationary density of probabilities $p_i(y)$, $i=\overline{2,n+1}$, that there are i demands in the system having the total volume y is given by the formula

$$p_i(y)=\int_0^y m_z p_{0,i}(z,y-z)\,dz,\quad i=\overline{2,n+1}.$$

The stationary probability a of demand arrival at a time slot (the demand may be rejected by the system because of the restriction on the volume) takes the form

$$a=a_0p_0+\sum_{i=1}^{n+1}a_i\int_0^Y p_i(y)\,dy.$$

The stationary density of probabilities $p^*_{k,1}(y)$, $k\ge 1$, that the arriving demand (which won't be necessarily accepted by the system) will find in the system one demand of (served) length k and volume y, and the stationary density of the probabilities $p^*_{k,i}(y,z)$, $i=\overline{2,n+1}$, $k\ge 1$, that the arriving demand will find in the system i other demands, where the length and volume of the demand at the server are equal to k and y respectively, and the aggregate volume of other demands residing in the system is equal to z, are determined by the expressions:

$$p^*_{k,1}(y)=B(k+1|y)p_{0,1}(y)=p_{k,1}(y),\quad k\ge 1,$$
$$p^*_{k,i}(y,z)=B(k+1|y)p_{0,i}(y,z)=p_{k,i}(y,z),$$
$$i=\overline{2,n+1},\quad k\ge 1.$$

The stationary probability p^*_0 that at the moment of a new demand arrival the server will finish serving a single demand situated in the system, and the stationary density of probabilities $p^*_i(y)$, $i=\overline{0,n}$, that at the moment of new demand arrival the server will finish

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serving a demand, and i other demands with the total volume y will remain in the system, are given by the formulae

$$p_0^* = \int_0^Y p_{0,1}(z)\,dz,$$

$$p_i^*(y) = \int_0^{Y-y} p_{0,i+1}(z,y)\,dz, \quad i=\overline{1,n}.$$

The stationary probabilities $\pi_y \quad (y<Y)$ and π that the arriving demand of volume y and the arriving demand of arbitrary volume will be admitted to the system, have the form

$$\pi_y = \frac{1}{a}\left(a_0p_0 + a_1p_0^* + \sum_{i=1}^n a_{i+1}\int_0^{Y-y} p_i^*(z)\,dz \right.$$
$$+ a_1\int_0^{Y-y}(m_z-1)p_{0,1}(z)\,dz$$
$$\left. + \sum_{i=2}^n a_i \iint\limits_{z+u\le Y-y}(m_z-1)p_{0,i}(z,u)\,dz\,du\right),$$

$$\pi = \frac{1}{a}\left((a_0p_0+a_1p_0^*)B(Y)\right.$$
$$+\sum_{i=1}^n a_{i+1}\int_0^Y B(Y-y)p_i^*(y)\,dy$$
$$+ a_1\int_0^Y B(Y-y)(m_y-1)p_{0,1}(y)\,dy$$
$$\left.+\sum_{i=2}^n a_i\iint\limits_{y+z\le Y} B(Y-y-z)(m_y-1)p_{0,i}(y,z)\,dy\,dz\right).$$

4 Stationary distribution of demand stay time in the system

In this section we shall present a calculation algorithm in terms of the generating function of the stationary distribution of the demand sojourn time in the system. We shall call an (i,z)-system a system similar to the initial one, but with $n-i, \ 0\le i\le n$, waiting places, restriction $z, \ 0\le z\le Y$, on the total volume of demands and probability $\hat{a}_m = a_{m+i}, \ m=\overline{0,n-i+1}$, of demand arrival at time slot given there are m demands in (i,z) system. It is easy to see that an (i,z)-system represents the initial queueing system but with the reqirement that it permanently contains i demands with the total volume $Y-z$.

It is convenient to consider that the busy period (BP) of an (i,z)-system ends at the moment of departure of the demand which had been the first at the server, even if at the same time a new demand is arriving in the system. Besides, we will consider that if a new demand having the volume greater than z arrives in the free (i,z)-system then it opens the BP of zero length.

Let:

$\varphi(s|k,y;i,z), \ i=\overline{0,n}, \ k\ge 1$, be the generating function of BP of (i,z)-system opened by the demand of (residual) length k and volume y;

$\varphi(s|i,z), \ i=\overline{0,n}$, be the generating function of BP of (i,z)-system opened by the demand of arbitrary length and arbitrary volume.

Then for $\varphi(s|k,y;i,z)$ and $\varphi(s|i,z)$ the following recurrent relations are valid:

$$\varphi(s|k,y;n,z) = s^k, \quad k\ge 1, \quad y\le z,$$

$$\varphi(s|k,y;i,z) = 1, \quad k\ge 1, \quad y>z, \quad i=\overline{0,n},$$

$$\varphi(s|n,z) = \sum_{k=1}^\infty \int_0^\infty \varphi(s|k,y;n,z)b_k(y)\,dy$$
$$= 1-B(z) + \int_0^z \beta(s|y)b(y)\,dy,$$

$$\varphi(s|k,y;i,z)$$
$$= s^k\sum_{l=0}^{k-1}\binom{k-1}{l}a^l\overline{a}^{k-1-l}\varphi^l(s|i+1,z-y)^l$$
$$= s^k(\overline{a}+a\varphi(s|i+1,z-y))^{k-1},$$
$$k\ge 1, \quad y\le z, \quad i=\overline{0,n-1},$$

$$\varphi(s|i,z) = \sum_{k=1}^\infty\int_0^\infty \varphi(s|k,y;i,z)b_k(y)\,dy$$
$$= 1-B(z) + \int_0^z \frac{\beta(s(\overline{a}+a\varphi(s|i+1,z-y))|y)}{\overline{a}+a\varphi(s|i+1,z-y)}b(y)\,dy,$$
$$i=\overline{0,n-1}.$$

Finally, denoting the generating function of sojourn time of the demand with length k and volume $y \quad (y\le Y)$ accepted for serving by $g(s|k,y)$, and the generating function of sojourn time of arbitrary demand accepted for serving by $g(s)$, we obtain the following result:

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$$g(s|k,y) = \frac{1}{a\pi_y}\Bigg((a_0 p_0 + a_1 p_0^*)\varphi(s|k,y;0,Y)$$

$$+ \sum_{i=1}^{n} a_{i+1} \int_0^{Y-y} \varphi(s|k,y;i,z) p_i^*(z)\,dz$$

$$+ a_1 \int_0^{Y-y} (m_z - 1)\varphi(s|k,y;i,z) p_{0,1}(z)\,dz$$

$$+ \sum_{i=2}^{n} a_i \iint\limits_{z+u\le Y-y} (m_z - 1)\varphi(s|k,y;i,z+u) p_{0,i}(z,u)\,dz\,du\Bigg),$$

$$k \ge 1,$$

$$g(s) = \frac{1}{\pi}\sum_{k=1}^{\infty}\int_0^{Y} \pi_y b_k(y) g(s|k,y)\,dy.$$

It is obvious that in the queueing system under consideration the waiting time for the start of a demand serving is equal to zero, and the sojourn time of a demand at the server (taking into account possible interruptions of service) coincides with the total sojourn time of a demand in the system.

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